Math 2250-004 Week 7 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes include material from 3.6, 4.1-4.3.

Tues Feb 20
3.6 Matrix inverses with determinants

Announcements:
- return exams tomorrow, everything will be posted tonight
- Start Chapter 4 tomorrow → "vectorspaces"
- HW this week: 3.6 & 4.1 → due in the lab

'til 10:47

Warm-up Exercise: Compute the following two determinants by expanding across the 1^{st} row. Notice that
1) the 1^{st} row cofactors are the same (because rows 2 & 3 are the same in each matrix
2) You know before you start that one of these determinants has to be zero

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix} = 1\cdot\left(+\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix}\right) + 2\left(-\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}\right) - 1\begin{vmatrix} 0 & 3 \\ 2 & -2 \end{vmatrix}$$

$$= 1\cdot 5 - 2(-2) - 1(-6) = 5+4+6 = 15$$

related to magic formula for matrix inverses

$$0 = \begin{vmatrix} 0 & 3 & 1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix} = 0\cdot\left(+\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix}\right) + 3\left(-\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}\right) + 1\left(+\begin{vmatrix} 0 & 3 \\ 2 & -2 \end{vmatrix}\right)$$

$$= 0 + 3(--2) + 1(-6) = 6-6 = 0$$

2 rows equal $\Rightarrow \det = 0$.

Cofactors were the same

Wednesday

On ~~Friday~~ we figured out what elementary row operations do to determinants, and used that reasoning to verify that a square matrix A is invertible exactly when $det(A) \neq 0$:

Theorem: Let $A_{n \times n}$. Then A^{-1} exists if and only if $det(A) \neq 0$.

proof: We already know that A^{-1} exists if and only if the reduced row echelon form of A is the identity matrix. Now, consider reducing A to its reduced row echelon form, and keep track of how the determinants of the corresponding matrices change: As we do elementary row operations,

- if we swap rows, the sign of the determinant switches.
- if we factor non-zero factors out of rows, we factor the same factors out of the determinants.
- if we replace a row by its sum with a multiple of another row, the determinant is unchanged.

Thus,

$$|A| = c_1|A_1| = c_1c_2|A_2| = \ldots = c_1c_2 \ldots c_N |rref(A)|$$

where the nonzero c_k's arise from the three types of elementary row operations. If $rref(A) = I$ its determinant is 1, and $|A| = c_1c_2 \ldots c_N \neq 0$. If $rref(A) \neq I$ then its bottom row is all zeroes and its determinant is zero, so $|A| = c_1c_2 \ldots c_N(0) = 0$. Thus $|A| \neq 0$ if and only if $rref(A) = I$ if and only if A^{-1} exists !

Today, we'll use the elementary row operation facts to understand the "magic" formulas for matrix inverses. The formula uses the determinant of A along with the cofactor matrix of A. To get started we first need to talk about matrix *transposes:*

Definition: Let $B_{m \times n} = \left[b_{i\,j} \right]$. Then the transpose of B, denoted by B^T is an $n \times m$ matrix defined by

$$entry_{i\,j}\left(B^T\right) := entry_{j\,i}(B) = b_{j\,i}.$$

$B_{m\times n}$ $\quad B^T_{n\times m}$

The effect of this definition is to turn the columns of B into the rows of B^T:

$$entry_i\left(col_j(B)\right) = b_{i\,j}.$$

$$entry_i\left(row_j\left(B^T\right)\right) = entry_{j\,i}\left(B^T\right) = b_{i\,j}.$$

$$col_j(B) = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{bmatrix}$$

And to turn the rows of B into the columns of B^T:

$$entry_j\left(row_i(B)\right) = b_{i\,j}$$

$$entry_j\left(col_i\left(B^T\right)\right) = entry_{j\,i}\left(B^T\right) = b_{i\,j}.$$

Exercise 1) explore these properties with the identity

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

$$row_j\left(B^T\right) = \left[b^T_{j1} \ \ b^T_{j2} \ \cdots \ b^T_{jm} \right] = \left[b_{1j} \ \ b_{2j} \ \ \ b_{mj} \right]$$

same entries as $col_j(B)$!

Theorem: Let $A_{n \times n}$, and denote its cofactor matrix by $cof(A) = [C_{i j}]$, with $C_{i j} = (-1)^{i+j} M_{i j}$, and $M_{i j} =$ the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A.
Define the adjoint matrix to be the transpose of the cofactor matrix:

$$Adj(A) := cof(A)^T$$

Then, when A^{-1} exists it is given by the formula

$$A^{-1} = \frac{1}{det(A)} Adj(A) \;. \quad = \frac{1}{|A|} (cof(A))^T$$

Fix this!

Exercise 2) Show that in the 2×2 case this reproduces the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} .$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad cof(A) = \begin{bmatrix} +d & -c \\ -b & +a \end{bmatrix} \qquad (cof(A))^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

general formula says

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \checkmark$$

Example) Last week for the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$ we worked out $cof(A) = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$:

$$cof(A) = \begin{vmatrix} +\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 3 \\ 2 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} \\ +\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} & +\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \end{vmatrix}$$

We take the transpose of the cofactor matrix to get the adjoint:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad adj(A) = (cof(A))^T = \begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix} \qquad det(A) = 15,$$

so according the magic formula,

$$A^{-1} = \frac{1}{15}\begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix}$$

check!

$$\frac{1}{15}\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{bmatrix} = I \text{ !!}$$

Let's understand why the magic worked. In other words, let's understand conceptually why

$$[A\,][adj(A)] = \begin{bmatrix} det(A) & 0 & 0 \\ 0 & det(A) & 0 \\ 0 & 0 & det(A) \end{bmatrix}$$

Exercise 3) Continuing with our example,

3a) The $(1, 1)$ entry of $(A)(Adj(A))$ is $15 = 1\cdot5 + 2\cdot2 + (-1)(-6)$. Explain why this is $det(A)$, expanded across the first row. (Similar reasoning for the other diagonal entries.)

$row_1(A) \cdot row_1(cof(A)) = |A|$ expanding across row_1

$(2,2)$ entry of $A(adj(A)) = row_2(A) \cdot row_2(cof(A)) = |A|$ expanding across row 2.

3b) The $(2, 1)$ entry of $(A)(Adj(A))$ is $0\cdot5 + 3\cdot2 + (1)(-6) = 0$. Notice that you're using the same cofactors as in (2a). What matrix, which is obtained from A by keeping two of the rows, but replacing a third one with one of those two, is this the determinant of?

cofactors from 1st row of A. factors were from 2nd row of A

$$= \begin{vmatrix} 0 & 3 & 1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix} \left.\vphantom{\begin{matrix} 0 \\ 0 \end{matrix}}\right\} \text{from } A \quad = 0$$

3c) The $(3, 2)$ entry of $(A)(Adj(A))$ is $\underline{2\cdot0 - 2\cdot3 + 1\cdot6} = 0$. What matrix (which uses two rows of A) is this the determinant of?

cofactors from 2nd row of A

$$= \begin{vmatrix} 1 & 2 & -1 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{vmatrix} = 0$$

from the previous page, for Exercise 3:

$$\begin{array}{c} 3a \\ 3a\ 3b \\ 3c \end{array} \; A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad cof(A) = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix} \begin{array}{c} 3a\ 3b \\ 3a \\ \end{array} \qquad adj(A) = \begin{array}{ccc} 3a\,3b & 3a\,3c & \\ \end{array}\begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix}$$

If you completely understand 3abc, then you have realized why

$$[A][Adj(A)] = det(A)[I]$$

for every square matrix, and so also why

$$A^{-1} = \frac{1}{det(A)} Adj(A) .$$

Precisely,

$$entry_{i\,i}\, A(Adj(A)) = row_i(A) \cdot col_i(Adj(A)) = row_i(A) \cdot row_i(cof(A)) = det(A),$$

expanded across the i^{th} row.

On the other hand, for $i \neq k$,

$$entry_{k\,i}\, A(Adj(A)) = row_k(A) \cdot col_i(Adj(A)) = row_k(A) \cdot row_i(cof(A)) .$$

This last dot produce is zero because it is the determinant of a matrix made from A by replacing the i^{th} row with the k^{th} row, expanding across the i^{th} row, and whenever two rows are equal, the determinant of a matrix is zero:

$$i^{th} \text{ row position} \begin{vmatrix} \mathscr{R}_1 \\ \mathscr{R}_2 \\ \mathscr{R}_k \\ \mathscr{R}_k \\ \mathscr{R}_n \end{vmatrix} . = 0$$

There's a related formula for solving for individual components of $\underline{\boldsymbol{x}}$ when $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ has a unique solution ($\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$). This can be useful if you only need one or two components of the solution vector, rather than all of it:

<u>Cramer's Rule</u>: Let $\underline{\boldsymbol{x}}$ solve $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$, for invertible A. Then

$$x_k = \frac{det(A_k)}{det(A)}$$

where A_k is the matrix obtained from A by replacing the k^{th} column with $\underline{\boldsymbol{b}}$.

proof: Since $\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$ the k^{th} component is given by

$$\begin{aligned} x_k &= entry_k\left(A^{-1}\underline{\boldsymbol{b}}\right) \\ &= entry_k\left(\frac{1}{|A|}Adj(A)\underline{\boldsymbol{b}}\right) \\ &= \frac{1}{|A|}row_k(Adj(A)) \cdot \underline{\boldsymbol{b}} \\ &= \frac{1}{|A|}col_k(cof(A)) \cdot \underline{\boldsymbol{b}}\,. \end{aligned}$$

read on your own (if you want).

Notice that $col_k(cof(A)) \cdot \underline{\boldsymbol{b}}$ is the determinant of the matrix obtained from A by replacing the k^{th} column by $\underline{\boldsymbol{b}}$, where we've computed that determinant by expanding down the k^{th} column! This proves the result. (See our text for another way of justifying Cramer's rule.)

$$4a):\quad x = \frac{\begin{vmatrix} 7 & -1 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 5 & -1 \\ 4 & 1 \end{vmatrix}} = \frac{9}{9} = 1$$

$$y = \frac{\begin{vmatrix} 5 & 7 \\ 4 & 2 \end{vmatrix}}{9} = \frac{-18}{9} = -2$$

<u>Exercise 4)</u> Solve $\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$.

<u>4a)</u> With Cramer's rule

<u>4b)</u> With A^{-1}, using the adjoint formula.

$$4b)\quad A\vec{x} = \vec{b}$$

$$\vec{x} = A^{-1}\vec{b} = \frac{1}{9}\begin{bmatrix} 1 & 1 \\ -4 & 5 \end{bmatrix}\begin{bmatrix} 7 \\ 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 \\ -18 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

(you would check).

Wed Feb 21
4.1-4.2 The vector spaces $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^n$.

Announcements:

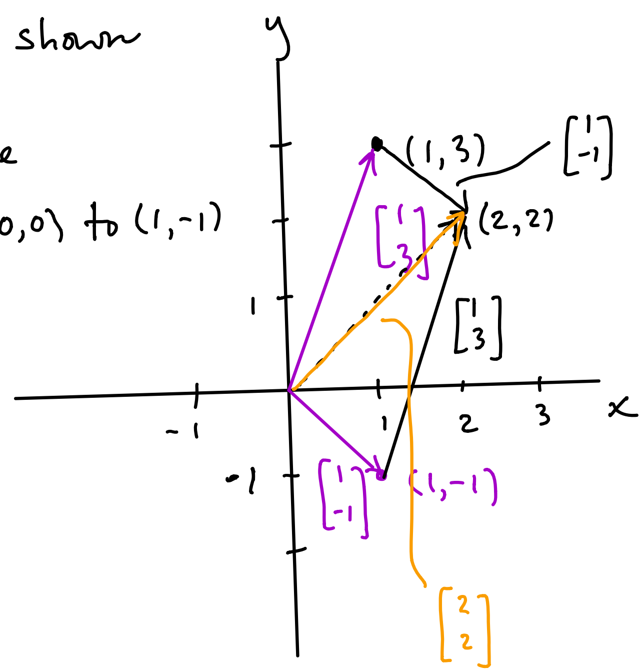
- Homework is due tomorrow in lab
- Quiz is a takehome quiz due in lab.

Warm-up Exercise: 'til 10:47

The point (1,3) is plotted below
The position vector displacement

2250 → $\begin{bmatrix}1\\3\end{bmatrix} = \underbrace{\hat{\imath}}_{2210,\,1320,\,1321} + 3\hat{\jmath}$ is also shown

a) plot the point (1,-1) and the position vector $\begin{bmatrix}1\\-1\end{bmatrix}$ from (0,0) to (1,-1)



b) compute $\begin{bmatrix}1\\3\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$

and plot the point for which it is the position vector

(2,2)

4.1-4.2 The vector space $\mathbb{R}^m$ and its subspaces; concepts related to "linear combinations of vectors."

Geometric interpretation of vectors

The space $\mathbb{R}^n$ may be thought of in two equivalent ways. In both cases, $\mathbb{R}^n$ consists of all possible $n - tuples$ of numbers:

(i) We can think of those $n - tuples$ as representing points, as we're used to doing for $n = 1, 2, 3$. In this case we can write

$$\mathbb{R}^n = \{ (x_1, x_2, ..., x_n), \ s.t. \ x_1, x_2, ..., x_n \in \mathbb{R} \}.$$

(ii) We can think of those $n - tuples$ as representing vectors that we can add and scalar multiply. In this case we can write

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_n \end{bmatrix}, \ s.t. \ x_1, x_2, ..., x_n \in \mathbb{R} \right\}.$$

Since algebraic vectors (as above) can be used to measure geometric displacement, one can identify the two models of $\mathbb{R}^n$ as sets by identifying each point $(x_1, x_2, ... x_n)$ in the first model with the displacement vector $\underline{\mathbf{x}} = [x_1, x_2, ... x_m]^T$ from the origin to that point, in the second model, i.e. the "position vector" of the point.

One of the key themes of Chapter 4 is the idea of *linear combinations*. These have an algebraic definition as well as a geometric interpretation as combinations of displacements, as we will review in our first few exercises.

Definition: If we have a collection of n vectors $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n\}$ in $\mathbb{R}^m$, then any vector $\underline{v} \in \mathbb{R}^m$ that can be expressed as a sum of scalar multiples of these vectors is called a linear combination of them. In other words, if we can write

"weighted sum"

$$\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + \ldots + c_n\underline{v}_n\,,$$

then $\underline{v}$ is a linear combination of $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$. The scalars $c_1, c_2, \ldots, c_n$ are called the *weights* or *linear combination coefficients*.

Example You've probably seen linear combinations in previous math/physics classes. For example you might have expressed the position vector $\underline{\mathbf{r}}$ of a point (x, y, z) as a linear combination

$$\underline{\mathbf{r}} = x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\boldsymbol{k}}$$

where $\underline{\mathbf{i}}, \underline{\mathbf{j}}, \underline{\mathbf{k}}$ represent the unit displacements in the Since we can express these displacements using Math 2250 notation as

$$\hat{\underline{\mathbf{i}}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \hat{\underline{\mathbf{j}}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \hat{\underline{\mathbf{k}}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we have

$$x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\boldsymbol{k}} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Remarks: When we had free parameters in our explicit solutions to linear systems of equations $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ back in Chapter 3, we sometimes rewrote the explicit solutions using linear combinations, where the scalars were the free parameters (which we often labeled with letters that were t, t_4, t_3 etc., rather than with "c's"). When we return to differential equations in Chapter 5 -studying higher order differential equations - then the explicit solutions will also be expressed using "linear combinations", just as we did in Chapters 1 -2, where we used the letter "C" for the single free parameter in first order differential equation solutions:

Definition: If we have a collection $\{y_1, y_2, \ldots, y_n\}$ of n functions $y(x)$ defined on a common interval I, then any function that can be expressed as a sum of scalar multiples of these functions is called a linear combination of them. In other words, if we can write

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n\,,$$

then y is a linear combination of $y_1, y_2, \ldots, y_n$.

The reason that the same words are used to describe what look like two quite different settings, is that there is a common fabric of mathematics (called vector space theory) that underlies both situations. We shall be exploring these concepts over the next several lectures, using a lot of the matrix algebra theory we've just developed in Chapter 3. This vector space theory will tie in directly to our study of differential equations, in Chapter 5 and subsequent chapters.

Exercise 1) Let $\underline{\boldsymbol{u}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

1a) Plot the points $(1, -1)$ and $(1, 3)$, which have position vectors $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}$. Draw these position vectors as arrows beginning at the origin and ending at the corresponding points.

1b) Compute $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}$ and then plot the point for which this is the position vector. Note that the algebraic operation of vector addition corresponds to the geometric process of composing horizontal and vertical displacements.

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

1c) Compute $\underline{\boldsymbol{u}}$ and $2\,\underline{\boldsymbol{v}}$, $\underline{\boldsymbol{u}} + 2\,\underline{\boldsymbol{v}}$ and plot the corresponding points for which these are the position vectors.

1d) Plot the parametric line whose points are the endpoints of the position vectors $\{\underline{\boldsymbol{u}} + t\,\underline{\boldsymbol{v}},\ t \in \mathbb{R}\}$. How else might you have expressed this parametric line in multivariable calculus class? What is the implicit equation of this line?

$\vec{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\quad 2\vec{v} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

$$\vec{u} + 2\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

1d) $\{\vec{u} + t\vec{v}\}_{t \in \mathbb{R}}$

endpoints are a line

$$\vec{u} + t\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+t \\ -1+3t \end{bmatrix}$$

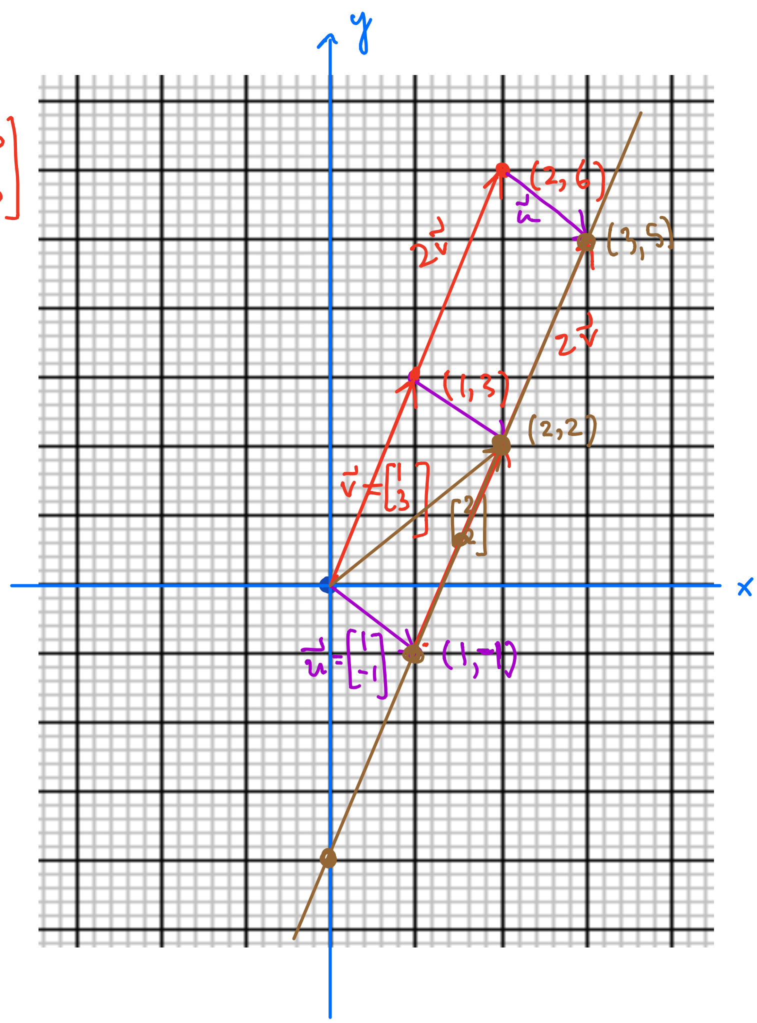
$x = 1 + t \rightarrow t = x - 1$

$y = -1 + 3t \rightarrow$

$y = -1 + 3(x-1)$

$$\boxed{y = 3x - 4}$$

agrees with graph

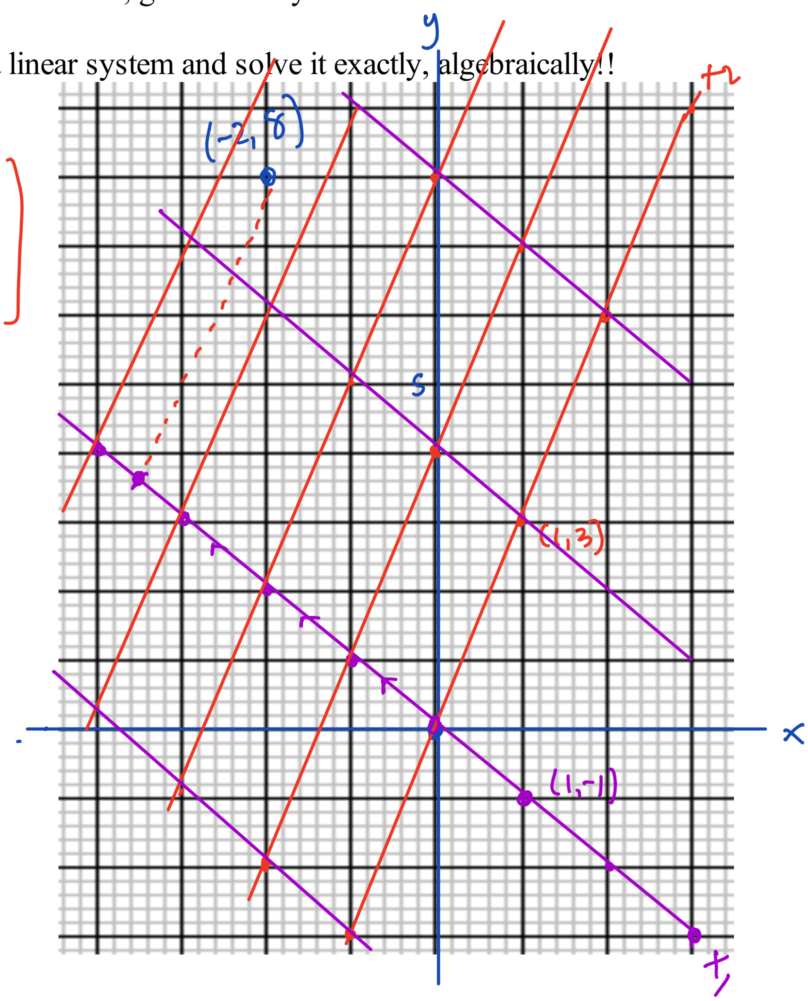


Exercise 2) Can you get to the point $(-2, 8) \in \mathbb{R}^2$, from the origin $(0, 0)$, by moving only in the $(\pm)$ directions of $\underline{\boldsymbol{u}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$? Algebraically, this means we want to solve the linear combination problem

$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}.$$

2a) Superimpose a grid related to the displacement vectors $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$ onto the graph paper below, and, recalling that vector addition yields net displacement, and scalar multiplication yields scaled displacement, try to approximately solve the linear combination problem above, geometrically.

2b) Rewrite the linear combination problem as a linear system and solve it exactly, algebraically!!



2 a) $-3.5 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1.5 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$

actually correct!

2 b) solve algebraically.

$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 3x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_2 &= -2 \\ -x_1 + 3x_2 &= 8 \end{aligned}$$

$$\begin{array}{cc|c} 1 & 1 & -2 \\ -1 & 3 & 8 \end{array}$$

$R_1 + R_2 \to R_2$
$$\begin{array}{cc|c} 1 & 1 & -2 \\ 0 & 4 & 6 \end{array}$$

$\frac{R_2}{4} \to R_2$
$$\begin{array}{cc|c} 1 & 1 & -2 \\ 0 & 1 & 1.5 \end{array}$$

$-R_2 + R_1 \to R_1$
$$\begin{array}{cc|c} 1 & 0 & -3.5 \\ 0 & 1 & 1.5 \end{array}$$

$x_1 = -3.5$
$x_2 = 1.5$

2c) Can you get to any point (x, y) in $\mathbb{R}^2$, starting at $(0, 0)$ and moving only in directions parallel to $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}$?

Yes. It's clear

Argue geometrically and algebraically. How many ways are there to express $\begin{bmatrix} x \\ y \end{bmatrix}$ as a linear combination of $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$?

only one way

Yes.

$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

solve for x_1, x_2.

$$\left[\begin{array}{cc|c} 1 & 1 & x \\ -1 & 3 & y \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & *_1 \\ 0 & 1 & *_2 \end{array}\right]$$

reason

$$\operatorname{rref}\left(\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\right) = I$$

$x_1 = *_1$

$x_2 = *_2$

Definition The *span* of a collection of vectors, written as $span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$, is the collection of all linear combinations of those vectors.

Examples: We showed in 2c that $span\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}\} = \mathbb{R}^2$. ~~Span~~

Remark: The mathematical meaning of the word *span* is related to the English meaning - as in "wing span" or "span of a bridge", but it's also different. The span of a collection of vectors goes on and on and does not "stop" at the vector or associated endpoint:

$\begin{bmatrix}1\\0\\0\end{bmatrix}$

Exercise 3) Consider the two vectors $\underline{\boldsymbol{v}}_1 = [1, 0, 0]^T$, $\underline{\boldsymbol{v}}_2 = [0, -1, 2]^T \in \mathbb{R}^3$.

3a) Sketch these two vectors as position vectors in $\mathbb{R}^3$, using the axes below.

$= \{ c\vec{v}_1 ,\ c \in \mathbb{R} \} = \{ c \begin{bmatrix}1\\0\\0\end{bmatrix} ,\ c \in \mathbb{R} \}$

3b) What geometric object is $span\{\underline{\boldsymbol{v}}_1\}$? (Remember, we are identifying position vectors with their endpoints.) Sketch a portion of this object onto your picture below. Remember though, the "span" continues beyond whatever portion you can draw.

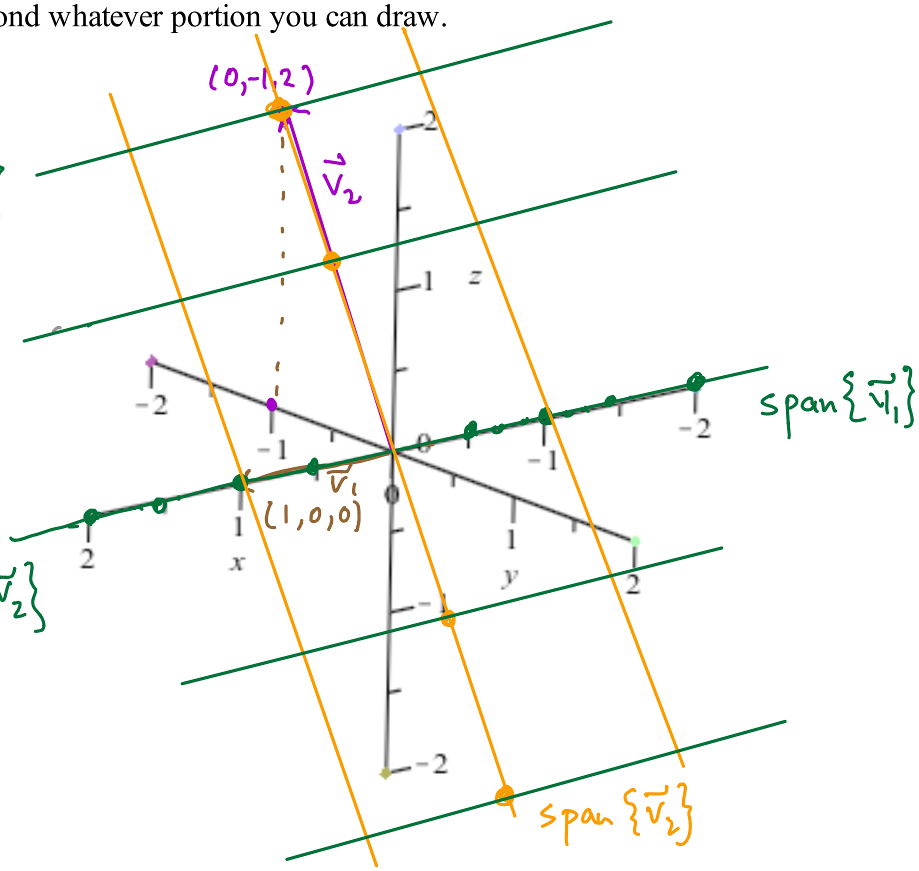
x-axis

3c) What geometric object is $span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2\}$? Sketch a portion of this object onto your picture below. Remember though, the "span" continues beyond whatever portion you can draw.

plane

$\{ c_1\vec{v}_1 + c_2\vec{v}_2 : c_1, c_2 \in \mathbb{R} \}$

partial grid for the plane that is $span\{\vec{v}_1, \vec{v}_2\}$



$$span\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\} = \{ c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n : c_1, c_2, \ldots c_n \in \mathbb{R} \}$$

= collection of all linear combinations

you have a lab problem like this.

3d) What implicit equation must vectors $[b_1, b_2, b_3]^T$ satisfy in order to be in $span\{\underline{v}_1, \underline{v}_2\}$? Hint: For what $[b_1, b_2, b_3]^T$ can you solve the system

$$c_1\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{\vec{v}_1}+c_2\underbrace{\begin{bmatrix}0\\-1\\2\end{bmatrix}}_{\vec{v}_2}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

for c_1, c_2? Write this an augmented matrix problem and use row operations to reduce it, to see when you get a consistent system for c_1, c_2.

$$\left[\begin{array}{cc|c}1 & 0 & b_1\\0 & -1 & b_2\\0 & 2 & b_3\end{array}\right]\begin{array}{l}=x\\=y\\=z\end{array}$$

$$\begin{array}{r}\\-R_2\to R_2\\ \\\end{array}\left[\begin{array}{cc|c}1 & 0 & b_1\\0 & 1 & -b_2\\0 & 2 & b_3\end{array}\right]$$

$$\begin{array}{r}\\ \\-2R_2+R_3\to R_3\end{array}\left[\begin{array}{cc|c}1 & 0 & b_1\\0 & 1 & -b_2\\0 & 0 & 2b_2+b_3\end{array}\right]$$

Solutions c_1, c_2 if and only if

$2b_2+b_3=0$

i.e. $\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ must lie on the plane

$2y+z=0$

compare to picture!

eg the points $(0,0,0)$ in plane

$(1,0,0)$ in plane

$(0,-1,2)$ in plane.

Fri Feb 23
4.1-4.3 The vector spaces $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^n$.

Announcements: finish Thursday example. & today's notes.
We are investigating linear combinations, algebraically & geometrically.

Warm-up Exercise: write the vector equation (a.k.a. a linear combination equation)

'til 10:47

$$* \quad x_1\begin{bmatrix}1\\-1\\2\end{bmatrix} + x_2\begin{bmatrix}1\\3\\0\end{bmatrix} + x_3\begin{bmatrix}0\\2\\7\end{bmatrix} = \begin{bmatrix}-1\\2\\-3\end{bmatrix}$$

as a matrix equation

$$A\vec{x} = \vec{b}.$$

(i.e. what is the matrix A?)

ans
$$\begin{bmatrix}1 & 1 & 0\\-1 & 3 & 2\\2 & 0 & 7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_2\end{bmatrix} = \begin{bmatrix}-1\\2\\-3\end{bmatrix}$$

long cut. rewrite * in steps

$$\begin{bmatrix}x_1\\-x_1\\2x_1\end{bmatrix} + \begin{bmatrix}x_2\\3x_2\\0\end{bmatrix} + \begin{bmatrix}0\\2x_3\\7x_3\end{bmatrix} = \begin{bmatrix}-1\\2\\-3\end{bmatrix}$$

$$\begin{bmatrix}x_1 + x_2 \\ -x_1 + 3x_2 + 2x_3\\ 2x_1 + 7x_3\end{bmatrix} = \begin{bmatrix}-1\\2\\-3\end{bmatrix}$$

On Wednes~~dy~~day we interpreted linear combinations geometrically. And, we noticed that to answer natural questions we ended up using matrix theory from Chapter 3 This is because

Exercise 1) By carefully expanding the linear combination below, check than in $\mathbb{R}^m$, the linear combination

$$\cancel{c_1}\begin{bmatrix} c_1 a_{11} \\ c_1 a_{21} \\ \vdots \\ c_1 a_{m1} \end{bmatrix} + \cancel{c_2}\begin{bmatrix} c_2 a_{12} \\ c_2 a_{22} \\ \vdots \\ c_2 a_{m2} \end{bmatrix} + \dots + \cancel{c_n}\begin{bmatrix} c_n a_{1n} \\ c_n a_{2n} \\ \vdots \\ c_n a_{mn} \end{bmatrix} = \begin{bmatrix} c_1 a_{11} + c_2 a_{12} + \cdots + c_n a_{1n} \\ c_1 a_{21} + c_2 a_{22} + \cdots + c_n a_{2n} \\ \vdots \\ c_1 a_{m1} + c_2 a_{m2} + \cdots + c_n a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} c_1 + a_{12} c_2 + \cdots + a_{1n} c_n \\ a_{21} c_1 + a_{22} c_2 + \cdots + a_{2n} c_n \\ \vdots \\ a_{m1} c_1 + a_{m2} c_2 + \cdots + a_{mn} c_n \end{bmatrix}$$

is always just the matrix times vector product

$$= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ : & : & : & : \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ : \\ c_n \end{bmatrix}.$$

vectors in the linear combo are the columns in the matrix. The "weights" $c_1 \dots c_n$ are the vector $\vec{c}$.

Thus linear combination problems in $\mathbb{R}^m$ can usually be answered using the linear system and matrix techniques we've just been studying in Chapter 3. This will be the main theme of Chapter 4.

Wed example, finished on Friday

$\begin{bmatrix}1\\0\\0\end{bmatrix}$

Exercise 3) Consider the two vectors $\underline{v}_1 = [1, 0, 0]^T$, $\underline{v}_2 = [0, -1, 2]^T \in \mathbb{R}^3$.

3a) Sketch these two vectors as position vectors in $\mathbb{R}^3$, using the axes below.

$= \{ c\vec{v}_1 ,\ c \in \mathbb{R} \} = \{ c \begin{bmatrix}1\\0\\0\end{bmatrix},\ c \in \mathbb{R} \}$

3b) What geometric object is $span\{\underline{v}_1\}$? (Remember, we are identifying position vectors with their endpoints.) Sketch a portion of this object onto your picture below. Remember though, the "span" continues beyond whatever portion you can draw.

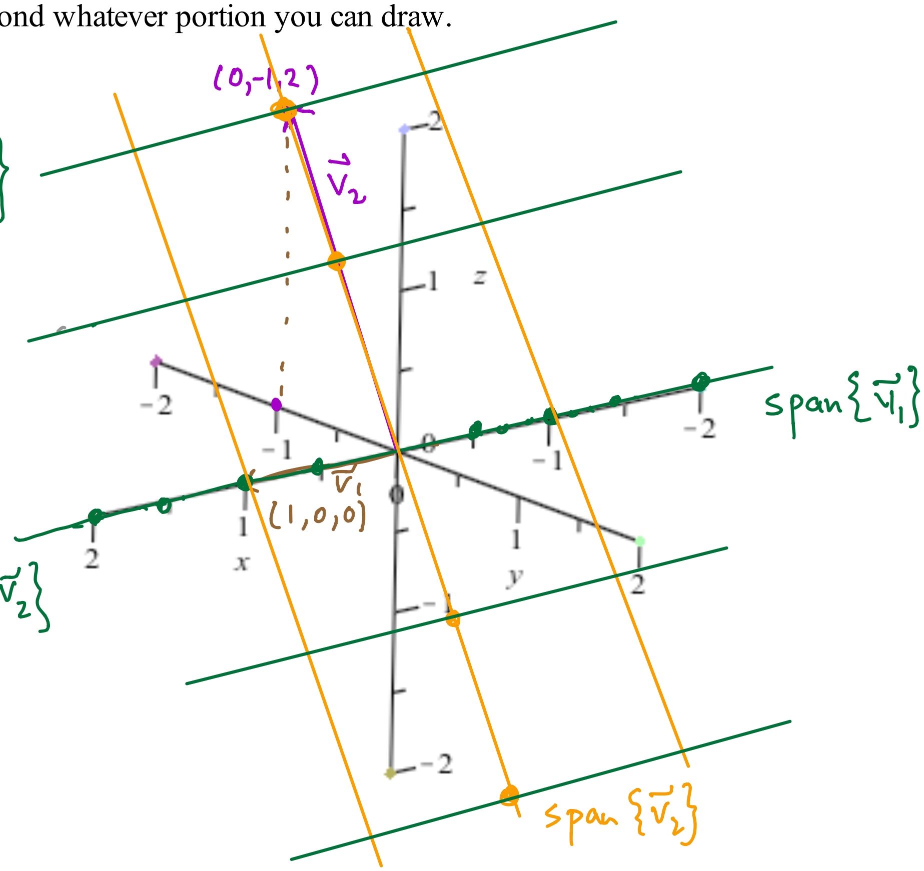
x-axis

3c) What geometric object is $span\{\underline{v}_1, \underline{v}_2\}$? Sketch a portion of this object onto your picture below. Remember though, the "span" continues beyond whatever portion you can draw.

plane

$\{ c_1\vec{v}_1 + c_2\vec{v}_2 : c_1, c_2 \in \mathbb{R} \}$

partial grid for the plane that is $span\{\vec{v}_1, \vec{v}_2\}$



$span\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\} = \{ c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n : c_1, c_2, \ldots c_n \in \mathbb{R} \}$

= collection of all linear combinations

you have a lab problem like this.

3d) What implicit equation must vectors $[b_1, b_2, b_3]^T$ satisfy in order to be in $span\{\underline{v}_1, \underline{v}_2\}$? Hint: For what $[b_1, b_2, b_3]^T$ can you solve the system

$$c_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{v}_1} + c_2 \underbrace{\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}}_{\vec{v}_2} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

for c_1, c_2? Write this an augmented matrix problem and use row operations to reduce it, to see when you get a consistent system for c_1, c_2.

$$\left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & -1 & b_2 \\ 0 & 2 & b_3 \end{array}\right] \begin{array}{l} = x \\ = y \\ = z \end{array}$$

$$-R_2 \rightarrow R_2 \quad \left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & -b_2 \\ 0 & 2 & b_3 \end{array}\right]$$

$$-2R_2 + R_3 \rightarrow R_3 \quad \left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & -b_2 \\ 0 & 0 & 2b_2 + b_3 \end{array}\right]$$

Solutions c_1, c_2 if and only if

$$2b_2 + b_3 = 0$$

i.e. $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ must lie on the plane

$$2y + z = 0$$

compare to picture!

eg the points $(0,0,0)$ in plane
$(1,0,0)$ in plane
$(0,-1,2)$ in plane.

When we are discussing the span of a collection of vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ we would like to know that we are being efficient in describing this collection, and not wasting any free parameters because of redundancies. This has to do with the concept of "linear independence":

Definition: ($\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is linearly independent)

a) The vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly independent if no one of the vectors is a linear combination of (some) of the other vectors. The logically equivalent concise way to say this is that the only way $\underline{\mathbf{0}}$ can be expressed as a linear combination of these vectors, not 1

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n = \underline{\mathbf{0}},$$

is for all the weights $c_1 = c_2 = \ldots = c_n = 0$. not 2

1

b) $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly dependent if at least one of these vectors *is* a linear combination of (some) of the other vectors. The concise way to say this is that there is some way to write $\underline{\mathbf{0}}$ as a linear combination of these vectors

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n = \underline{\mathbf{0}} \quad 2$$

where not all of the $c_j = 0$. (We call such an equation a linear dependency. Note that if we have any such linear dependency, then any $\underline{\boldsymbol{v}}_j$ with $c_j \neq 0$ is a linear combination of the remaining $\underline{\boldsymbol{v}}_k$ with $k \neq j$. We say that such a $\underline{\boldsymbol{v}}_j$ is linearly dependent on the remaining $\underline{\boldsymbol{v}}_k$.)

start here: 1st way: Some $\vec{v}_j = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n$ 1

(but no $\vec{v}_j$ term on right side)

$\Rightarrow \vec{0} = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n - \vec{v}_j$ 2

i.e. some combo of vectors adds up to $\vec{0}$, where not all coeff's $= 0$.

conversely, if

2 $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$

then I can solve for any $\vec{v}_j$ in terms of the others, if its coeff $c_j \neq 0$ 1

Note: Two non-zero vectors are linearly independent precisely when they are not multiples of each other. For more than two vectors the situation is more complicated.

two vectors are dependent means at least one is a linear combo of the other,

i.e. $\vec{v}_1 = c\vec{v}_2$ (or $\vec{v}_2 = d\vec{v}_1$)

Example (Refer to Exercise 2 Wednesday):

The vectors $\underline{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\underline{v}_3 = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$ in $\mathbb{R}^2$ are linearly dependent because, as we showed on ~~Tuesday~~ Wednesday and as we can quickly recheck,

$$-3.5\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1.5\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}.$$

We can also write this linear dependency as

$$-3.5\underline{v}_1 + 1.5\underline{v}_2 - \underline{v}_3 = \underline{\mathbf{0}}$$

(or any non-zero multiple of that equation.)

Exercise 2) Are the vectors $\underbrace{\underline{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{\text{ind., not scalar mults}}$ linearly independent? How about $\underline{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\underline{v}_3 = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$?

Same for $\vec{v}_1$ & $\vec{v}_3$.

Exercise 3) For linearly independent vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$, every vector $\underline{\boldsymbol{v}}$ in their span can be written as $\underline{\boldsymbol{v}} = d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + \ldots + d_n\underline{\boldsymbol{v}}_n$ uniquely, i.e. for exactly one choice of linear combination coefficients $d_1, d_2, \ldots d_n$. This is not true if vectors are dependent. Explain these facts. (You can illustrate these facts with the vectors in Exercise 2.)

return to this!

Exercise 5) (Recall Exercise 3 in Wednesday's notes):

5a) Are the vectors

$$\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \underline{\boldsymbol{v}}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

linearly independent? yes

not scalar multiples... which is all I need to check with just two vectors.

5b) Show that the vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \ \underline{v}_3 = \begin{bmatrix} 3 \\ 4 \\ -8 \end{bmatrix}$$

are linearly dependent (even though no two of them are scalar multiples of each other). What does this mean geometrically about the span of these three vectors?
Hint: You might find this computation useful:

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & 2 & -8 \end{array}\right] \quad \text{reduces to} \quad \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{array}\right]$$

dependent: $\vec{v}_3 = 3\vec{v}_1 - 4\vec{v}_2$

is $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{v}_3$?

$c_1 = 3$
$c_2 = -4$

$$\begin{bmatrix} 3 \\ 4 \\ -8 \end{bmatrix} \overset{?}{=} 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 4\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \quad \checkmark$$

OR $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 2 & -8 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

∞'ly many soln's

Exercise 6) Are the vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \ \underline{w}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

linearly independent? What is their span? Hint:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right] \quad \text{reduces to} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0} \implies c_1 = 0$
$c_2 = 0$
$c_3 = 0.$