Math 2250-004
Week 9: March 5-9 5.1-5.3 linear differential equations of arbitrary order.

Mon March 5:
5.1 Second order linear differential equations, and vector space theory connections.

In Chapter 5 we'll be using vector space theory to understand solutions to differential equations!

Announcements: • Quiz is on Wednesday, as usual (early version of HW said Friday... again)

'til 10:47
Warm-up Exercise: Consider the 2^{nd} order differential equation for $y(x)$:

$$y'' - 2y' - 3y = 0$$

"homogeneous linear differential eqtn"

Show that
$$y_1(x) = e^{3x}$$
$$y_2(x) = e^{-x}$$
each solve this differential equation.

$(e^{3x})'' - 2(e^{3x})' - 3e^{3x} = 9e^{3x} - 2(3e^{3x}) - 3e^{3x} \overset{!}{=} 0$

$(e^{-x})'' - 2(e^{-x})' - 3e^{-x} = e^{-x} + 2e^{-x} - 3e^{-x} \overset{!}{=} 0$

(substitute the functions into the DE. Is the resulting identity true?)

In fact $y(x) = c_1 e^{3x} + c_2 e^{-x}$ also solves this DE!

$$\begin{array}{lrl} -3\,E_1 & -3(y(x) & = c_1 e^{3x} + c_2 e^{-x}) \\ -2\,E_2 & -2(y'(x) & = 3c_1 e^{3x} - c_2 e^{-x}) \\ +1\,E_3 & +1(y''(x) & = 9c_1 e^{3x} + c_2 e^{-x}) \end{array}$$

$$y'' - 2y' - 3y = c_1 e^{3x}(-3-6+9) + c_2 e^{-x}(-3+2+1) = 0 + 0 = 0!$$

In fact the solution space to
$y'' - 2y' - 3y$ is
$$\operatorname{span}\{e^{3x}, e^{-x}\} = \{c_1 e^{3x} + c_2 e^{-x} : c_1, c_2 \in \mathbb{R}\}$$

$\{e^{3x}, e^{-x}\}$ is a basis for the solution space

The two main goals in Chapter 5 are

(1) to learn the structure of solution sets to n^{th} order linear DE's, including how to solve the corresponding initial value problems with n initial values:

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$
$$y''(x_0) = b_2$$
$$\vdots$$
$$y^{(n-1)}(x_0) = b_{n-1}$$

<u>and</u>

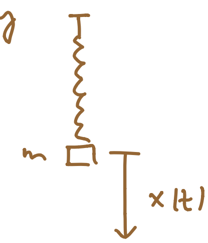
(2) to learn important physics/engineering applications of these general techniques. The applications we learn about in this course will be for second order linear differential equations ($n = 2$), as below.

<u>Applications Example</u> (sections 5.4 and 5.6): The forced-damped-oscillator differential equation. Here $\boxed{x(t)}$ is the function, instead of $\boxed{y(x)}$, as often happens in our textbook when we move between theory and applications. This particular differential equation arises in a multitude of contexts besides just the mass-spring model, as we shall see.

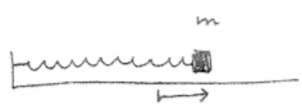
$$\boxed{m x'' + c x' + k x = F(t)}$$
$$\left.\begin{aligned} x(0) &= x_0 \\ x'(0) &= v_0 \end{aligned}\right\} \text{get to specify}$$

x(0) & v(0) before releasing the mass

or



model



x(t) displacement from equilibrium.

equilibrium location

possible external forcing function

Newton's 2^{nd} Law:
$$\boxed{\begin{aligned} m x''(t) &= \text{net forces} \\ &= -c x' - k x + F(t) \end{aligned}}$$

linear drag, proportional to velocity

force from compression or stretching of spring, relative to equilibrium $x = 0$

Start with second order linear differential equations - we'll see that differential equations with (higher) order n follow the same conceptual outline.

<u>Definition:</u> A general *second order linear differential equation* for a function $y(x)$ is a differential equation that can be written in the form

$$A(x)y'' + B(x)y' + C(x)y = F(x)\,.$$

We search for solution functions $y(x)$ defined on some specified interval I of the form $a < x < b$, or (a, ∞), $(-\infty, a)$ or (usually) the entire real line $(-\infty, \infty)$. In this chapter we assume the function $A(x) \neq 0$ on I, and divide by it in order to rewrite the differential equation in the standard form

$$y'' + p(x)y' + q(x)y = f(x)\,.$$

analogous to first order linear differential equations in Chapters 1-2:

$$y' + p(x)y = q(x).$$

<u>Exercise 1)</u> Find all solutions to to second order differential equation for $y(x)$

$$y'' + \underbrace{2}_{P(x)}\, \underbrace{y'}_{} = \underbrace{0}_{Q(x)}$$

1st order DE for y' that we don't anyone to tell us the sol'ns

on the x-interval $-\infty < x < \infty$.

$v(x)$

$v' + 2v = 0$

$e^{\int P(x)\,dx}\left[v' + 2v\right] = 0\, e^{\int P(x)\,dx}$

linear algorithm for 1st order DE.

$$e^{2x}\left[y'' + 2y'\right] = e^{2x}\cdot 0 = 0$$

$$\frac{d}{dx}\left[e^{2x}y'\right] = 0$$

$$\Rightarrow\ e^{2x}y' = \int 0\,dx = C$$

$\div\, e^{2x}$

$$\Rightarrow\ y' = Ce^{-2x}$$

$$\Rightarrow\ y(x) = \int Ce^{-2x}\,dx = -\frac{C}{2}e^{-2x} + D$$

$$\boxed{y(x) = c_1 e^{-2x} + c_2}$$

Solution space : 2 free parameters

equals span $\{e^{-2x}, 1\}$.

Theorem 1 (Existence-Uniqueness Theorem): Let $p(x), q(x), f(x)$ be specified continuous functions on the interval I, and let $x_0 \in I$. Then there is a unique solution $y(x)$ to the initial value problem

$$y'' + p(x)y' + q(x)y = f(x)$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$

and $y(x)$ exists and is twice continuously differentiable on the entire interval I.

Example For the forced mass-spring model this would be saying that once you specify the initial displacement and velocity of the mass, and given the values of m, c, k and the known forcing function, the future motion of the mass is uniquely determined .... i.e. the experiment is repeatable with the same result. This make intuitive sense.

Exercise 2) Verify Theorem 1 for the interval $I = (-\infty, \infty)$ and the IVP

$$y'' + 2y' = 0$$
$$y(0) = b_0$$
$$y'(0) = b_1$$

Exercise1 $y(x) = c_1 e^{-2x} + c_2$ all solns

$y'(x) = -2c_1 e^{-2x}$

given b_0 & b_1, trying to find c_1 & c_2.

$$y(0) = c_1 + c_2 = b_0$$
$$y'(0) = -2c_1 = b_1$$

$$\begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \quad *$$

det = 2
so inverse exists

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

unique $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ for given $\begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$.

The real reason that differential equations of the form

$$y'' + p(x)y' + q(x)y = f(x)$$

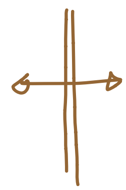
are called *linear* is that the "linear operator" L that operates on functions by the rule

$$L(y) \coloneqq y'' + p(x)y' + q(x)y$$

satisfies the so-called *linearity properties*

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(2) $L(c\,y) = c\,L(y),\ c \in \mathbb{R}$.



Chapter 3-4.

$A(\vec{x} + \vec{z}) = A\vec{x} + A\vec{z}$

$A(c\vec{x}) = c A\vec{x}$.

(Recall that the matrix multiplication function $L(\underline{\boldsymbol{x}}) \coloneqq A\,\underline{\boldsymbol{x}}$ satisfies the analogous properties. Any time we have have a transformation L satisfying (1),(2), we say it is a *linear transformation.*)

<u>Example</u>: If

$$L(y) \coloneqq y'' + 2\,y',$$

and

$$y_1(x) = x^2,\ y_2(x) = e^{3x},\ y_3(x) = e^{-2x}$$

we can compute at x,

$$L(x^2) = (x^2)'' + 2(x^2)' = 2 + 2(2x) = 2 + 4x$$

$$L(y_1)(x) = 2 + 2\cdot 2\,x = 2 + 4\,x$$

$$L(e^{3x}) = (e^{3x})'' + 2(e^{3x})' = 9e^{3x} + 2(3e^{3x}) = 15e^{3x}$$

$$L(y_2)(x) = 9\,e^{3x} + 2\cdot 3\,e^{3x} = 15\,e^{3x}$$

$$L(e^{-2x}) = (e^{-2x})'' + 2(e^{-2x})' = 4e^{-2x} + 2(-2e^{-2x}) = 0$$

$$L(y_3)(x) = 4\,e^{-2x} + 2\cdot 2\,e^{-2x} = 0.$$ (<u>We knew that!</u>)

$$L(y_1 + y_2)(x) = (2 + 9\,e^{3x}) + 2(2\,x + 3\,e^{3x})$$
$$= L(y_1)(x) + L(y_2)(x)\ !$$

$$L(5\,y_1)(x) = (10 + 2\cdot 10\,x) = 5\,L(y_1)(x).$$

Exercise 3a) Check the linearity properties (1),(2) for the general second order differential operator L and general functions $y_1(x), y_2(x)$. Compare to matrix multiplication properties. In other words show that the operator L defined by

$$\boxed{L(y) := y'' + p(x)y' + q(x)y}$$

Chptr 3-4 analog.

$A(\vec{x}+\vec{z}) = A\vec{x} + A\vec{z}$

$A(c\vec{x}) = cA\vec{x}$

satisfies the so-called *linearity properties*

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(2) $L(cy) = cL(y),\ c \in \mathbb{R}$.

(1) $L(y_1+y_2) = (y_1+y_2)'' + p(x)(y_1+y_2)' + q(x)(y_1+y_2)$

$= (y_1''+y_2'') + p(x)(y_1'+y_2') + q(x)(y_1+y_2)$

$= y_1'' + p(x)y_1' + q(x)y_1 + y_2'' + p(x)y_2' + q(x)y_2$

$= L(y_1) + L(y_2)$

Tuesday:

(2) $L(cy) = (cy)'' + p(x)(cy)' + q(x)(cy)$

$= cy'' + p(x)cy' + q(x)cy = c\left[y'' + p(x)y' + q(x)y\right]$

$= cL(y)$

3b) Use these properties to show that what happened in Exercise 2b was no accident:

Theorem 2: the solution space to the *homogeneous* second order linear DE

$$L(y) = y'' + p(x)y' + q(x)y = 0$$

is a subspace. Notice that this is the analogous proof we used earlier to show that the solution space to a homogeneous matrix equation is a subspace.

α) closed under +

β) closed under scalar mult

proof α) let $L(y_1) = 0$

$L(y_2) = 0$

so $L(y_1+y_2) = L(y_1) + L(y_2) = 0 + 0 = 0$!

so y_1+y_2 solves the homog. DE too

β) Let $L(y) = 0$

$L(cy) = cL(y) = c\cdot 0 = 0$ so cy solves the DE too.

Chptr 4

Soln space to $A\vec{x} = \vec{0}$

Unlike in the previous example, and unlike what was true for the first order linear differential equation

$$y' + p(x)y = q(x)$$

there is not a clever integrating factor formula that will always work to find the general solution of the second order linear differential equation

$$y'' + p(x)y' + q(x)y = f(x).$$

Rather, we will usually resort to vector space theory and algorithms based on clever guessing, as in the following example:

Exercise 4) Consider the homogeneous linear DE for $y(x)$

$$y'' - 2y' - 3y = 0$$

4a) Find two exponential functions $y_1(x) = e^{rx}$, $y_2(x) = e^{\rho x}$ that solve this DE.

$y' + ay = 0$
$y' = -ay$
$y = Ce^{-ax}$

exponentials worked for 1st order, const coeff homog. linear DE's, so why not try them for higher order.

try e^{rx}

$y = e^{rx}$
$y' = re^{rx}$
$y'' = r^2e^{rx}$

$$y'' - 2y' - 3y = r^2e^{rx} - 2(re^{rx}) - 3e^{rx}$$
$$= e^{rx}\left[r^2 - 2r - 3\right] \overset{\text{want}}{=} 0$$

zero fcn

$$= e^{rx}\left[(r-3)(r+1)\right]$$

$r = 3, -1$ will work.

Monday warmup explained.

So e^{3x}, e^{-x} are solns.

Since soln space is sub vector space.
c_1e^{3x}, c_2e^{-x}, $c_1e^{3x} + c_2e^{-x}$ are solns!

$$y(x) = c_1e^{3x} + c_2e^{-x}$$

4b) Show that every IVP

Tuesday's warm-up

$$y'' - 2y' - 3y = 0$$
$$y(0) = b_0$$
$$y'(0) = b_1$$

can be solved with a unique linear combination $y(x) = c_1 y_1(x) + c_2 y_2(x)$, (where c_1, c_2 depend on b_0, b_1).

$= c_1 e^{3x} + c_2 e^{-x}$

Then use the uniqueness theorem to deduce that y_1, y_2 span the solution space to this homogeneous differential equation. Since these two functions are not constant multiples of each other, they are linearly independent and a basis for the *2-dimensional* solution space!

Some important facts about spanning sets, independence, bases and dimension follow from one key fact, and then logic. We will want to use these facts going forward, in our study of differential equations.

key fact: If n vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ span a vector space W then any collection $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_N$ of vectors in W with $N > n$ will always be linearly dependent. (This is explained on pages 254-255 of the text, and has to do with matrix facts that we already know.) Notice too that this fact fits our intuition based on what we know in the special cases that we've studied, in particular $W = \mathbb{R}^n$.)

Thus:

1) If a finite collection of vectors in W is linearly independent, then no collection with fewer vectors can span all of W. (This is because if the smaller collection did span, the larger collection wouldn't have been linearly independent after all, by the key fact.)

2) Every basis of W has the same number of vectors, so the concept of dimension is well-defined and doesn't depend on choice of basis. (This is because if $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are a basis for W then every larger collection of vectors is dependent by the key fact and every smaller collection fails to span by (1), so only collections with exactly n vectors have a chance to be bases.)

3) Let the dimension of W be the number n, i.e. there is some basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ for W. Then if vectors $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$ span W then they're automatically linearly independent and thus a basis. (If they were dependent we could delete one of the $\underline{\boldsymbol{w}}_j$ that was a linear combination of the others and still have a spanning set. This would violate (1) since $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly independent.)

4) Let the dimension of W be the number n, i.e. there is some basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ for W. Then if $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$ are in W and are linearly independent, they automatically span W and thus are a basis. (If they didn't span W we could augment with a vector $\boldsymbol{w}_{n+1}$ not in their span and have a collection of $n+1$ still independent* vectors in W, violating the key fact.

* Check: If $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$ are linearly independent, and $\underline{\boldsymbol{w}}_{n+1} \notin span\{\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n\}$, then $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n, \underline{\boldsymbol{w}}_{n+1}$ are also linearly independent. This fact generalizes the ideas we used when we figured out all possible subspaces of $\mathbb{R}^3$. Here's how it goes:

To show the larger collection is still linearly independent study the equation

$$c_1\underline{\boldsymbol{w}}_1 + c_2\underline{\boldsymbol{w}}_2 + \ldots + c_n\underline{\boldsymbol{w}}_n + d\,\underline{\boldsymbol{w}}_{n+1} = \underline{\boldsymbol{0}}.$$

Since $\underline{\boldsymbol{w}} \notin span\{\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n\}$ it must be that $d = 0$ (since otherwise we could solve for $\underline{\boldsymbol{w}}_{n+1}$ as a linear combination of $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$). But once $d = 0$, we have

$$c_1\underline{\boldsymbol{w}}_1 + c_2\underline{\boldsymbol{w}}_2 + \ldots + c_n\underline{\boldsymbol{w}}_n = \underline{\boldsymbol{0}}$$

which implies $c_1 = c_2 = \ldots = c_n = 0$ by the independence of $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$.

Tues March 6:

5.1-5.2 Second order and n^{th} order linear differential equations, and vector space theory connections.

Announcements:
- office hours 4:30-6:00 JWB 240 (Chapter 4 is hard).
- finish Monday's & Tuesday's notes

'til 10:47

Warm-up Exercise: a) Solve the IVP

a lot like §5.1 HW

$$\begin{cases} y'' - 2y' - 3y = 0 \\ y(0) = 1 \\ y'(0) = -5 \end{cases}$$

using the D.E. solutions we verified on Monday,

$$y(x) = c_1 e^{3x} + c_2 e^{-x} \quad \boxed{= -e^{3x} + 2e^{-x}}$$ warmup

(find c_1, c_2)

b) Would you have been able to solve with arbitrary initial values

$$y(0) = b_1$$
$$y'(0) = b_2$$

? yes.

$$y(x) = c_1 e^{3x} + c_2 e^{-x}$$
$$y'(x) = c_1(3e^{3x}) + c_2(-e^{-x})$$

(reducing augmented matrix faster)

$$\begin{bmatrix} y(x) \\ y'(x) \end{bmatrix} = \begin{bmatrix} e^{3x} & e^{-x} \\ 3e^{3x} & -e^{-x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

this would've been any $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Wronskian Matrix, $W(e^{3x}, e^{-x})$

$$W(y_1, y_2) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

For IVP $x=0$ @ $x=0$

$$\begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

inverse matrix

$$\frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} = -\frac{1}{4} \begin{bmatrix} 4 \\ -8 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

so $\boxed{y(x) = -e^{3x} + 2e^{-x}}$

Since each IVP can be solved this way, using $c_1 e^{3x} + c_2 e^{-x} = y$, the existence-uniqueness thm for IVP's tells us these are all to solns to DE!!

Theorem 3: The solution space to the second order homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = 0$$

is 2-dimensional on any interval I for which the hypotheses of the existence-uniqueness theorem hold.

proof:

Pick any $x_0 \in I$. Find solutions $y_1(x), y_2(x)$ to initial value problems at x_0 so that the so-called Wronskian matrix for y_1, y_2 at x_0

$$W(y_1, y_2)(x_0) = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}$$

is invertible (i.e. $\begin{bmatrix} y_1(x_0) \\ y_1'(x_0) \end{bmatrix}, \begin{bmatrix} y_2(x_0) \\ y_2'(x_0) \end{bmatrix}$ are a basis for $\mathbb{R}^2$, or equivalently so that the determinant of the Wronskian matrix (called just the Wronskian) is non-zero at x_0).

- You may be able to find suitable y_1, y_2 by a method like we used in the previous example, but the existence-uniqueness theorem guarantees they exist even if you don't know how to find formulas for them.

Under these conditions, the solutions y_1, y_2 are actually a basis for the solution space! Here's why:

• span: the condition that the Wronskian matrix is invertible at x_0 means we can solve each IVP there with a linear combination $y = c_1y_1 + c_2y_2$: In that case, to solve the IVP

$$y'' + p(x)y' + q(x)y = 0$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$

we set

$$y(x) = c_1y_1(x) + c_2y_2(x).$$

$$y'(x) = c_1y_1' + c_2y_2'$$

At x_0 we wish to find c_1, c_2 so that

$$c_1y_1(x_0) + c_2y_2(x_0) = b_0$$
$$c_1y_1'(x_0) + c_2y_2'(x_0) = b_1.$$

This system is equivalent to the the matrix equation

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}.$$

When the Wronskian matrix at x_0 has an inverse, the unique solution $[c_1, c_2]^T$ is given by

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}^{-1} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}.$$

Since the uniqueness theorem says each IVP has a unique solution, we've found it!

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

- Span: Since each solution $y(x)$ to the differential equation solves *some* initial value problem at x_0, this gives all solutions, as we let $[b_0, b_1]^T$ vary freely in $\mathbb{R}^2$. So each solution $y(x)$ *is* a linear combination of y_1, y_2. Thus $\{y_1, y_2\}$ spans the solution space.

- Linear independence: If we have the identity

$$c_1 y_1(x) + c_2 y_2(x) = 0$$

then by differentiating each side with respect to x we also have

$$c_1 y_1'(x) + c_2 y_2'(x) = 0.$$

Evaluating at $x = x_0$ this is the system

$$c_1 y_1(x_0) + c_2 y_2(x_0) = 0$$
$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = 0$$

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Theorem 4: All solutions to the nonhomogeneous second order linear DE

post pone until Wed.

$$* \quad L(y) = y'' + p(x)y' + q(x)y = f(x)$$

helps with last part of lab problem

are of the form $y = y_P + y_H$ where y_P is any single particular solution and y_H is some solution to the homogeneous DE. (y_H is called y_c, for complementary solution, in the text). Thus, if you can find a single particular solution to the nonhomogeneous DE, and all solutions to the homogeneous DE, you've actually found all solutions to the nonhomogeneous DE. (You had a homework problem related to this idea, 3.4.40, in homework 5, but in the context of matrix equations, a week or two ago. The same idea reappears in your current lab, in the last problem.)

proof: Make use of the fact that

$$L(y) := y'' + p(x)y' + q(x)y$$

is a linear operator. In other words, use the *linearity properties*

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(2) $L(c\,y) = c\,L(y),\ c \in \mathbb{R}$.

proof. Let $L(y_P) = f$
Let $L(y_H) = 0$

Then $L(y_P + y_H) = L(y_P) + L(y_H)$ (1)
$= f + 0$
$= f$

so $y_P + y_H$ solves $*$

Conversely, let
$L(y_Q) = f$
then $y_Q = y_P + (y_Q - y_P)$

$L(y_Q - y_P) = L(y_Q) + L(-y_P)$ (1)
$= L(y_Q) - L(y_P)$ (2)
$= f - f$
$= 0$

so $y_Q - y_P$ was a homog. soln!

In Monday's notes we found that the general solution to the homogeneous differential equation

$$y'' - 2y' - 3y = 0$$

is

$$y_H = c_1 e^{-x} + c_2 e^{3x}.$$

Now consider the non-homogeneous differential equation

$$L(y) = y'' - 2y' - 3y = 6.$$

$y_p = A$
$y_p' = 0$
$y_p'' = 0$
want
$L(y_p) = 0 - 0 - 3A = 6$
$A = -2$

Notice that

$$y_P = -2$$

is one particular solution to the differential equation.

Exercise 1a) Solve the initial value problem

$$y'' - 2y' - 3y = 6.$$
$$y(0) = -1$$
$$y'(0) = -5$$

with a solution to the differential equation of the form

$$y = y_P + y_H = -2 + c_1 e^{-x} + c_2 e^{3x}.$$

by Thm 4!!

$$y' = 0 + -c_1 e^{-x} + 3c_2 e^{3x}$$

$$y(0) = -1 = -2 + c_1 + c_2$$
$$y'(0) = -5 = -c_1 + 3c_2$$

$c_1 + c_2 = 1$
$-c_1 + 3c_2 = -5$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 3 & -5 \end{array}\right]$$

$$R_1 + R_2 \to R_2 \quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 4 & -4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \end{array}\right]$$

$$-R_2 + R_1 \to R_1 \quad \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$$

$c_1 = 2$
$c_2 = -1$

$$\boxed{y(x) = -2 + 2e^{-x} - 1e^{3x}}$$

$y(0) = -2 + 2 - 1 = -1$ ✓
$y'(0) = 0 - 2 - 3 = -5$ ✓

1b) Notice that the same algebra shows you could solve every initial value problem

$$y'' - 2y' - 3y = 6.$$
$$y(0) = b_0$$
$$y'(0) = b_1$$

with a solution of the form

$$y = y_P + y_H = -2 + c_1 e^{-x} + c_2 e^{3x}$$

so by the uniqueness theorem for initial value problems, these ARE all the solutions to the differential equation even though we did not get them a direct method like we used for first order linear differential equations.

The theory for n^{th} order linear differential equations is conceptually the same as for second order...

Definition: An n^{th} order linear differential equation for a function $y(x)$ is a differential equation that can be written in the form

$$A_n(x)y^{(n)} + A_{n-1}(x)y^{(n-1)} + ... + A_1(x)y' + A_0(x)y = F(x) .$$

We search for solution functions $y(x)$ defined on some specified interval I of the form $a < x < b$, or (a, ∞), $(-\infty, a)$ or (usually) the entire real line $(-\infty, \infty)$. In this chapter we assume the function $A_n(x) \neq 0$ on I, and divide by it in order to rewrite the differential equation in the standard form

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = f.$$

($a_{n-1},\ldots a_1, a_0, f$ are all functions of x, and the DE above means that equality holds for all value of x in the interval I.)

Theorem 1 (Existence-Uniqueness Theorem): Let $a_{n-1}(x), a_{n-2}(x),\ldots a_1(x), a_0(x), f(x)$ be specified continuous functions on the interval I, and let $x_0 \in I$. Then there is a unique solution $y(x)$ to the initial value problem

$$\begin{gathered}
y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = f \\
y(x_0) = b_0 \\
y'(x_0) = b_1 \\
y''(x_0) = b_2 \\
\vdots \\
y^{(n-1)}(x_0) = b_{n-1}
\end{gathered}$$

and $y(x)$ exists and is n times continuously differentiable on the entire interval I.

The differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = f$$

is called <u>linear</u> because the operator L defined by

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y$$

satisfies the so-called <u>linearity properties</u>

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(2) $L(c\,y) = c\,L(y),\ c \in \mathbb{R}$.

• *The proof that L satisfies the linearity proporties is just the same as it was for the case when n* = 2, which we checked.

The following two theorems only use the linearity properties of the operator L. I've kept the same numbering we used for the case $n = 2$.

<u>Theorem 2:</u> The solution space to the homogeneous linear DE

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = 0$$

is a subspace.

<u>Theorem 4:</u> The general solution to the <u>nonhomogeneous</u> n^{th} order linear DE

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = f$$

is $y = y_P + y_H$ where y_P is any single particular solution and y_H is the general solution to the homogeneous DE. (y_H is called y_c, for complementary solution, in the text).

Theorem 3: The solution space to the n^{th} order homogeneous linear differential equation

$$y^{(n)} + a_{n-1}\,y^{(n-1)} + ... + a_1\,y' + a_0\,y \equiv 0$$

is n-dimensional. Thus, any n independent solutions $y_1, y_2, ... y_n$ will be a basis, and all homogeneous solutions will be uniquely expressible as linear combinations

$$y_H = c_1 y_1 + c_2 y_2 + ... + c_n y_n.$$

<u>proof:</u> By the existence half of <u>Theorem 1</u>, we know that there are solutions for each possible initial value problem for this (homogenenous case) of the IVP for n^{th} order linear DEs. So, pick solutions $y_1(x), y_2(x), ... y_n(x)$ so that their vectors of initial values (which we'll call initial value vectors)

$$\begin{bmatrix} y_1(x_0) \\ y_1'(x_0) \\ y_1''(x_0) \\ \vdots \\ y_1^{(n-1)}(x_0) \end{bmatrix}, \begin{bmatrix} y_2(x_0) \\ y_2'(x_0) \\ y_2''(x_0) \\ \vdots \\ y_2^{(n-1)}(x_0) \end{bmatrix}, ..., \begin{bmatrix} y_n(x_0) \\ y_n'(x_0) \\ y_n''(x_0) \\ \vdots \\ y_n^{(n-1)}(x_0) \end{bmatrix}$$

are a basis for $\mathbb{R}^n$ (i.e. these n vectors are linearly independent and span $\mathbb{R}^n$. (Well, you may not know how to "pick" such solutions, but you know they exist because of the existence theorem.)

<u>Claim:</u> In this case, the solutions $y_1, y_2, ... y_n$ are a basis for the solution space. In particular, every solution to the homogeneous DE is a unique linear combination of these n functions and the dimension of the solution space is n .... discussion on next page.

- Check that $y_1, y_2, \ldots y_n$ **span** the solution space: Consider any solution $y(x)$ to the DE. We can compute its vector of initial values

$$\begin{bmatrix} y(x_0) \\ y'(x_0) \\ y''(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix} := \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}.$$

Now consider a linear combination $z = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$. Compute its initial value vector, and notice that you can write it as the product of the Wronskian matrix at x_0 times the vector of linear combination coefficients:

$$\begin{bmatrix} z(x_0) \\ z'(x_0) \\ \vdots \\ z^{(n-1)}(x_0) \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y'_n(x_0) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

We've chosen the $y_1, y_2, \ldots y_n$ so that the Wronskian matrix at x_0 has an inverse, so the matrix equation

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y'_n(x_0) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}$$

has a unique solution $\underline{\mathbf{c}}$. For this choice of linear combination coefficients, the solution $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ has the same initial value vector at x_0 as the solution $y(x)$. By the uniqueness half of the existence-uniqueness theorem, we conclude that

$$y(x) = z(x) = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n .$$

Thus $y_1, y_2, \ldots y_n$ <u>span</u> the solution space.

- **<u>linear independence</u>**: If a linear combination $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \equiv 0$, then differentiate this identity $n - 1$ times, and then substitute $x = x_0$ into the resulting n equations. This yields the Wronskian matrix equation above, with $\left[b_0, b_1, \ldots b_{n-1}\right]^T = [0, 0, \ldots, 0]^T$. So the matrix equation above implies that $\left[c_1, c_2, \ldots c_n\right]^T = \underline{\mathbf{0}}$. So $y_1, y_2, \ldots y_n$ are also <u>linearly independent</u>.

- Thus $y_1, y_2, \ldots y_n$ are a <u>basis</u> for the solution space and the general solution to the homogeneous DE can be written as

$$y_H = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n .$$

Let's do some new exercises that tie these ideas together. (We may do these exercises while or before we wade through the general discussions on the previous pages!)

Exercise 2) Consider the 3^{rd} order linear homogeneous DE for $y(x)$:

$$L(y) = y''' + 3y'' - y' - 3y = 0.$$

Find a basis for the 3-dimensional solution space, and the general solution. Use the Wronskian matrix (or determinant) to verify you have a basis. Hint: try exponential functions.

try

$$y = e^{rx}$$
$$y' = re^{rx}$$
$$y'' = r^2 e^{rx}$$
$$y''' = r^3 e^{rx}$$

$$L(y) = r^3 e^{rx} + 3r^2 e^{rx} - re^{rx} - 3e^{rx} \equiv 0$$
$$= e^{rx}\left[r^3 + 3r^2 - r - 3\right]$$
$$= e^{rx}\left[r^2(r+3) - (r+3)\right]$$

"cheat"

$$= \cancel{e^{rx}}\left[(r+3)(r^2-1)\right] \equiv 0$$
$$(r+3)(r-1)(r+1) = 0$$
$$r = 1, -1, -3$$

e^{x}, e^{-x}, e^{-3x} are soltns

so $y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{-3x}$ are soltns

these are all!

Exercise 3a) Find the general solution to

$$y''' + 3y'' - y' - 3y = 6 .$$

Hint: First try to find a particular solution ... try a constant function.

3b) Set up the linear system to solve the initial value problem for this DE, with $y(0) = -1, y'(0) = 2, y''(0) = 7$. Does the form agree with the actual solution?

WolframAlpha computational knowledge engine.

{y'''(x)+3y''(x)-y'(x)-3y(x)=6, y(0)=-1, y'(0)=2, y''(0)=7}

Web Apps Examples Random

Input:

$$\{y^{(3)}(x) + 3\,y''(x) - y'(x) - 3\,y(x) = 6,\ y(0) = -1,\ y'(0) = 2,\ y''(0) = 7\}$$

Open code

Autonomous equation:

$$3\,y^{(3)}(x) = 6 + 3\,y(x) + y'(x) - y^{(3)}(x)$$

Autonomous equation »

ODE classification:

third-order linear ordinary differential equation

Alternate form:

$$\{y'(x) + 3\,y(x) + 6 = y^{(3)}(x) + 3\,y''(x),\ y(0) = -1,\ y'(0) = 2,\ y''(0) = 7\}$$

Differential equation solution:

Approximate form Step-by-step solution

$$y(x) = \frac{3\,e^{-3x}}{4} - 2\,e^{-x} + \frac{9\,e^{x}}{4} - 2$$

Wed March 7:
5.3 Solving constant coefficient homogeneous linear differential equations

Announcements:
- quiz today.
- Finish Tuesday's notes. Do today's!

'til 10:47

Warm-up Exercise: Find all solutions $y(x)$ to the homogeneous linear differential equation

like §5.1

$$L(y) = y'' + 5y' + 6y = 0$$

Hint: look for a basis of the 2-dimensional solution space made out of exponential functions, $y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}$.

try $y = e^{rx}$ find r's that work

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

want

$$L(y) = r^2 e^{rx} + 5(re^{rx}) + 6e^{rx} = 0$$

$$= e^{rx}\underbrace{[r^2 + 5r + 6]}_{\text{"characteristic polynomial" for the D.E.}} = 0$$

need $r^2 + 5r + 6 = 0$

$(r+3)(r+2) = 0$

$r = -2, -3$.

So e^{-2x}, e^{-3x} are solns

So since soln space is closed under + & scalar mult. (i.e. subspace),

$$y(x) = c_1 e^{-2x} + c_2 e^{-3x}$$

are solns.

Since soln space is 2-dim'l and since e^{-2x}, e^{-3x} are independent, they're a basis and these are all the solns!!

For the next two sections we focus homogneous linear differential equations with constant coefficients. Section 5.3 contains the algorithms we'll need and in section 5.4 we'll apply the general theory to the unforced mass-spring differential equation.

5.3: Algorithms for the basis and general (homogeneous) solution to

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

when the coefficients $a_{n-1}, a_{n-2}, \ldots a_1, a_0$ are all constant.

step 1) Try to find a basis for the $\underbrace{\text{solution space}}_{n\text{-dim'l}}$ made of exponential functions....try $y(x) = e^{rx}$. In this case

$y' = re^{rx}$
$y'' = r^2 e^{rx}$
$y''' = r^3 e^{rx}$

$$L(y) = e^{rx}\left(r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0\right) = e^{rx} p(r) .$$

We call this polynomial $p(r)$ the characteristic polynomial for the differential equation, and can read off what it is directly from the expression for $L(y)$ if we want. For each root r_j of $p(r)$, we get a solution $e^{r_j x}$ to the homogeneous DE.

Case 1) If $p(r)$ has n distinct (i.e. different) real roots $r_1, r_2, \ldots, r_n$, then

$$e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}$$

is a basis for the solution space; i.e. the general solution is given by

$$y_H(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_n e^{r_n x} .$$

Example: The differential equation

$$y''' + 3y'' - y' - 3y = 0$$

has characteristic polynomial

$$p(r) = r^3 + 3r^2 - r - 3 = (r+3)\cdot\left(r^2 - 1\right) = (r+3)(r+1)\cdot(r-1)$$

so the general solution to

$$y''' + 3y'' - y' - 3y = 0$$

is

$$y_H(x) = c_1 e^x + c_2 e^{-x} + c_3 e^{-3x} .$$

$r = -3, -1, 1$

solns e^x, e^{-x}, e^{-3x}

up to now, you would've checked that $\{e^x, e^{-x}, e^{-3x}\}$ are a basis for soln space, with Wronskian.

$y(x) = c_1 e^x + c_2 e^{-x} + c_3 e^{-3x}$

$y'(x) = c_1 e^x + c_2(-e^{-x}) + c_3(-3e^{-3x})$

$y''(x) = c_1 e^x + c_2 e^{-x} + c_3(9e^{-3x})$

$$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = \underbrace{\begin{bmatrix} e^x & e^{-x} & e^{-3x} \\ e^x & -e^{-x} & -3e^{-3x} \\ e^x & e^{-x} & 9e^{-3x} \end{bmatrix}}_{W(e^x, e^{-x}, e^{-3x})} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$y''' + 3y'' - y' - 3y = 0$

$y(0) = b_0$

$y'(0) = b_1$

$y''(0) = b_2$

det $\neq 0$!

IVP
$$\begin{bmatrix} y(0) \\ y'(0) \\ y''(0) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -3 \\ 1 & 1 & 9 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}$$

Exercise 1) By construction, $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}$ all solve the differential equation. Show that they're linearly independent. This will be enough to verify that they're a basis for the solution space, since we know the solution space is n-dimensional. Hint: The easiest way to show this is to list your roots so that $r_1 < r_2 < \ldots < r_n$ and to use a limiting argument.

better way to show basis

- if n fcns are indep. in n-dim'l space, then they're a basis.

$$c_1 e^{x} + c_2 e^{-x} + c_3 e^{-3x} = 0 \quad x \in \mathbb{R}.$$

$$\div e^{x} \quad c_1 + c_2 e^{-2x} + c_3 e^{-4x} = 0 \quad x \in \mathbb{R}$$

$$\lim_{x \to \infty}: \quad c_1 + 0 + 0 = 0 \quad \text{so } c_1 = 0.$$

$$c_2 e^{-x} + c_3 e^{-3x} = 0$$

$\div e^{-x}$
(mult. by e^{x}) : $c_2 + c_3 e^{-2x} = 0$

$$\lim_{x \to \infty} \quad c_2 + 0 = 0 \quad \text{so } c_2 = 0$$

$$c_3 e^{-3x} = 0 \implies c_3 = 0$$

Case 2) Repeated real roots. In this case $p(r)$ has all real roots $r_1, r_2, \ldots r_m$ $(m < n)$ with the r_j all different, but some of the factors $(r - r_j)$ in $p(r)$ appear with powers bigger than 1. In other words, $p(r)$ factors as

$$p(r) = (r - r_1)^{k_1} (r - r_2)^{k_2} \ldots (r - r_m)^{k_m}$$

with some of the $k_j > 1$, and $k_1 + k_2 + \ldots + k_m = n$.

Start with a small example: The case of a second order DE for which the characteristic polynomial has a double root.

Exercise 2) Let r_1 be any real number. Consider the homogeneous DE

$$L(y) := y'' - 2\, r_1\, y' + r_1^2\, y = 0.$$

with $p(r) = r^2 - 2\, r_1\, r + r_1^2 = (r - r_1)^2$, i.e. r_1 is a double root for $p(r)$. Show that $e^{r_1 x}, x\, e^{r_1 x}$ are a basis for the solution space to $L(y) = 0$, so the general homogeneous solution is $y_H(x) = c_1 e^{r_1 x} + c_2\, x\, e^{r_1 x}$. Start by checking that $x\, e^{r_1 x}$ actually (magically?) solves the DE. (We may wish to study a special case $y'' + 6\, y' + 9\, y = 0$.)

$y'' + 6y' + 9y = 0$

you checked one like this in your HW!

Here's the general algorithm: If

$$p(r) = (r - r_1)^{k_1}(r - r_2)^{k_2}\ldots(r - r_m)^{k_m}$$

then (as before) $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_m x}$ are independent solutions, but since $m < n$ there aren't enough of them to be a basis. Here's how you get the rest: For each $k_j > 1$, you actually get independent solutions

$$e^{r_j x}, x e^{r_j x}, x^2 e^{r_j x}, \ldots, x^{k_j - 1} e^{r_j x}.$$

This yields k_j solutions for each root r_j, so since $k_1 + k_2 + \ldots + k_m = n$ you get a total of n solutions to the differential equation. There's a good explanation in the text as to why these additional functions actually do solve the differential equation, see pages 316-318 and the discussion of "polynomial differential operators". I've also made a homework problem in which you can explore these ideas. Using the limiting method we discussed earlier, it's not too hard to show that all n of these solutions are indeed linearly independent, so they are in fact a basis for the solution space to $L(y) = 0$.

Exercise 3) Explicitly antidifferentiate to show that the solution space to the differential equation for $y(x)$

$$y^{(4)} - y^{(3)} = 0$$

agrees with what you would get using the repeated roots algorithm in Case 2 above. Hint: first find $v = y'''$, using $v' - v = 0$, then antidifferentiate three times to find y_H. When you compare to the repeated roots algorithm, note that it includes the possibility $r = 0$ and that $e^{0\,x} = 1$.

$L(y) = y^{(4)} - y^{(3)} = 0$

$y = e^{rx}$: $\quad L(e^{rx}) = (e^{rx})^{(4)} - (e^{rx})^{(3)}$

$= r^4 e^{rx} - r^3 e^{rx}$

$= e^{rx}(r^4 - r^3) \overset{\text{want}}{=} 0$

$r^4 - r^3 = 0$

$r^3(r-1)$

$(r-0)^3(r-1) = 0$

basis of solutions:

recipe: $\quad e^x, \; 1, \; x, \; x^2$

$r=1 \quad e^{0x} \quad xe^{0x} \quad x^2e^{0x}$

Chapter 1

$v = y'''$ then DE says

$v' - v = 0$

$e^{-x}(v' - v) = 0$

$\frac{d}{dx}(e^{-x}v) = 0$

$e^{-x}v = C$

$v = Ce^x$

$y''' = Ce^x$

S: $y'' = Ce^x + D$

$y' = Ce^x + Dx + E$

$y = Ce^x + \frac{D}{2}x^2 + Ex + F$

Case 3) Complex number roots - this will be our surprising and fun topic on Friday. Our analysis will explain exactly how and why trig functions and mixed exponential-trig-polynomial functions show up as solutions for some of the homogeneous DE's you worked with in your homework and lab for this past week. This analysis depends on Euler's formula, one of the most beautiful and useful formulas in mathematics:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$
$$\text{for } i^2 = -1.$$

Fri Mar 9

5.3 Solving constant coefficient homogeneous linear differential equations: complex roots in the characteristic polynomial

Announcements: another magic day... your labs yesterday told you the recipe for finding solns to homog. linear const. coeff DE's, when the roots of characteristic polynomial are complex

$r = a \pm ib$ root $\longrightarrow y(x) = e^{ax}\cos bx,\ e^{ax}\sin bx$

$y(x) = e^{rx}$

• first, finish Wed

'til 10:47

Warm-up Exercise: Recall Taylor - McLaurin series from Calculus 2:

$$f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + \ldots$$

(The series on the right is created to match $f(0), f'(0), f''(0), \ldots$)

Recover the McLaurin series for

a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ $\qquad 3! = 3\cdot2\cdot1$ etc.

See notes below

b) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

even function
so even powers

c) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

odd function
so odd powers

5.3 continued. How to find the solution space for n^{th} order linear homogeneous DE's with constant coefficients, and why the algorithms work.

$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$

Strategy: In all cases we first try to find a basis for the n-dimensional solution space made of or related to exponential functions....trying $y(x) = e^{rx}$ yields

$$L(y) = e^{rx}\left(r^n + a_{n-1}r^{n-1} + \dots + a_1 r + a_0\right) = e^{rx}p(r).$$

The characteristic polynomial $p(r)$ and how it factors are the keys to finding the solution space to $L(y) = 0$. There are three cases, of which the first two (distinct and repeated real roots) are in yesterday's notes.

Case 3) $\boxed{p(r) \text{ has complex number roots.}}$ This is the hardest, but also most interesting case.The punch line is that exponential functions e^{rx} still work, except that $r = a \pm b\,i$; but, rather than use those complex exponential functions to construct solution space bases we decompose them into real-valued solutions that are products of exponential and trigonometric functions.

punchline in lab: if $r = a \pm bi$ is complex root then $y_1 = e^{ax}\cos bx$, $e^{ax}\sin bx$ are real soltns.

Magic What do these have to do with $e^{(a+bi)x}$, $e^{(a-bi)x}$

To understand how this all comes about, we need to learn Euler's formula . This also lets us review some important Taylor's series facts from Calc 2. As it turns out, complex number arithmetic and complex exponential functions actually are important in many engineering and science applications.

Recall the Taylor-Maclaurin formula from Calculus

$$f(x) \sim f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + \ldots + \frac{1}{n!}f^{(n)}(0)x^n + \ldots$$

(Recall that the partial sum polynomial through order n matches f and its first n derivatives at $x_0 = 0$. When you studied Taylor series in Calculus you sometimes expanded about points other than $x_0 = 0$. You also needed error estimates to figure out on which intervals the Taylor polynomials actually coverged back to f.)

Warm-up.

Exercise 1) Use the formula above to recall the three very important Taylor series for

converge for all x

1a) $e^x = 1 + 1x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots + \frac{1}{n!}x^n + \ldots$

$f(0)=1 \quad f'(0)=1, \quad f''(0)=1, \ldots$

$f(x)=e^x, \quad f'(x)=e^x, \quad f''(x)=e^x, \ldots$

1b) $\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots$

$f(0)=1, \quad f'(0)=0, \quad f''(0)=-1 \quad f'''(0)=0 \quad f^{(4)}(0)=1$, repeat

$f(x)=\cos x, \quad f'(x)=-\sin x, \quad f''(x)=-\cos x, \quad f'''(x)=\sin x, \quad f^{(4)}(x)=\cos x$, repeat!

1c) $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

$f(0)=0 \quad f'(0)=1 \quad f''(0)=0 \quad f'''(0)=-1 \quad f^{(4)}(0)=0$, repeat

$f(x)=\sin x, \quad f'(x)=\cos x, \quad f''(x)=-\sin x, \quad f'''(x)=-\cos x, \quad f^{(4)}(x)=\sin x$, repeat

In Calculus you checked that these series actually converge and equal the given functions, for all real numbers x.

Exercise 2) Let $x = i\,\theta$ and use the Taylor series for e^x as the definition of $e^{i\,\theta}$ in order to derive Euler's formula:

$i = \sqrt{-1} \quad i^2 = -1$

$i^3 = i^2 i = -i$

$$e^{i\,\theta} = \cos(\theta) + i\sin(\theta) .$$

$i^4 = (i^2)(i^2) = 1$

$x = i\theta \qquad e^{i\theta} = 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \ldots$

$= 1 + i\theta - \frac{1}{2!}\theta^2 - \frac{i}{3!}\theta^3 + \frac{1}{4!}\theta^4 + \frac{i}{5!}\theta^5 - \frac{1}{6!}\theta^6$

$i^3 = i^2 \cdot i = (-1)i$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

e.g. $\theta = \pi$: $e^{i\pi} = -1$

the best formula there is.

From Euler's formula it makes sense to define

$$e^{a+bi} := e^a e^{bi} = e^a(\cos(b) + i\sin(b))$$

for $a, b \in \mathbb{R}$. So for $x \in \mathbb{R}$ we also get

$$e^{(a+bi)x} = e^{ax}(\cos(bx) + i\sin(bx)) = e^{ax}\cos(bx) + i\,e^{ax}\sin(bx) .$$

For a complex function $f(x) + i\,g(x)$ we define the derivative by

$$D_x(f(x) + i\,g(x)) := f'(x) + i\,g'(x) .$$

It's straightforward to verify (but would take some time to check all of them) that the usual differentiation rules, i.e. sum rule, product rule, quotient rule, constant multiple rule, all hold for derivatives of complex functions. The following rule pertains most specifically to our discussion and we should check it:

Exercise 3) Check that $D_x\left(e^{(a+bi)x}\right) = (a+bi)e^{(a+bi)x}$, i.e.

$$D_x e^{rx} = r\,e^{rx}$$

even if r is complex. (So also $D_x^2 e^{rx} = D_x r\,e^{rx} = r^2 e^{rx}$, $D_x^3 e^{rx} = r^3 e^{rx}$, etc.)

$D_x e^{rx} = re^{rx}$

$D_x\left(e^{(a+bi)x}\right) = D_x\left(e^{ax}(\cos bx + i\sin bx)\right)$ Euler.

$= D_x\left(e^{ax}\cos bx\right) + i\,D_x\left(e^{ax}\sin bx\right)$

$= -b\,e^{ax}\sin bx + a e^{ax}\cos bx$

$+ i\left(a e^{ax}\sin bx + b e^{ax}\cos bx\right)$

$\overset{?}{=} (a+bi)\,e^{(a+bi)x}$

$= (a+bi)(e^{ax})(\cos bx + i\sin bx)$

$(a\cos bx - b\sin bx)\,e^{ax}$

$i\left[e^{ax}(a\sin bx) + b e^{ax}\cos bx\right]$

Now return to our differential equation questions, with

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y .$$

Then even for complex $r = a + b\,i$ $\;(a, b \in \mathbb{R})$, our work above shows that

$$L\left(e^{rx}\right) = e^{rx}\left(r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0\right) = e^{rx}\,p(r) .$$

So if $r = a + b\,i$ is a complex root of $p(r)$ then e^{rx} is a complex-valued function solution to $L(y) = 0$. But L is linear, and because of how we take derivatives of complex functions, we can compute in this case that

$e^{(a+bi)x} = e^{ax}\cos bx$
$+ i e^{ax}\sin bx$

$$0 + 0\,i = L\left(e^{rx}\right) = L\left(e^{ax}\cos(bx) + i\,e^{ax}\sin(bx)\right)$$

$$= L\left(e^{ax}\cos(bx)\right) + i\,L\left(e^{ax}\sin(bx)\right) .$$

Equating the real and imaginary parts in the first expression to those in the final expression (because that's what it means for complex numbers to be equal) we deduce

$$0 = L\left(e^{ax}\cos(bx)\right)$$
$$0 = L\left(e^{ax}\sin(bx)\right) .$$

<u>Upshot:</u> If $r = a + b\,i$ is a complex root of the characteristic polynomial $p(r)$ then

$$y_1 = e^{ax}\cos(bx)$$
$$y_2 = e^{ax}\sin(bx)$$

are two solutions to $L(y) = 0$. (The conjugate root $a - b\,i$ would give rise to y_1, $-y_2$, which have the same span.

$z_1 = e^{ax}\cos(-bx) = e^{ax}\cos bx$
$z_2 = e^{ax}\sin(-bx) = -e^{ax}\sin bx$

downside : algebra messier
upside : got solns from $a+bi$, don't worry ab't $a-bi$

Case 3) Let L have characteristic polynomial

$$p(r) = r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0$$

with real constant coefficients $a_{n-1}, \ldots, a_1, a_0$. If $(r - (a + b\,i))^k$ is a factor of $p(r)$ then so is the conjugate factor $(r - (a - b\,i))^k$. Associated to these two factors are $2\,k$ real and independent solutions to $L(y) = 0$, namely

$$e^{a\,x}\cos(b\,x),\ e^{a\,x}\sin(b\,x)$$
$$x\,e^{a\,x}\cos(b\,x),\ x\,e^{a\,x}\sin(b\,x)$$
$$\vdots \qquad \vdots$$
$$x^{k-1}e^{a\,x}\cos(b\,x),\ x^{k-1}e^{a\,x}\sin(b\,x)$$

Combining cases 1,2,3, yields a complete algorithm for finding the general solution to $L(y) = 0$, as long as you are able to figure out the factorization of the characteristic polynomial $p(r)$.

Exercise 4) Find a basis for the solution space of functions $y(x)$ that solve

$$y'' + 9\,y = 0 .$$

(You were told a basis in the last problem of last week's lab....now you know where it came from.)

$p(r) = r^2 + 9 = 0$ $\quad r^2 = -9$

$r = \pm 3i$

$= a \pm bi$ $\quad a = 0$, $b = 3$

$r = a \pm bi$

$y_1 = e^{ax}\cos bx$

$y_2 = e^{ax}\sin bx$

$e^{0x}\cos 3x = \cos 3x$

$e^{0x}\sin 3x = \sin 3x$

Exercise 5) Find a basis for the solution space of functions $y(x)$ that solve

$$y'' + 6\,y' + 13\,y = 0 .$$

Exercise 6) Suppose a 7^{th} order linear homogeneous DE has characteristic polynomial

$$p(r) = \left(r^2 + 6\,r + 13\right)^2 (r - 2)^3 .$$

What is the general solution to the corresponding homogeneous DE?