Name...
I.D. number..

Math 2270-2
Sample Final Exam
December 2001

This exam is closed-book and closed-note. You may not use a calculator which is capable of doing linear algebra computations. In order to receive full or partial credit on any problem, you must show all of your work and **justify your conclusions.** There are 200 points possible, and the point values for each problem are indicated in the right-hand margin. Of course, this exam counts for 30% of your final grade even though it is scaled to 200 points. Good Luck!

1) Let

$$A := \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Then T(**x**)=A**x** is a matrix map from **R**^3 to **R**^2.
1a) Find the four fundamental subspaces associated to the matrix A.

(20 points)

1b) State and verify the theorem which relates rank and nullity of A, in this particular case

(5 points)

1c) Find an orthonormal basis for R^3 in which the first two vectors are a basis for the rowspace of A and the last vector spans its nullspace.

(10 points)

2a) Exhibit the rotation matrix which rotates vectors in R^2 by an angle of α radians in the counter-clockwise direction.

(5 points)

2b) Verify that the product of an α-rotation matrix with a β-rotation matrix is an (α+β)-rotation matrix.

(10 points)

3) Let T be the linear map from the complex plane to the complex plane, defined by

$$\mathrm{T}(z) = (1+i)\, z$$

We can consider the complex numbers as a real vector space of dimension 2, with basis β={1,i}. Find the matrix for T with respect to this basis, and explain what T does geometrically.

(10 points)

4a) Explain the procedure which allows one to convert a general quadratic equation in n-variables

$$x^T A\, x + B\, x + c = 0$$

into one without any ''cross terms''. Be precise in explaining the change of variables, and the justification for why such a change of variables exists.

(10 points)

4b) Apply the procedure from part (4a) to put the conic section

$$6\, x^2 + 9\, y^2 - 4\, x\, y + 4\sqrt{5}\, x - 18\sqrt{5}\, y = 5$$

into standard form. Along the way, identify the conic section.

(20 points)

5) Let

$$C := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

5a) Find the inverse of C using elementary row operations.

(15 points)

5b) Find the inverse of C using the adjoint formula.

(15 points)

6a) Define what it means for a tranformation (function) T:V-->W between vector spaces to be a **linear transformation**.

(4 points)

6b) Define what it means for a set S={$\mathbf{v}_1,\mathbf{v}_2, \ldots \mathbf{v}_n$} to be a **basis** for a vector space V.

(3 points)

6c) For a linear map T as in part (6a), define the **kernel** of T.

(3 points)

6d) Let β={$\mathbf{v}_1,\mathbf{v}_2, \ldots \mathbf{v}_n$} be a basis for V, and let T:V-->V be linear. Explain what the matrix for T with respect to β is, and what it does.

(5 points)

7) Let

$$A := \begin{bmatrix} -7 & -6 & 5 \\ 4 & 4 & -2 \\ -6 & -5 & 6 \end{bmatrix}$$

7a) Let

$$\beta := \left[\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right]$$

be a basis for R^3. (Note these are the columns of your matrix C from #5.) Let T(x)=Ax, a linear map from R^3 to R^3. Thus A is the matrix of T with respect to the standard basis of R^3. Find the matrix of T with respect to the basis β.

(15 points)

7b) Let

$$v := 2\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Compute T(v) two ways: once using the matrix for T with respect to β, and once using the matrix A. Verify that your answers agree.

(10 points)

8) True-False. Four points each (2 points for answer, two points for justification.)

(40 points)

8i) Let A be an n by n matrix. Then the equation Ax=Ay implies that x=y.

8ii) If A and B are n by n matrices, then

$$(A+B)^2 = A^2 + 2\,A\,B + B^2$$

8iii) If A is any matrix with (real entries) then the product

$$B = A^T A$$

has only real eigenvalues, and is in fact similar to a diagonal matrix.

8iv) Every set of five orthonormal vectors in **R**^5 is automatically a basis for **R**^5.

8v) The number of linearly independent eigenvectors of a matrix is always greater than or equal to the number of distinct eigenvalues.

8vi) If the rows of a 4 by 6 matrix are linearly dependent then the nullspace is at least three dimensional.

8vii) A diagonalizable n by n matrix must always have n distinct eigenvalues.

8viii) The equation

$$2\,x^2 + 5\,x\,y + 3\,y^2 = 1$$

defines an ellipse.

8ix) If A and B are orthogonal matrices then so is AB.

8x) If A is a square matrix and A^2 is singular, then so is A.