2270-2

Practice Exam #2

October 26, 2001

This exam is closed-book and closed-note. You may not use a calculator which is capable of doing linear algebra computations. In order to receive full or partial credit on any problem, you must show all of your work and **justify your conclusions.** There are 100 points possible, and the point values for each problem are indicated in the right-hand margin. Good Luck!

Well, actually, for this practice exam there are more than 100 points.

1a) Show that the following matrix equation has no solution

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$$

(5 points)

I could write down the augmented matrix, row reduce, and show that I get an inconsistent equation 0=1. Or, for this simple system I can argue directly: he last equation says x2=4, the second equation says x1=0, but then the first equation would say that 1(0)+2(4)=2, which is false.

1b) Find the least-squares solution to the problem in part (a).

(10 points)

I solve the equation (A^T)Ax=(A^T)b:

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

2) Let W be the span of the vectors {[1,-1,0],[2,0,1]} in 3-space.

2a) Find an orthonormal basis for the plane W.

(10 points)

Use Gram-Schmidt. I'm in Maple, you would do this by hand.

```
> v1:=vector([1,-1,0]);
```

```
v2:=vector([2,0,1]);
w1:=1/sqrt(2)*v1;
   #first orthogonal element
z2:=v2-(dotprod(v2,w1))*w1;
   #second orthogonal element;
w2:=1/norm(z2,2)*z2;
   #second orthonormal element
[evalm(w1),evalm(w2)];#orthonormal basis,
   #I used brackets because maple disorders
   #sets, i.e. {x,y}={y,x}
```

$$v1 := [1, -1, 0]$$

$$v2 := [2, 0, 1]$$

$$w1 := \frac{1}{2}\sqrt{2}\ v1$$

$$z2 := v2 - v1$$

$$w2 := \frac{1}{3}\sqrt{3}\ (v2 - v1)$$

$$\left[\left[\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}, 0\right], \left[\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}\right]\right]$$

So, writing things vertically as we are used to doing, this would read

$$\left\{\frac{1}{2}\sqrt{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{3}\sqrt{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$$

2b) Let v=[2,0,4]. Find the projection of v onto the subspace W, using your answer from (2 a)

(10 points)

```
> v:=vector([2,0,4]);
  proj(v):=dotprod(v,w1)*w1+dotprod(v,w2)*w2;
  evalm(proj(v));
```

$$v := [2, 0, 4]$$

$$\mathrm{proj}(v) := -v1 + 2\ v2$$

$$[3, 1, 2]$$

So the projection vector is [3,1,2]

2c) If you did your computations correctly, your answer to part (2b) should equal the matrix from number (1) multiplied by your least squares solution [x1,x2] to part (1b),

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \end{bmatrix}$$

Explain why.

(5 points)

Check:

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

The reason for this is that the least squares solution to an inconsistent matrix system Ax=b IS the solution to Ax=proj(b) onto the image of A. The image of A in #1 is the matrix A's column space, which is the subspace W of #2. Thus Ax gives the projection vector worked out in #2b

3a) Define what a linear transformation L:V->W is.

(5 points)

L is ***linear*** *means that for all u,v in V and all scalars c*
(A) L(u+v)=L(u)+L(v), and
(B) L(cu)=cL(u).

3b) Define the kernel and image, for a linear transformation L

(5 points)

kernel(L):={v in V such that L(v)=0}
image(L):={w in W such that w=L(v) for some v in V}

3c) Prove that the kernel of a linear transformation is a subspace.

(5 points)

We need to verify that kernel(L) is closed under addition and scalar multiplication:
(A) Closure under addition: Let u and v be in ker(L). This means L(u)=0, L(v)=0. Therefore, L(u+v)=L(u)+L(v)=0, so u+v is also in the kernel of L.
(B) Closure under scalar multiplication: Let u be in ker(L). Then L(cu)=cL(u)=c0=0, so cu is in ker(L).

4) Let P2 be the space of polynomials in "t" of degree at most 2. Define T:P2->P2 by

T(f) = f'' + 4f

4a) Find the matrix for T, with respect to the standard basis {1,t,t^2}.

(5 points)

Write B for the matrix of T with respect to our basis. The jth column of B is the coordinate vector for T of the jth basis element. So we compute:
T(1)=4 .... coord vector is [4,0,0]
T(t)=4t .....coord vector is [0,4,0]
T(t^2)=2+4t^2 ....coord vector is [2,0,4].
Thus

$$B := \begin{bmatrix} 4 & 0 & 2 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

>

4b) Find bases for image(T) and Kernel(T).

(10 points)

Since the matrix of T has the property that when we multiply it by the coordinates of a vector v, we get back the coordinates of T(v), we work with B:

For ker(T) we want to know what input coords give output coords equal to zero. Since B row reduces to the identity, the only such input coords are c1=c2=c3=0. Thus ker(T)={0}. The zero-dimensional subspace has empty basis.

For image(T) we want to know what output coords be can get, i.e. we want all possible Bc. Since B row reduces to the identity, we can solve Bc=b for c, given any b. Thus image(T) is all of P2. So we may take our P2 basis {1,t,t^2} as a basis for image(T).

4c) verify the rank+nullity theorem for T

(5 points)

rank + nullity = domain dimension.
3+0=3

5) Let

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

be a linear (matrix) map from R^2 to R^2. Let B={[1,1],[-1,1]} be a non-standard basis for R^2. Find the matrix for L with respect to the basis B.

(20 points)

Method 1:
Write v1=[1,1], v2=[-1,1]. The columns of the matrix are the B-coords of L(v1), and L(v2).

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}=\begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Since the B-coords of [4,4] are [4,0], and the B-coords of [2,-2] are [0,-2], the matrix of L with respect to our non-standard basis is

$$\begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$$

Method 2:
Use transition matrices. If we call the matrix with respect to the nonstandard matrix B, then the transition matrix which converts B-coords to E-coords, S=$\mathbf{P}_{E\leftarrow B}$, just has the E-coords of the B basis in its columns, i.e.

$$S := \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

The matrix with respect to the standard basis, A, was given in the problem:

$$A := \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

*The relation between A and B is (check!) B=inverse(S)*A*S:*

$$B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$$

6) Find the least squares line fit for the following collection of 4 points: {[0,0],[1,2],[-1,-1],[-2,-3]}.
(10 points)

We seek the best approximate solution to

```
> m*matrix(4,1,[0,1,-1,-2])+b*matrix(4,1,[1,1,1,1])=matrix(4,1,[0,2,
  -1,-3]);
```

$$m \begin{bmatrix} 0 \\ 1 \\ -1 \\ -2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \\ -3 \end{bmatrix}$$

This is the matrix equation

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \\ -3 \end{bmatrix}$$

The least squares solution solves

$$\begin{bmatrix} 0 & 1 & -1 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -1 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{10} \\ \frac{1}{10} & \frac{3}{10} \end{bmatrix} \begin{bmatrix} 9 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \frac{8}{5} \\ \frac{3}{10} \end{bmatrix}$$

So the least squares line fit is y=1.6x + 0.3

7) Find the orthogonal complement in R^4 to the span of

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(10 points)

We want vectors perpendicular to the basis of V given above. This is exactly the kernel of

$$A := \begin{bmatrix} 1 & 2 & -2 & 1 \\ 1 & -2 & 0 & 1 \end{bmatrix}$$

$$RREF := \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & \frac{-1}{2} & 0 \end{bmatrix}$$

Backsolving for the homogeneous equation we see x4=s, x3=t, x2=(1/2)t, x1=t-s, in vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

So a basis for the orthogonal complement is given by

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

7) True-False: 4 points for each problem; two points for the answer and two points for the reason.

7a) Any collection of 3 polynomials of degree at most 2 must be linearly dependent.

FALSE: P2 is 3 dimensional, so has lots of bases consisting of 3 vectors

7b) If the columns of a square matrix are orthonormal then so are the rows.

*TRUE: We discovered this magic in our discussion of orthogonal matrices ... if A has orthonormal columns, then transpose(A)*A=I, from which it follows that A*(transpose(A))=I, which says that the rows of a are orthonormal.*

7c) The formula

$$[AB]^T = A^T B^T$$

holds for all square matrices

FALSE:

$$[AB]^T = B^T A^T$$

7d) If x and y are any two vectors in R^n, then ||x+y||^2 = ||x||^2 + ||y||^2.

*FALSE: Using the dot product you see that ||x+y||^2 = ||x||^2 + ||y||^2 + 2*dotprod(x,y). You only get the Pythagorean Theorem when x and y are orthogonal.*

7e) If V is a 3-dimensional subspace of R^6, then the orthogonal complement to V is also 3-dimensional.

TRUE: In general the dimension of V plus the dimension of its orthogonal complement add up to the dimension of the Euclidean space. This is because if you take a basis for V and augment it with a basis for V perp, it is "easy" to show you have a basis for R^n.

7f) If A is symmetric and B is symmetric, then so is A+B

TRUE: if aij=aji and bij=bji then aij+bij=aji+bji

7g) The functions f(t)=t^2 and g(t)=t^3 are orthogonal, with respect to the inner product

$$[f, g] = \int_{-1}^{1} \mathrm{f}(t)\, \mathrm{g}(t)\, dt$$

TRUE. If you compute [f,g] you get

$$\int_{-1}^{1} t^5\, dt = 0$$

7h) If a collection of vectors is dependent, then any one of the vectors in the collection may be deleted without shrinking the span of the collection.

FALSE: You may only delete a vector which is dependent on the remaining ones.