Math 2250-004 Week 8 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes include material from 4.2-4.4 and an introduction to Chapter 5.

Mon Feb 26
4.3, 4.2 linear combinations and independence of vectors, vector subspaces of $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^n$.

Announcements:

Warm-up Exercise:

We've been discussing ideas related to linear combinations of vectors. After the weekend, we should review, to recall these concepts:

A *linear combination* of the vectors in the set $\{\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \ldots \underline{\mathbf{v}}_n\}$ is
any vector $\underline{\mathbf{v}}$ *that is a sum of scalar multiples of those vectors,*

i.e. any $\underline{\mathbf{v}}$ *expressible as*

$$\underline{\mathbf{v}} = c_1\underline{\mathbf{v}}_1 + c_2\,\underline{\mathbf{v}}_2 + \ \ldots + c_n\underline{\mathbf{v}}_n.$$

The *span* of the vectors in the set $\{\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \ldots \underline{\mathbf{v}}_n\}$ is
the collection of all possible linear combinations:

$$span\{\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \ldots \underline{\mathbf{v}}_n\} := \{\underline{\mathbf{v}} = c_1\underline{\mathbf{v}}_1 + c_2\,\underline{\mathbf{v}}_2 + \ \ldots + c_n\underline{\mathbf{v}}_n \text{ such that each } c_i \in \mathbb{R},\ i = 1, 2, \ldots n\}$$

The vectors in the set $\{\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \ldots \underline{\mathbf{v}}_n\}$ are *linearly dependent* means

it is possible to satisfy $c_1\underline{\mathbf{v}}_1 + c_2\,\underline{\mathbf{v}}_2 + \ \ldots + c_n\underline{\mathbf{v}}_n = \underline{\mathbf{0}}$ *with not all of the weights* $c_1, c_2, \ldots c_n = 0$.

(Equivalently, at least one $\underline{\mathbf{v}}_j$ *can be expressed as a linear combination of some of the other vectors in the collection.)*

The vectors in the set $\{\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \ldots \underline{\mathbf{v}}_n\}$ are *linearly independent* means

$$c_1\underline{\mathbf{v}}_1 + c_2\,\underline{\mathbf{v}}_2 + \ \ldots + c_n\underline{\mathbf{v}}_n = 0 \quad \Rightarrow \quad c_1 = c_2 = \ldots = c_n = 0.$$

(Equivalently, no $\underline{\mathbf{v}}_j$ *can be expressed as a linear combination of some of the other vectors in the collection.)*

NEW:

<u>Definition</u>: Let $V = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ \dots \ \underline{\boldsymbol{v}}_n\}$. If the vectors in the set $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ \dots \ \underline{\boldsymbol{v}}_n\}$ are linearly independent, then we say that they are a *basis* for *V*. And, we say that the *dimension* of *V* is n.

<u>Example</u> The set of vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ spans $\mathbb{R}^3$ and is linearly independent, so is a *basis* for $\mathbb{R}^3$. The *dimension* of $\mathbb{R}^3$ is 3!

Let's review what we were doing at the end of class on Friday, in light of these new definitions...

Monday Feb 26
span, basis, dimension

Wed example, finished on Friday

Exercise 3) Consider the two vectors $\underline{v}_1 = [1, 0, 0]^T$, $\underline{v}_2 = [0, -1, 2]^T \in \mathbb{R}^3$.

3a) Sketch these two vectors as position vectors in $\mathbb{R}^3$, using the axes below.

3b) What geometric object is $span\{\underline{v}_1\}$? (Remember, we are identifying position vectors with their endpoints.) Sketch a portion of this object onto your picture below. Remember though, the "span" continues beyond whatever portion you can draw.

$= \{c\vec{v}_1, c \in \mathbb{R}\} = \{c\begin{bmatrix}1\\0\\0\end{bmatrix}, c \in \mathbb{R}\}$

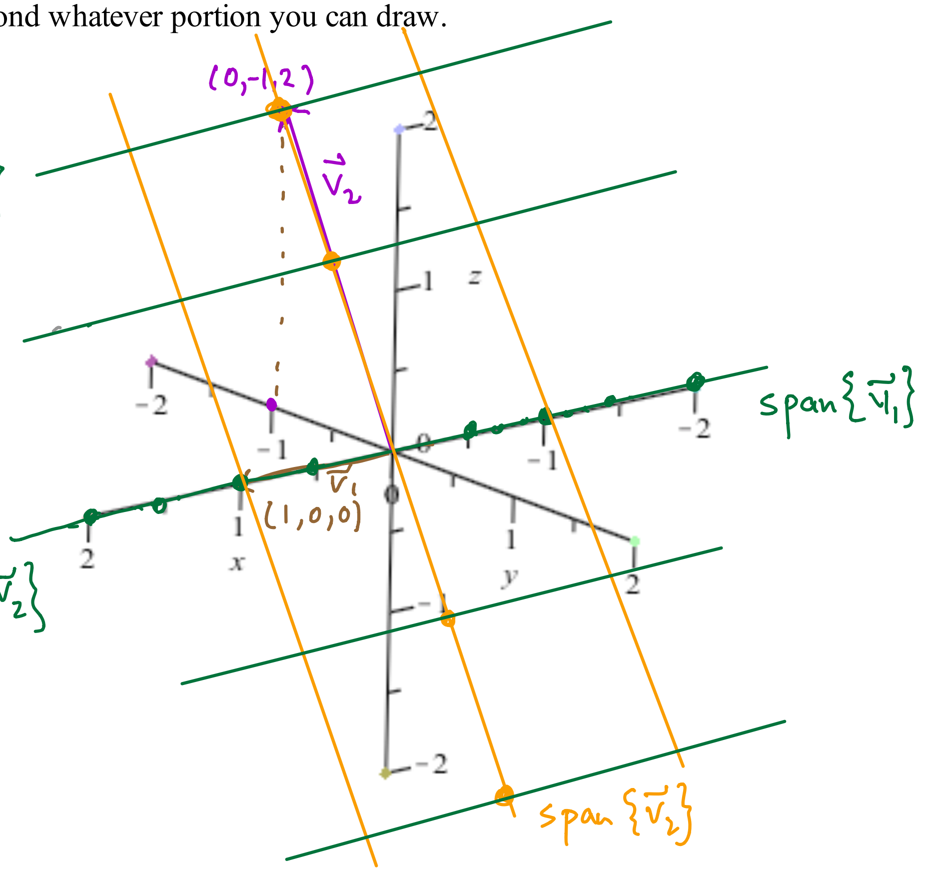
x-axis

3c) What geometric object is $span\{\underline{v}_1, \underline{v}_2\}$? Sketch a portion of this object onto your picture below. Remember though, the "span" continues beyond whatever portion you can draw.

plane

$\{c_1\vec{v}_1 + c_2\vec{v}_2 : c_1, c_2 \in \mathbb{R}\}$

partial grid for the plane that is $span\{\vec{v}_1, \vec{v}_2\}$



$$span\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n : c_1, c_2, \ldots c_n \in \mathbb{R}\}$$

= collection of all linear combinations

Monday Feb 26
span, basis, dimension

you have a lab problem like this.

3d) What implicit equation must vectors $[b_1, b_2, b_3]^T$ satisfy in order to be in $span\{\underline{v}_1, \underline{v}_2\}$? Hint: For what $[b_1, b_2, b_3]^T$ can you solve the system

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$\vec{v}_1$ $\vec{v}_2$

for c_1, c_2? Write this an augmented matrix problem and use row operations to reduce it, to see when you get a consistent system for c_1, c_2.

$$\left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & -1 & b_2 \\ 0 & 2 & b_3 \end{array}\right] \quad \begin{array}{l} = x \\ = y \\ = z \end{array}$$

$-R_2 \to R_2$

$$\left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & -b_2 \\ 0 & 2 & b_3 \end{array}\right]$$

$-2R_2 + R_3 \to R_3$

$$\left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & -b_2 \\ 0 & 0 & 2b_2 + b_3 \end{array}\right]$$

Solutions c_1, c_2 if and only if

$2b_2 + b_3 = 0$

i.e. $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ must lie on the plane

$2y + z = 0$

compare to picture!

e.g. the points $(0,0,0)$ in plane
$(1,0,0)$ in plane
$(0,-1,2)$ in plane.

Monday Feb 26
span, basis, dimension

5b) Show that the vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \underline{v}_3 = \begin{bmatrix} 3 \\ 4 \\ -8 \end{bmatrix}$$

are linearly dependent (even though no two of them are scalar multiples of each other). What does this mean geometrically about the span of these three vectors?
Hint: You might find this computation useful:

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & 2 & -8 \end{array}\right] \text{ reduces to } \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{array}\right]$$

dependent: $\vec{v}_3 = 3\vec{v}_1 - 4\vec{v}_2$

is $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{v}_3$?

$c_1 = 3$
$c_2 = -4$

$$\begin{bmatrix} 3 \\ 4 \\ -8 \end{bmatrix} \stackrel{?}{=} 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 4\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \checkmark$$

OR $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 2 & -8 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

∞'ly many soln's

Exercise 6) Are the vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \underline{w}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

linearly independent? What is their span? Hint:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right] \text{ reduces to } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0} \implies c_1 = 0, \; c_2 = 0, \; c_3 = 0.$

Exercise 1) Use properties of reduced row echelon form matrices to answer the following questions:

1a) Why must more than 3 vectors in $\mathbb{R}^3$ always be linearly dependent?

1b) Why can fewer than 3 vectors never span $\mathbb{R}^3$?
(So every basis of $\mathbb{R}^3$ must have exactly three vectors.)

1c) If you are given a set of 3 vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \underline{\boldsymbol{v}}_3\}$ in $\mathbb{R}^3$, what is the condition on the reduced row echelon form of the 3 × 3 matrix that has these vectors as columns, $[\underline{\boldsymbol{v}}_1\ \underline{\boldsymbol{v}}_2\ \underline{\boldsymbol{v}}_3]$ that guarantees they're linearly independent? That guarantees they span $\mathbb{R}^3$? That guarantees they're a basis of $\mathbb{R}^3$?

1d) What is the dimension of $\mathbb{R}^3$?

1e) How does this discussion generalize to $\mathbb{R}^n$?

New concepts for this week, related to linear combinations

In addition to vectors in $\mathbb{R}^n$, functions are objects that we can add and scalar multiply. (So are matrices). A lot of our discussions about linear combination concepts - such as "span" and "linear independence" apply in these more general settings, and so we introduce the abstract idea of a "vector space". (Trust me, this will help us in Chapter 5, when we return to differential equations.)

Definition: A *vector space* V is a collection of objects together with and "addition" operation "+", and a scalar multiplication operation, so that the rules below all hold.

(α) Whenever $f, g \in V$ then $f + g \in V$. (closure with respect to addition)
(β) Whenever $f \in V$ and $c \in \mathbb{R}$, then $c \cdot f \in V$. (closure with respect to scalar multiplication)

As well as:

(a) $f + g = g + f$ (commutative property)

(b) $f + (g + h) = (f + g) + h$ (associative property)

(c) $\exists\, 0 \in V$ so that $f + 0 = f$ is always true.

(d) $\forall f \in V \, \exists -f \in V$ so that $f + (-f) = 0$ (additive inverses)

(e) $c \cdot (f + g) = c \cdot f + c \cdot g$ (scalar multiplication distributes over vector addition)

(f) $(c_1 + c_2) \cdot f = c_1 \cdot f + c_2 \cdot f$ (scalar addition distributes over scalar multiplication)

(g) $c_1 \cdot (c_2 \cdot f) = (c_1 c_2) \cdot f$ (associative property)

(h) $1 \cdot f = f,\ (-1) \cdot f = -f,\ 0 \cdot f = 0$ (these last two actually follow from the others).

Notice that $V = \mathbb{R}^m$, $m = 1, 2, 3, \ldots$, with the usual vector addition and scalar multiplication, defined component-wise, satisfy these axioms. But we've actually already been seeing some vector spaces which are *not* $\mathbb{R}^m$, but are hiding in plain sight within $\mathbb{R}^m$. These are *subspaces* (of $\mathbb{R}^m$), which are their "own" vector spaces, contained within $\mathbb{R}^m$. (When you read the word *subspace* it might help if you say to yourself, *sub-vector-space*, for that's what these are).

What makes a subset of $\mathbb{R}^m$ (or any other vector space) into its own vector space? The answer is on the next page

Definition A subset W of a vector space V is a *subspace* if W is closed under addition and scalar multiplication, i.e. if the two vector space closure axioms hold:

$$(\alpha) \qquad f, g \in W \Rightarrow f + g \in W,$$
$$(\beta) \qquad f \in W, c \in \mathbb{R} \Rightarrow c f \in W.$$

In this case W inherits the other vector space axioms (a)-(h) just because they hold for all the vectors in V. (Check!) One axiom in particular is worth keeping in mind, namely from closure under scalar multiplication, (β), and the last axiom in (h), we see that every subspace contains the zero vector. So if you're checking whether a collection of objects is a subspace and the zero vector is not in the collection, you immediately know that your collection is not a subspace. You can see how W inherits the rest of the vector spaces axioms, once (α), (β) hold:

(a) $f + g = g + f$ (commutative property)

(b) $f + (g + h) = (f + g) + h$ (associative property)

(c) $\exists\ 0 \in V$ so that $f + 0 = f$ is always true.

(d) $\forall f \in V\ \exists -f \in V$ so that $f + (-f) = 0$ (additive inverses)

(e) $c \cdot (f + g) = c \cdot f + c \cdot g$ (scalar multiplication distributes over vector addition)

(f) $(c_1 + c_2) \cdot f = c_1 \cdot f + c_2 \cdot f$ (scalar addition distributes over scalar multiplication)

(g) $c_1 \cdot (c_2 \cdot f) = (c_1 c_2) \cdot f$ (associative property)

(h) $1 \cdot f = f$, $(-1) \cdot f = -f$, $0 \cdot f = 0$ (these last two actually follow from the others).

Definition: Let V be a vector space. Suppose $V = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ AND that the set of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ is linearly independent. Then $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ is called a *basis* for V. The number of vectors in this set, i.e. "n", is called the *dimension* of V. (It turns out that all bases for a vector space have the same number of vectors.)

Exercise 2) Most subsets of $\mathbb{R}^m$ are *not* subspaces. (Remember, "sub vector spaces"). Show that

2a) $W = \{[x, y]^T \in \mathbb{R}^2 \ s.t.\ x^2 + y^2 = 4\}$ is not a subspace of $\mathbb{R}^2$.

2b) $W = \{[x, y]^T \in \mathbb{R}^2 \ s.t.\ y = 3x + 1\}$ is not a subspace of $\mathbb{R}^2$.

2c) $W = \{[x, y]^T \in \mathbb{R}^2 \ s.t.\ y = 3x\}$ is a subspace of $\mathbb{R}^2$. Can you express this subspace as the span of a single vector in $\mathbb{R}^2$?

Tues Feb 27
4.2 - 4.3 subspaces and bases.

Announcements:

Warm-up Exercise:

We've been talking about vector spaces and subspaces, with examples in $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^n$.

Key facts about how subspaces (sub vector spaces) DO arise:

There are two main ways that subspaces arise: (These ideas will be important when we return to differential equations, in Chapter 5, although it's probably difficult to envision what they have to do with differential equations right now.)

1) $W = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n\}$ is always a subspace.
Expressing a subspace this way is an explicit way to describe the subspace W, because you are "listing" all of the vectors in it.

Why $W = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n\}$ is a subspace: Let $\underline{\boldsymbol{v}}, \underline{\boldsymbol{w}} \in W$. In other words, we can express

$$\underline{\boldsymbol{v}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n$$
$$\underline{\boldsymbol{w}} = d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + \ldots + d_n\underline{\boldsymbol{v}}_n \,.$$

So,

$$\underline{\boldsymbol{v}} + \underline{\boldsymbol{w}} = \left(c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n\right) + \left(d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + \ldots + d_n\underline{\boldsymbol{v}}_n\right).$$

After using the vector space axioms (addition is commuative and associative, and scalar addition distributes over scalar multiplication), we can rewrite

$$\underline{\boldsymbol{v}} + \underline{\boldsymbol{w}} = \left(c_1 + d_1\right)\underline{\boldsymbol{v}}_1 + \left(c_2 + d_2\right)\underline{\boldsymbol{v}}_2 + \ldots + \left(c_n + d_n\right)\underline{\boldsymbol{v}}_n \in W \,.$$

This verifies (α).

Now let $c \in \mathbb{R}$. Then

$$c\,\underline{\boldsymbol{v}} = c\left(c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n\right) = cc_1\underline{\boldsymbol{v}}_1 + cc_2\underline{\boldsymbol{v}}_2 + \ldots + cc_n\underline{\boldsymbol{v}}_n \in W$$

which verifies (β).

(And, notice that $0\,\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}} \in W$.)

Example: From yesterday's discussion, an example subspace in $\mathbb{R}^3$:

$$W = span\left\{\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \underline{\boldsymbol{v}}_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}\right\}$$

is a 2-dimensional subspace, i.e. plane through the origin.

The other way subspaces arise (in $\mathbb{R}^n$) :

2) Let A be an $m \times n$ matrix. Let $V = \{\underline{\mathbf{x}} \in \mathbb{R}^n \textit{ such that } A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}\}$. Then V is a subspace. (We call this collection of vectors the "homogeneous solution space" or "null space" of A.

Note that this is an <u>implicit way</u> to describe the subspace V because we're only specifying a homogeneous matrix equation that the vectors in V must satisfy, but you're not saying what the vectors are.

Why V is a subspace: Let $\underline{\mathbf{v}}, \underline{\mathbf{w}} \in V \Rightarrow$

$$A\underline{\mathbf{v}} = \underline{\mathbf{0}},\ A\underline{\mathbf{w}} = \underline{0} \quad \Rightarrow \quad A(\underline{\mathbf{v}} + \underline{\mathbf{w}}) = A\underline{\mathbf{v}} + A\underline{\mathbf{w}} = \underline{\mathbf{0}} + \underline{\mathbf{0}} = \underline{\mathbf{0}}, \quad \Rightarrow \quad \underline{\mathbf{v}} + \underline{\mathbf{w}} \in V \ \text{(verifies } \alpha)$$

and let $c \in \mathbb{R} \Rightarrow$

$$A\underline{\mathbf{v}} = \underline{0} \quad \Rightarrow \quad A(c\,\underline{\mathbf{v}}) = c\,A\,\underline{\mathbf{v}} = c\,\underline{\mathbf{0}} = \underline{\mathbf{0}}, \quad \Rightarrow \quad c\,\underline{\mathbf{v}} \in V \ \text{(verfies } \beta).$$

(and $\underline{\mathbf{0}} \in V$, since $A\,\underline{\mathbf{0}} = \underline{\mathbf{0}}$.)

....

<u>Example:</u> Continuing the example from the previous page, the plane

$$W = span\left\{\underline{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \underline{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}\right\}$$

could have been described implicitly as the collection of position vectors for points (x, y, z) satisfying the very small homogeneous matrix equation

$$0\,x + 2\,y + z = 0.$$

$$\begin{bmatrix} 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ x \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}.$$

Exercise 1) Use geometric reasoning to argue why the *only* subspaces of $\mathbb{R}^2$ are as follows. Also, what are the dimensions of these subspaces?

(0) The single vector $[0, 0]^T$, or

(1) A line through the origin, i.e. $span\{\underline{\boldsymbol{u}}\}$ for some non-zero vector $\underline{\boldsymbol{u}}$, or

(2) All of $\mathbb{R}^2$.

Exercise 2) Use matrix theory to show that the *only* subspaces of $\mathbb{R}^3$ are

(0) The single vector $[0, 0, 0]^T$, or

(1) A line through the origin, i.e. $span\{\underline{\boldsymbol{u}}\}$ for some non-zero vector $\underline{\boldsymbol{u}}$, or

(2) A plane through the origin, i.e.$span\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}\}$ where $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$ are linearly independent, or

(3) All of $\mathbb{R}^3$.

Wed Feb 28
4.3 - 4.4. linear independence, bases and dimension for vector spaces.

Announcements:

Warm-up Exercise:

Definition Let V be a vector space. Recall that a *basis* for V is

a collection of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ *that is linearly independent and that spans V.*

We use the word "basis", because from just the vectors in this collection we are able to reconstruct the entire space - by taking linear combinations. *AND* each vector in V can be expressed in one and only one way as a linear combination of the vectors in the basis, so there is no unneccesary reduncancy:

Theorem For a collection linearly independent vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$, every vector $\underline{\boldsymbol{v}}$ in their span can be written as $\underline{\boldsymbol{v}} = d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + \ldots + d_n\underline{\boldsymbol{v}}_n$ uniquely, i.e. for exactly one choice of linear combination coefficients $d_1, d_2, \ldots d_n$. This is not true if vectors are dependent.

proof:

Usually in applications we do not start with a basis for a subspace - rather this is the goal we search for.

For example, if we wish to find a basis for the homogeneous solution space $W = \{\underline{\mathbf{x}} \in \mathbb{R}^n \text{ such that } A_{m \times n}\underline{\mathbf{x}} = \underline{\mathbf{0}}\}$, then the algorithm we already learned in Chapter 3 will work, if we interpret the results correctly: Reduce the augmented matrix, backsolve and write the explicit solution in linear combination form. The vectors that you are taking linear combinations of will always span the solution space, by construction. If you follow this algorithm the generating vectors will automatically be linearly independent, so they will be a basis for the solution space. This is illustrated in the large example below:

Exercise 1 Consider the matrix equation $A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$, with the matrix A (and its reduced row echelon form) shown below:

$$A := \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 2 & 4 & 1 & 0 & 7 \\ -1 & -2 & 1 & -3 & 1 \end{bmatrix} \qquad \text{reduces to} \qquad \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a basis for the solution space $W = \{\underline{\mathbf{x}} \in \mathbb{R}^5 \text{ s.t. } A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}\}$ by backsolving, writing your explicit solutions in linear combination form, and extracting a basis. Explain why these vectors span the solution space and verify that they're linearly independent.

Exercise 2) Now we're going to study a different subspace that turns out in this case to be connected to the same matrix A. Suppose we wish to understand the subspace of $\mathbb{R}^3$ that is given by

$$V = span\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}2\\4\\-2\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}2\\7\\1\end{bmatrix}\right\}.$$

This is a subspace of $\mathbb{R}^3$ - so what kind is it? And can we find a basis? Clearly the original five vectors can't be a basis, because more than three vectors in $\mathbb{R}^3$ are always linearly dependent. If we can find dependencies we'll be able to shrink the number of vectors in our generating set for V until we reduce it to a basis. This is how:

Let A be the matrix that has those five vectors as columns. (In this case it's the same matrix A as on the previous page.)

$$A := \begin{bmatrix}1 & 2 & 0 & 1 & 2\\ 2 & 4 & 1 & 0 & 7\\ -1 & -2 & 1 & -3 & 1\end{bmatrix}$$

Focus on the idea that solutions to homogeneous matrix equations correspond exactly to linear dependencies between the columns of the matrix. (If it helps, think of renaming the vector $\underline{\boldsymbol{x}}$ in the example above, with a vector $\underline{\boldsymbol{c}}$ of linear combination coefficients; then recall the prime Chapter 4 algebra fact that

$$A\,\underline{\boldsymbol{c}} = c_1 col_1(A) + c_2 col_2(A) + \ldots + c_n col_n(A)\ .$$

So any solution $\underline{\boldsymbol{c}}$ to $A\,\underline{\boldsymbol{c}} = \underline{\boldsymbol{0}}$ is secretly a columns dependency, and vise-verse.)

Now, since the solution set to a homogeneous linear system does not change as you do elementary row operations to the augmented matrix, column dependencies also do not change.

2a) Check how column dependencies don't change as you reduce a matrix, by reading off "easy" column dependencies in the reduced matrix; seeing that they are also dependencies in the original matrix; also, that these dependencies actually correspond to the basis of the homogeneous solution space that you were studying in exercise 1. Magic! We will use this magic in important interesting ways, later in the course.

$$A := \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 2 & 4 & 1 & 0 & 7 \\ -1 & -2 & 1 & -3 & 1 \end{bmatrix} \qquad \text{reduces to} \qquad \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2b) Find a basis for

$$V = span\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix} \right\}.$$

(And identify which type of subspace of $\mathbb{R}^3$ we're dealing with here.)

Exercise 3) (This exercise explains why any given matrix has only one reduced row echelon form, no matter what sequence of elementary row operations one uses to find it. We didn't have the tools to explain why this fact was true earlier, back in Chapter 3.) Let $B_{4\times 5}$ be a matrix whose columns satisfy the following dependencies:

$$col_1(B) \neq \underline{\mathbf{0}} \quad \text{(i.e. is independent)}$$
$$col_2(B) = 3\ col_1(B)$$
$$col_3(B) \text{ is independent of column 1}$$
$$col_4(B) \text{ is independent of columns 1,3.}$$
$$col_5(B) = -3\ col_1(B) + 2\ col_3(B) - col_4(B).$$

What does the reduced row echelon form of B have to be?

Some important facts about spanning sets, independence, bases and dimension follow from one key fact, and then logic. We will want to use these facts going forward, as we return to studying differential equations on Friday.

key fact: If n vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ span a subspace W then any collection $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_N$ of vectors in W with $N > n$ will always be linearly dependent. (This is explained on pages 254-255 of the text, and has to to do with matrix facts that we already know.) Notice too that this fact fits our intuition based on what we know in the special cases that we've studied, in particular $W = \mathbb{R}^n$.)

Thus:
1) If a finite collection of vectors in W is linearly independent, then no collection with fewer vectors can span all of W. (This is because if the smaller collection did span, the larger collection wouldn't have been linearly independent after all, by the key fact.)

2) Every basis of W has the same number of vectors, so the concept of dimension is well-defined and doesn't depend on choice of basis. (This is because if $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are a basis for W then every larger collection of vectors is dependent by the key fact and every smaller collection fails to span by (1), so only collections with exactly n vectors have a chance to be bases.)

3) Let the dimension of W be the number n, i.e. there is some basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ for W. Then if vectors $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$ span W then they're automatically linearly independent and thus a basis. (If they were dependent we could delete one of the $\underline{\boldsymbol{w}}_j$ that was a linear combination of the others and still have a spanning set. This would violate (1) since $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly independent.)

4) Let the dimension of W be the number n, i.e. there is some basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ for W. Then if $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$ are in W and are linearly independent, they automatically span W and thus are a basis. (If they didn't span W we could augment with a vector $\boldsymbol{w}_{n+1}$ not in their span and have a collection of $n+1$ still independent* vectors in W, violating the key fact.

* Check: If $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$ are linearly independent, and $\underline{\boldsymbol{w}}_{n+1} \notin span\{\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n\}$, then $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n, \underline{\boldsymbol{w}}_{n+1}$ are also linearly independent. This fact generalizes the ideas we used when we figured out all possible subspaces of $\mathbb{R}^3$. Here's how it goes:

To show the larger collection is still linearly independent study the equation
$$c_1\underline{\boldsymbol{w}}_1 + c_2\underline{\boldsymbol{w}}_2 + \ldots + c_n\underline{\boldsymbol{w}}_n + d\,\underline{\boldsymbol{w}}_{n+1} = \underline{\boldsymbol{0}}.$$
Since $\underline{\boldsymbol{w}} \notin span\{\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n\}$ it must be that $d = 0$ (since otherwise we could solve for $\underline{\boldsymbol{w}}_{n+1}$ as a linear combination of $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$). But once $d = 0$, we have
$$c_1\underline{\boldsymbol{w}}_1 + c_2\underline{\boldsymbol{w}}_2 + \ldots + c_n\underline{\boldsymbol{w}}_n = \underline{\boldsymbol{0}}$$
which implies $c_1 = c_2 = \ldots = c_n = 0$ by the independence of $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$.

Fri Mar 2
5.1 Second order linear differential equations, and vector space theory connections.

In Chapter 5 we'll be using vector space theory to understand solutions to differential equations!

Announcements:

Warm-up Exercise:

Exercise 0) In Chapter 5 we focus on the vector space

$$V = C(\mathbb{R}) := \{f: \mathbb{R} \to \mathbb{R} \text{ s.t. } f \text{ is a continuous function}\}$$

and its subspaces. Verify that the vector space axioms for linear combinations are satisfied for this space of functions. Recall that the function $f + g$ is defined by $(f + g)(x) := f(x) + g(x)$ and the scalar multiple $c f(x)$ is defined by $(c f)(x) := c f(x)$. What is the zero vector for functions?

Definition: A vector space is a collection of objects together with and "addition" operation "+", and a scalar multiplication operation, so that the rules below all hold.

(α) Whenever $f, g \in V$ then $f + g \in V$. (closure with respect to addition)

(β) Whenever $f \in V$ and $c \in \mathbb{R}$, then $c \cdot f \in V$. (closure with respect to scalar multiplication)

As well as:

(a) $f + g = g + f$ (commutative property)

(b) $f + (g + h) = (f + g) + h$ (associative property)

(c) $\exists\ 0 \in V$ so that $f + 0 = f$ is always true.

(d) $\forall f \in V\ \exists -f \in V$ so that $f + (-f) = 0$ (additive inverses)

(e) $c \cdot (f + g) = c \cdot f + c \cdot g$ (scalar multiplication distributes over vector addition)

(f) $(c_1 + c_2) \cdot f = c_1 \cdot f + c_2 \cdot f$ (scalar addition distributes over scalar multiplication)

(g) $c_1 \cdot (c_2 \cdot f) = (c_1 c_2) \cdot f$ (associative property)

(h) $1 \cdot f = f$, $(-1) \cdot f = -f$, $0 \cdot f = 0$ (these last two actually follow from the others).

Because the vector space axioms are exactly the arithmetic rules we used to work with linear combination equations, all of the concepts and vector space theorems we talked about for $\mathbb{R}^m$ and its subspaces make sense for the function vector space V and its subspaces. In particular we can talk about

- the span of a finite collection of functions $f_1, f_2, \ldots f_n$, $span\{f_1, f_2, \ldots f_n\}$.

- linear independence/dependence for a collection of functions $\{f_1, f_2, \ldots f_n\}$.

- subspaces of V

- bases and dimension for finite dimensional subspaces. (The function space V in Exercise 0 itself is infinite dimensional, meaning that no finite collection of functions spans it.)

Exercise 1 Consider the three functions with domain $\mathbb{R}$, given by

$$f_1(x) = 1, \;\; f_2(x) = x, \; f_3(x) = x^2.$$

1a) Describe $span\{f_1, f_2, f_3\}$.

1b) Is the set $\{f_1, f_2, f_3\}$ linearly dependent or linearly independent?

Definition: A *second order linear differential equation* for a function $y(x)$ is a differential equation that can be written in the form

$$A(x)y'' + B(x)y' + C(x)y = F(x) .$$

We search for solution functions $y(x)$ defined on some specified interval I of the form $a < x < b$, or (a, ∞), $(-\infty, a)$ or (usually) the entire real line $(-\infty, \infty)$. In this chapter we assume the function $A(x) \neq 0$ on I, and divide by it in order to rewrite the differential equation in the standard form

$$y'' + p(x)y' + q(x)y = f(x) .$$

One reason this DE is called *linear* is that the "operator" L defined by

$$L(y) := y'' + p(x)y' + q(x)y$$

satisfies the so-called *linearity properties*

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$
(2) $L(c\,y) = c\,L(y)$, $c \in \mathbb{R}$.

(Recall that the matrix multiplication function $L(\underline{\mathbf{x}}) := A\,\underline{\mathbf{x}}$ satisfies the analogous properties. Any time we have have a transformation L satisfying (1),(2), we say it is a *linear transformation.*)

Exercise 2a) Check the linearity properties (1),(2) for the differential operator L .

2b) Use these properties to show that

Theorem 1: the solution space to the *homogeneous* second order linear DE

$$y'' + p(x)y' + q(x)y = 0$$

is a subspace. Notice that this is the analogous proof we used earlier to show that the solution space to a homogeneous matrix equation is a subspace.

Exercise 3) Find subspace of solutions to to homogeneous differential equation for $y(x)$

$$y'' + 2y' = 0$$

on the x-interval $-\infty < x < \infty$. Notice that the solution space is the span of two independent functions, so the solution space is *2-dimensional*. (It will turn out that the solution space to second order linear homogeneous DE's is always *2-dimensional.*) Hint: This is really a first order DE for $v = y'$.

Theorem 2 (Existence-Uniqueness Theorem): Let $p(x), q(x), f(x)$ be specified continuous functions on the interval I, and let $x_0 \in I$. Then there is a unique solution $y(x)$ to the initial value problem

$$y'' + p(x)y' + q(x)y = f(x)$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$

and $y(x)$ exists and is twice continuously differentiable on the entire interval I.

Exercise 4) Verify Theorem 2 for the interval $I = (-\infty, \infty)$ and the IVP

$$y'' + 2y' = 0$$
$$y(0) = b_0$$
$$y'(0) = b_1$$

Unlike in the previous example, and unlike what was true for the first order linear differential equation

$$y' + p(x)y = q(x)$$

there is not a clever integrating factor formula that will always work to find the general solution of the second order linear differential equation

$$y'' + p(x)y' + q(x)y = f(x).$$

Rather, we will usually resort to vector space theory and algorithms based on clever guessing, as in the following example:

Exercise 5) Consider the homogeneous linear DE for $y(x)$

$$y'' - 2y' - 3y = 0$$

5a) Find two exponential functions $y_1(x) = e^{rx}$, $y_2(x) = e^{\rho x}$ that solve this DE.

5b) Show that every IVP

$$y'' - 2y' - 3y = 0$$
$$y(0) = b_0$$
$$y'(0) = b_1$$

can be solved with a unique linear combination $y(x) = c_1 y_1(x) + c_2 y_2(x)$, (where c_1, c_2 depend on b_0, b_1).

Then use the uniqueness theorem to deduce that y_1, y_2 span the solution space to this homogeneous differential equation. Since these two functions are not constant multiples of each other, they are linearly independent and a basis for the *2-dimensional* solution space!