Math 2250-004 Week 6 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes include material from 3.5-3.6.

Mon Feb 12
- 3.5 matrix inverses

Announcements:
- Exam 1 is Friday.
  - I'll post 2 previous exams & sol'ts on CANVAS
  - Your lab sessions will be exam review
  - exam from 10:40 – 11:40 a.m. (extra 5 minutes at each end.)
  - it covers thru §3.5 (matrix inverses)
  - on T, W I'll start §3.6 determinants — not on exam

Warm-up Exercise: 'til 10:47

Compute the reduced row echelon form of

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{bmatrix}.$$

Then review our last page from Friday, about matrix inverses, which is on first page of this week's notes. MAGIC

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{bmatrix}$$

NOT MAGIC

Matrix inverses: A square matrix $A_{n \times n}$ is invertible if there is a matrix $B_{n \times n}$ so that

$$AB = BA = I\,,$$

where I is the $n \times n$ identity matrix. In this case we call B the inverse of A, and write $B = A^{-1}$.

Remark: A matrix A can have at most one inverse, because if we have two candidates B, C with

$$AB = BA = I \quad \text{and also} \quad AC = CA = I$$

then

$$(BA)C = IC = C$$
$$B(AC) = BI = B$$

so since the associative property $(BA)C = B(AC)$ is true, it must be that

$$B = C.$$

Exercise 1a) Verify that for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ the inverse matrix is $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$BA = I$ too!

Inverse matrices are very useful in solving algebra problems. For example

Theorem: If A^{-1} exists then the only solution to $\boxed{A\mathbf{x} = \mathbf{b}}$ is $\mathbf{x} = A^{-1}\mathbf{b}$. $\Rightarrow$

$$A\vec{x} = \vec{b}$$
$$A^{-1}(A\vec{x}) = A^{-1}\vec{b}$$
$$(A^{-1}A)\vec{x} = A^{-1}\vec{b}$$
$$I\vec{x} = A^{-1}\vec{b}$$
$$\boxed{\vec{x} = A^{-1}\vec{b}}$$

double check

$$A(A^{-1}\vec{b}) \overset{?}{=} \vec{x}$$
$$= (AA^{-1})\vec{b}$$
$$= I\vec{b}$$
$$= \vec{b}\ !$$

Exercise 1b) Use the theorem and A^{-1} in 1a, to write down the solution to the system

$$x + 2y = 5$$
$$3x + 4y = 6$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 9/2 \end{bmatrix}.$$

check $x + 2y = -4 + 9 = 5$ ✓

$3x + 4y = -12 + 18 = 6$ ✓

Exercise 2a) Use matrix algebra to verify why the Theorem on the previous page is true. Notice that the correct formula is $\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$ and not $\underbrace{\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}\, A^{-1}}$ (this second product can't even be computed because the dimensions don't match up!).

see above

$a\,x = b$ scalar eqtn

$x = \frac{b}{a}$

$A\vec{x} = \vec{b}$

$A_{n\times n}\,\vec{x}_{n\times 1} = \vec{b}_{n\times 1}$

but $\vec{b}_{n\times 1}\,A^{-1}_{n\times n}$ isn't even defined!!

2b) Assuming A is a square matrix with an inverse A^{-1}, and that the matrices in the equation below have dimensions which make for meaningful equation, solve for X in terms of the other matrices:

$$XA + C = B$$

$$-C = -C$$

$$XA = B - C$$

$$XAA^{-1} = (B-C)A^{-1}$$

$$XI = (B-C)A^{-1}$$

$$X = (B-C)A^{-1}$$

NOTE When A^{-1} exists for an $n \times n$ matrix A, every linear system $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ has a unique solution (given by the formula $\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$). That means that the reduced row echelon form of A must be the identity matrix in these cases! (Because if A doesn't reduce to the identity then there will be fewer than n pivots, so many systems $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ will be inconsistent and the ones that are consistent will have free parameters in the solutions.)

if A^{-1} exists, rref(A) must equal I

(otherwise solns to $A\vec{x} = \vec{b}$ would not necessarily exist, or be unique)

But where did that formula for A^{-1} come from?

Answer: Consider A^{-1} as an unknown matrix, $A^{-1} = X$. We want

$$A\,X = I.$$

We can break this matrix equation down by the columns of *X*. In the two by two case we get:

$$A\left[\begin{array}{c|c} col_1(X) & col_2(X) \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array}\right].$$

To be more concrete, in this example we may write the unknown columns as

$$col_1(X) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad col_2(X) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Then we want to solve the two systems

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We can solve for both of these mystery columns at once, as we've done before when we had different right hand sides, with a double-augmented matrix

Exercise 3: Reduce the double augmented matrix

$$\left[\begin{array}{cc|c|c} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

to find the two columns of A^{-1} for the previous example.

$\downarrow$ rref

$$\left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

backsolve $\quad x_1 = -2 \quad y_1 = 1$

$x_2 = 3/2 \quad y_2 = -1/2$

Conclude: $A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$

if A reduces to I

$A \,\vdots\, b \to I \,\vdots\, \vec{c}$

then soln to $A\vec{x} = \vec{b}$ is $\vec{c}$

$A \,\vdots\, I$

$\downarrow$

$I \,\vdots\, B$

$B = A^{-1}$

Exercise 4: Will this always work? Can you find A^{-1} for

$$A := \begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix} ?$$

Try to solve $A\,X = I$ for the mystery matrix, and do it column by column. In other words, if we write

$$col_1(X) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad col_2(X) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad col_3(X) = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Solve

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

with a triple-augmented matrix reduction!

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 & 1 & 0 \\ 2 & 7 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1+R_2 \to R_2 \\ -2R_1+R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -5 & -2 & -2 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right]$$

- could do $\frac{R_2}{-5} \to R_2$
- or $-2R_3 + R_2 \to R_2$

$$-2R_3+R_2 \to R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$3R_2+R_3 \to R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & -1 & 4 & 3 & -5 \end{array}\right]$$

$$-R_3 \to R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{array}\right]$$

$$-R_3+R_1 \to R_1 \quad \left[\begin{array}{ccc|ccc} 1 & 5 & 0 & 5 & 3 & -5 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{array}\right]$$

$$-5R_2+R_1 \to R_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -2 & 5 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -5 & -2 & 5 \\ 2 & 1 & -2 \\ -4 & -3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix}\begin{bmatrix} -5 & -2 & 5 \\ 2 & 1 & -2 \\ -4 & -3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

Exercise 5) Will this always work? Try to find B^{-1} for $B := \begin{bmatrix} 1 & 5 & 5 \\ 2 & 5 & 0 \\ 2 & 7 & 4 \end{bmatrix}$.

Well, if we set up the 3 × 6 matrix where we have augmented B with the identity matrix, and then reduce, this is what happens:

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 5 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 & 1 & 0 \\ 2 & 7 & 4 & 0 & 0 & 1 \end{array}\right] \text{ reduces to } \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 0 & \frac{7}{4} & -\frac{5}{4} \\ 0 & 1 & 2 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{3}{4} & -\frac{5}{4} \end{array}\right]$$

What happened, and what does this mean?

- when we tried to find the columns of B^{-1} we got inconsistent systems. (last eqtns ~~a~~ said $0 = 1$).
- $\text{rref}(B) \neq I$ matrix.

as we discussed, B^{-1} exists means B must reduce to I

Theorem: Let $A_{n \times n}$ be a square matrix. Then A has an inverse matrix if and only if its reduced row echelon form is the identity. In this case the algorithm illustrated on the previous page will always yield the inverse matrix.

explanation: By the previous theorem, when A^{-1}exists, the solutions to linear systems

$$A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$$

are unique ($\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$). So, the reduced row echelon form of A must be the identity $I_{n \times n}$ in these cases.

In the case that A does reduce to I, we search for A^{-1} as the solution matrix X to the matrix equation

$$A\,X = I$$

i.e.

$$A\left[\begin{array}{c|c|c|c} col_1(X) & col_2(X) & \ldots. & col_n(X) \end{array}\right] = \left[\begin{array}{c|c|c|c} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & \ldots. & 0 \\ 0 & 0 & & 1 \end{array}\right]$$

Because A reduces to the identity matrix, we may solve for X column by column as in the examples we just worked, by using a chain of elementary row operations:

$$[\,A\,|\,I\,] \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow [\,I\,|\,B\,],$$

and deduce that the columns of X are exactly the columns of B, i.e. $X = B$. Thus we know that

$$A\,B = I\,.$$

To realize that $B\,A = I$ as well, we would try to solve $B\,Y = I$ for Y, and hope $Y = A$. But we can actually verify this fact by reordering the columns of $[\,I\,|\,B\,]$ to read $[\,B\,|\,I\,]$ and then reversing each of the elementary row operations in the first computation, in reverse order, i.e. create the chain

$$[\,B\,|\,I\,] \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow [\,I\,|\,A\,]\,.$$

So $B\,A = I$ also holds! (This is one of those rare times when matrix multiplication actually is commuative.)

To summarize: If A^{-1} exists, then solutions $\underline{\boldsymbol{x}}$ to $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ always exist and are unique, so the reduced row echelon form of A is the identity. If the reduced row echelon form of A is the identity, then A^{-1} exists, because we have an algorithm to find it. That's exactly what the Theorem claims.

There's a nice formula for the inverses of 2×2 matrices, and it turns out this formula will lead to the next text section 3.6 on determinants:

Theorem: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ exists if and only if the determinant $D = ad - bc$ of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-zero. And in this case,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(Notice that the diagonal entries have been swapped, and minus signs have been placed in front of the off-diagonal terms. This formula should be memorized.)

Exercise 6a) Check that this formula for the inverse works, for $D \neq 0$. (We could have derived it with elementary row operations, but it's easy to check since we've been handed the formula.)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \overset{?}{=} \begin{bmatrix} -2 & 1 \\ +3/2 & -1/2 \end{bmatrix}$$

A

$\det(A) = 1\cdot 4 - 3\cdot 2 = -2.$

formula:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \checkmark$$

6b) Even with systems of two equations in two unknowns, unless they come from very special problems the algebra is likely to be messier than you might expect (without the formula above). Use the magic formula to solve the system

$$3x + 7y = 5$$
$$5x + 4y = 8 .$$

← this was Tuesday warmup.

6a).
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & \underbrace{-ab+ba}_{0} \\ \underbrace{cd-dc}_{0} & -bc+ad \end{bmatrix} = \det(A)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

so if $|A| \neq 0$, divide both sides by it, to get magic formula.

Remark: For a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the reduced row echelon form will be the identity if and only if the two rows are not multiples of each other: And if a, b are both non-zero then saying that the second row is not a multiple of the first row is the same as saying that $\frac{c}{a} \neq \frac{d}{b}$ (the ratios of the second row entries to the corresponding first row entries). Cross multiplying we see this is the same as $ad \neq bc$, i.e. $ad - bc \neq 0$. The "determinant not equal to zero" condition is also the correct condition for the rows not being multiples, even if one or both of a, b are zero. So this is the condition that A reduces to the identity and has an inverse matrix. knowing the inverse matrix exists.

Remark: Determinants are defined for square matrices $A_{n \times n}$ and the resulting number determines whether or not the inverse matrices exist, (i.e. whether the reduced row echelon form of A is the identity matrix). And, when $n > 2$ there are analogous (but more complicated) magic formulas for the inverse matrices when the inverses exist, that generalize the one you're memorizing for $n = 2$. This is section 3.6 material that we'll discuss carefully tomorrow, Wednesday, and next week Monday.

Tues Feb 13

- 3.6 determinants

Announcements:
- no quiz tomorrow (exam on Friday)
- I posted 2 practice exams (& solns) on CANVAS (one has a det. question, we won't).
- In HW this week, I included some determinant questions that are part of next week's HW
- a couple odds & ends from Monday, then proceed.

Warm-up Exercise: recall that for a 2×2 matrix A that has an inverse,

'til 10:47

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}; \quad \det(A) := a_{11}a_{22} - a_{12}a_{21}$$

- $$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

use the magic formula for A^{-1} to solve the system

$$3x + 7y = 5$$
$$5x + 4y = 8$$

ans $= \begin{bmatrix} x \\ y \end{bmatrix} =$

$$A = \begin{bmatrix} 3 & 7 \\ 5 & 4 \end{bmatrix}.$$

$$\begin{bmatrix} 3 & 7 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

$$A\vec{x} = \vec{b}$$
$$\underbrace{A^{-1}A}_{I}\vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}\vec{b}.$$

$$\det(A) \text{ "=" } |A| = 3\cdot4 - 5\cdot7 = 12 - 35 = -23.$$

so $$A^{-1} = \frac{1}{-23}\begin{bmatrix} 4 & -7 \\ -5 & 3 \end{bmatrix} = -\frac{1}{23}\begin{bmatrix} 4 & -7 \\ -5 & 3 \end{bmatrix}$$

soln is

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{23}\begin{bmatrix} 4 & -7 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

$= -\frac{1}{23}\begin{bmatrix} -36 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{36}{23} \\ \frac{1}{23} \end{bmatrix}$

- Determinants are scalars defined for square matrices $A_{n \times n}$. They always <u>determine</u> whether or not the inverse matrix A^{-1} exists, (i.e. whether the reduced row echelon form of A is the identity matrix): In fact, the determinant of A is non-zero if and only if A^{-1} exists. The determinant of a 1×1 matrix $[a_{11}]$ is defined to be the number a_{11}; determinants of 2×2 matrices are defined as in yesterday's notes; and in general determinants for $n \times n$ matrices are defined recursively, in terms of determinants of $(n-1) \times (n-1)$ submatrices:

<u>Definition:</u> Let $A_{n \times n} = [a_{ij}]$. Then the determinant of A, written $det(A)$ or $|A|$, is defined by

cofactor

$$det(A) := \sum_{j=1}^{n} a_{1j}(-1)^{1+j}M_{1j} = \sum_{j=1}^{n} a_{1j}C_{1j}.$$

$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & & \end{bmatrix}$

Here M_{1j} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and the j^{th} column, and C_{1j} is simply $(-1)^{1+j}M_{1j}$.

More generally, the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A is called the <u>ij Minor</u> M_{ij} of A, and $C_{ij} := (-1)^{i+j}M_{ij}$ is called the <u>ij Cofactor</u> of A.

<u>Exercise 1</u> Check that the messy looking definition above gives the same answer we talked about yesterday in the 2×2 case, namely

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$(-1)^{i+j}$ $\pm$: $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$.

$|A| = a_{11}(-1)^{1+1}M_{11} + a_{12}(-1)^{1+2}M_{12}$

$= a_{11} \cdot 1 \cdot a_{22} + a_{12} \cdot (-1) \cdot a_{21}$

$= a_{11}a_{22} - a_{12}a_{21}$

There's a nice formula for the inverses of 2×2 matrices, and it turns out this formula will lead to the next text section 3.6 on determinants:

<u>Theorem:</u> $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ exists if and only if the <u>determinant</u> $D = ad - bc$ of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-zero. And in this case,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(Notice that the diagonal entries have been swapped, and minus signs have been placed in front of the off-diagonal terms. This formula should be memorized.)

<u>Exercise 6a)</u> Check that this formula for the inverse works, for $D \neq 0$. (We could have derived it with elementary row operations, but it's easy to check since we've been handed the formula.)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \overset{?}{=} \begin{bmatrix} -2 & 1 \\ +\frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

A

$\det(A) = 1\cdot 4 - 3\cdot 2 = -2.$

formula:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \checkmark$$

<u>6b)</u> Even with systems of two equations in two unknowns, unless they come from very special problems the algebra is likely to be messier than you might expect (without the formula above). Use the magic formula to solve the system

$$3x + 7y = 5$$
$$5x + 4y = 8\,.$$

← this was Tuesday warmup.

6a). $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & \underbrace{-ab+ba}_{0} \\ \underbrace{cd-dc}_{0} & -bc+ad \end{bmatrix} = \det(A)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

so if $|A| \neq 0$, divide both sides by it, to get magic formula.

from the last page, for our convenience:

Definition: Let $A_{n \times n} = [a_{ij}]$. Then the determinant of A, written $det(A)$ or $|A|$, is defined by

$$det(A) := \sum_{j=1}^{n} a_{1j}(-1)^{1+j} M_{1j} = \sum_{j=1}^{n} a_{1j} C_{1j}.$$

Here M_{1j} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and the j^{th} column, and C_{1j} is simply $(-1)^{1+j} M_{1j}$.

Exercise 2 Work out the expanded formula for the determinant of a 3×3 matrix. It's not worth memorizing (as opposed to the recursive formula above), but it's good practice to write out at least once, and we might point to it later.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$[(-1)^{i+j}] = \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

sum of 6 terms.

each is $\pm$ product : each ~~column~~ row is used exactly once in each term.

see wikipedia formula for more info.

Theorem: (proof is in text appendix) $det(A)$ can be computed by expanding across any row, say row i:

$$det(A) := \sum_{j=1}^{n} a_{ij}(-1)^{i+j}M_{ij} = \sum_{j=1}^{n} a_{ij}C_{ij}$$ ← across $\text{row}_i(A)$

or by expanding down any column, say column j:

$$det(A) := \sum_{i=1}^{n} a_{ij}(-1)^{i+j}M_{ij} = \sum_{i=1}^{n} a_{ij}C_{ij}.$$ ← down $\text{col}_j(A)$

Exercise 3a) Let $A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$. Compute $det(A)$ using the definition. (On the next page we'll use other rows and columns to do the computation.)

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$|A| = 1\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 3 \\ 2 & -2 \end{vmatrix}$$

$$= 1\cdot 5 - 2(-2) - 1(-6)$$
$$= 5 + 4 + 6 = 15$$

From previous page,

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}.$$

3b) Verify that the matrix of all the cofactors of A is given by $[C_{ij}] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$. Then expand $det(A)$ down various columns and rows using the a_{ij} factors and C_{ij} cofactors. Verify that you always get the same value for $det(A)$, as the Theorem on the previous page guarantees. Notice that in each case you are taking the dot product of a row (or column) of A with the corresponding row (or column) of the cofactor matrix.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$[C_{ij}] = \begin{bmatrix} +\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 3 \\ 2 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} \\ +\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad [C_{ij}] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

$row_1(A) \cdot row_1(cof(A))$

$$|A| = 1 \cdot 5 + 2 \cdot 2 + (-1)(-6) = 5 + 4 + 6 = 15$$

$col_2(A) \cdot col_2(cof(A))$

$$|A| = 4 + 9 + 2 = 15.$$

middle rows : $|A| = 0 + 9 + 6 = 15.$

3c) What happens if you take dot products between a row of A and a *different* row of $\left[C_{ij}\right]$? A column of A and a *different* column of $\left[C_{ij}\right]$? The answer may seem magic. We'll come back to this example when we talk about the magic formula for the inverses of 3×3 (or $n \times n$) invertible matrices.

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad \left[C_{ij}\right] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

$row_1(A) \cdot row_3(cof(A)) = 5 - 2 - 3 = 0$

$col_3(A) \cdot col_2(cof(A)) = -2 + 3 - 1 = 0$

we will return.

Exercise 4) Compute the following determinants by being clever about which rows or columns to use:

4a) $\begin{vmatrix} 1 & 38 & 106 & 3 \\ 0 & 2 & 92 & -72 \\ 0 & 0 & 3 & 45 \\ 0 & 0 & 0 & -2 \end{vmatrix}$; 4b) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ \pi^2 & 2 & 0 & 0 \\ 0.476 & 88 & 3 & 0 \\ 1 & 22 & 33 & -2 \end{vmatrix}$.

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

4a). $|A| = 1 \begin{vmatrix} 2 & 92 & -72 \\ 0 & 3 & 45 \\ 0 & 0 & -2 \end{vmatrix} - 0 + 0 - 0$

$= 1 \left(2 \begin{vmatrix} 3 & 45 \\ 0 & -2 \end{vmatrix} - 0 + 0 \right)$

$= 1 \cdot 2 \cdot \left(3(-2) - 0 \right)$

$= 1 \cdot 2 \cdot 3 \cdot (-2)$. = product of diagonal entries (upper triangular)

Exercise 5) Explain why it is always true that for an upper triangular matrix (as in 2a), or for a lower triangular matrix (as in 2b), the determinant is always just the product of the diagonal entries.

Wed Feb 14

- 3.6 determinants
- midterm Friday!

Announcements:

1) test Friday, 10:40 ~ 11:40
2) review for at least last 20 minutes today (labs tomorrow are review).
3) HW due next week!, due in lab, though.

Warm-up Exercise:

Compute the determinant!

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 4 & 8 \\ -3 & 1 & 6 & -9 \\ 4 & 0 & 2 & 10 \end{vmatrix}$$

'til 10:47

can expand across any row or down any column

using first row $\det = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$

$$= 1 \cdot \begin{vmatrix} 0 & 4 & 8 \\ 1 & 6 & -9 \\ 0 & 2 & 10 \end{vmatrix} + 0 + 0 + 0$$

$$= 1 \cdot (\cancel{0 \cdot C_{11}} + 1 \cdot C_{21} + \cancel{0 \cdot C_{31}})$$

$$= 1 \cdot \left(- \begin{vmatrix} 4 & 8 \\ 2 & 10 \end{vmatrix} \right)$$

$$= -(40 - 16) = \underline{-24}$$

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

The effective way to compute determinants for larger-sized matrices without lots of zeroes is to not use the definition, but rather to use the following facts, which track how elementary row operations affect determinants:

- (1a) Swapping any two rows changes the sign of the determinant.

  proof: This is clear for 2×2 matrices, since

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \qquad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad\,.$$

For 3×3 determinants, expand across the row *not* being swapped, and use the 2×2 swap property to deduce the result. Prove the general result by induction: once it's true for $n \times n$ matrices you can prove it for any $(n+1) \times (n+1)$ matrix, by expanding across a row that wasn't swapped, and applying the $n \times n$ result.

swap rows 1 & 3:

expand unswapped matrix across 2nd row

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{21} \cdot \left(- \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \right) + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{33} \end{vmatrix}$$

what happens if we swap 1st & 3rd row of original matrix, on the right side?

ans rows swap in 2×2 matrices

so by fact for 2×2 dets, the sign on the right side (& left side) changes

(1b) Thus, if two rows in a matrix are the same, the determinant of the matrix must be zero: on the one hand, swapping those two rows leaves the matrix and its determinant unchanged; on the other hand, by (1a) the determinant changes its sign. The only way this is possible is if the determinant is zero.

- (2a) If you factor a constant out of a row, then you factor the same constant out of the determinant.
  Precisely, using $\mathscr{R}_i$ for i^{th} row of A, and writing $\mathscr{R}_i = c\,\mathscr{R}_i^*$

$$\begin{vmatrix} \mathscr{R}_1 \\ \mathscr{R}_2 \\ : \\ \mathscr{R}_i \\ \mathscr{R}_n \end{vmatrix} = \begin{vmatrix} \mathscr{R}_1 \\ \mathscr{R}_2 \\ : \\ c\,\mathscr{R}_i^* \\ \mathscr{R}_n \end{vmatrix} = c \begin{vmatrix} \mathscr{R}_1 \\ \mathscr{R}_2 \\ : \\ \mathscr{R}_i^* \\ \mathscr{R}_n \end{vmatrix} .$$

proof: expand across the i^{th} row, noting that the corresponding cofactors don't change, since they're computed by deleting the i^{th} row to get the corresponding minors:

$$det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} c\, a_{ij}^* C_{ij} = c \sum_{j=1}^{n} a_{ij}^* C_{ij} = c\, det\left(A^*\right) .$$

row$_i$

$$\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} \stackrel{?}{=} 2 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$6-4=2 \stackrel{?}{=} 2(3-2)=2$$

$$\underbrace{\begin{vmatrix} 3 & 6 \\ 4 & 12 \end{vmatrix}}_{36-24=12\ \checkmark} = 3\cdot 4 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 12$$

(2b) Combining (2a) with (1b), we see that if one row in A is a scalar multiple of another, then $det(A) = 0$. !

- (3) If you replace row i of A, $\mathcal{R}_i$ by its sum with a multiple of another row, say $\mathcal{R}_k$ then the determinant is unchanged! Expand across the i^{th} row:

i^{th} rows location

$$\begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_k \\ \mathcal{R}_i + c\,\mathcal{R}_k \\ \mathcal{R}_n \end{vmatrix} = \sum_{j=1}^{n} \underbrace{\left(a_{ij} + c\,a_{kj}\right)} C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij} + c \sum_{j=1}^{n} a_{kj} C_{ij} = det(A) + c \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_k \\ \mathcal{R}_k \\ \mathcal{R}_n \end{vmatrix} = det(A) + 0\,.$$

<u>Remark:</u> The analogous properties hold for corresponding "elementary column operations". In fact, the proofs are almost identical, except you use column expansions.

<u>Exercise 1)</u> Recompute $\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix}$ from yesterday (using row and column expansions we always got an answer of 15 then.) This time use elementary row operations (and/or elementary column operations).

$$\underset{-2R_1+R_3}{=} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & -6 & 3 \end{vmatrix} \underset{2R_2+R_3 \rightarrow R_3}{=} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{vmatrix} = 1 \cdot 3 \cdot 5 = 15$$

<u>Exercise 2)</u> Compute $\begin{vmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ -1 & 0 & -2 & 1 \end{vmatrix}$.

$$= -0 + 1 \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ -1 & -2 & 1 \end{vmatrix} - 0 + 0$$

$$= 1 \cdot \begin{vmatrix} 1 & -1 & 2 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{vmatrix} \quad \begin{matrix} -2R_1+R_2 \rightarrow R_2 \\ R_1+R_3 \rightarrow R_3 \end{matrix}$$

$= 0$ two rows are multiples!

Theorem: Let $A_{n \times n}$. Then A^{-1} exists if and only if $det(A) \neq 0$.

proof: We already know that A^{-1} exists if and only if the reduced row echelon form of A is the identity matrix. Now, consider reducing A to its reduced row echelon form, and keep track of how the determinants of the corresponding matrices change: As we do elementary row operations,

- if we swap rows, the sign of the determinant switches.
- if we factor non-zero factors out of rows, we factor the same factors out of the determinants.
- if we replace a row by its sum with a multiple of another row, the determinant is unchanged.

Thus,

$$|A| = c_1|A_1| = c_1c_2|A_2| = ... = c_1c_2 \,...\, c_N|rref(A)|$$

where the nonzero c_k's arise from the three types of elementary row operations. If $rref(A) = I$ its determinant is 1, and $|A| = c_1c_2 \,...\, c_N \neq 0$. If $rref(A) \neq I$ then its bottom row is all zeroes and its determinant is zero, so $|A| = c_1c_2 \,...\, c_N(0) = 0$. Thus $|A| \neq 0$ if and only if $rref(A) = I$ if and only if A^{-1} exists !

Remark: Using the same ideas as above, you can show that $det(A\,B) = det(A)det(B)$. This is an important identity that gets used, for example, in multivariable change of variables formulas for integration, using the Jacobian matrix. (It is not true that $det(A + B) = det(A) + det(B)$.) Here's how to show $det(A\,B) = det(A)det(B)$: The key point is that if you do an elementary row operation to AB, that's the same as doing the elementary row operation to A, and then multiplying by B. With that in mind, if you do exactly the same elementary row operations as you did for A in the theorem above, you get

$$|A\,B| = c_1|A_1B| = c_1c_2|A_2B| = ... = c_1c_2 \,...\, c_N|rref(A)B|.$$

If $rref(A) = I$, then from the theorem above, $|A| = c_1c_2 \,...\, c_N$, and we deduce $|A\,B| = |A||B|$. If $rref(A) \neq I$, then its bottom row is zeroes, and so is the bottom row of $rref(A)B$. Thus $|A\,B| = 0$ and also $|A||B| = 0$.

There is a "magic" formula for the inverse of square matrices A (called the "adjoint formula") that uses the determinant of A along with the cofactor matrix of A. We'll talk about the magic formula on Monday next week, after the midterm.

Exam notes:

The exam is this Friday February 16, from 10:40-11:40 a.m. Note that it will start 5 minutes before the official start time for this class, and end 5 minutes afterwards, so you should have one hour to work on the exam. Get to class early, and bring your University I.D. card, which we might ask you to show if we don't recognize you from sections or lecture.

2.1-2.3 + 2.4 not 2.5, 2.6

This exam will cover textbook material from 1.1-1.5, 2.1-2.4, 3.1-3.5. The exam is closed book and closed note. You may use a scientific (but not a graphing) calculator, although symbolic answers are accepted for all problems, so no calculator is really needed. (Using a graphing calculator which can do matrix computations for example, is grounds for receiving grade of 0 on your exam. So please ask before the exam if you're unsure about your calculator. And of course, your cell phones must be put away.)

I recommend trying to study by organizing the conceptual and computational framework of the course so far. Only then, test yourself by making sure you can explain the concepts and do typical problems which illustrate them. The class notes and text should have explanations for the concepts, along with worked examples. Old homework assignments and quizzes are also a good source of problems. It could be helpful to look at quizzes/exams from my previous Math 2250 classes, which go back several years from the link

http://www.math.utah.edu/~korevaar/oldclasses.html.

Your lab meetings tomorrow will be exam review sessions.

Chapter 3:

3a) Can you recognize an algebraic linear system of equations?

- $a_{11}x_1 + a_{12}x_2 + \cdots = b_1$
  $\vdots$
  $a_{m1}x_1 + \cdots \quad + a_{mn}x_n = b_m$

$A\vec{x} = \vec{b}$

3b) Can you interpret the solution set geometrically when there are 2 or three unknowns?

common intersection set for lines

intersection set of planes.

3c) Can you use Gaussian elimination to compute reduced row echelon form for matrices? Can you apply this algorithm to augmented matrices to solve linear systems ?

you better be able to!

*3d) What does the shape of the reduced row echelon form of a matrix A tell you about the possible solution sets to A**x** = **b** (perhaps depending, and perhaps not depending on **b**)? Focus especially on whether each row of the reduced matrix has a pivot or not; and on whether each column of the reduced matrix has a pivot or not.*

if each row of rref(A) has a pivot, you can always solve $A\vec{x} = \vec{b}$

if each col. of rref(A) has a pivot then solutions that exist are unique (because no free parameters).

3e) What properties do (and do not) hold for the matrix algebra of addition, scalar multiplication, and matrix multiplication?

division does not hold.

$AB \neq BA$ usually.

the other algebra rules work, as long as matrices have the right #'s of rows and columns to allow the indicated operations

3f) What is the matrix inverse, A^{-1} for a square matrix A? Does every square matrix have an inverse? How can you tell whether or not a matrix has an inverse, using reduced row echelon form? What's the row operations way of finding A^{-1}, when it exists? Can you use matrix algebra to solve matrix equations for unknown vectors $\underline{\mathbf{x}}$ *or matrices* $\boldsymbol{X}$*, possibly using matrix inverses and other algebra manipulations?*

- A has an inverse means there is a matrix B so that $AB = I$, $BA = I$. We write A^{-1} for B
- $A_{n \times n}$ has an inverse if and only if $\text{rref}(A) = I$
- In this case we find the columns of A^{-1} & so all of A^{-1} by augmenting A with the identity matrix and reducing: $A \,\vdots\, I \longrightarrow I \,\vdots\, A^{-1}$
- When solving matrix eqtns such as

$$AX = B$$

or

$$XA = C$$

multiply both sides of the eqtn by A^{-1}, but since multiplication doesn't commute put the A^{-1} where it can cancel the A

$$AX = B \Rightarrow A^{-1}AX = A^{-1}B$$
$$\Rightarrow X = A^{-1}B$$
$$XA = C \Rightarrow XAA^{-1} = CA^{-1}$$
$$\Rightarrow X = CA^{-1}$$

Chapters 1-2:

highest order deriv. in the eqtn

1a) What is a differential equation? What is its order? What is an initial value problem, for a first or second order DE?

→ an eqtn involving a function & its derivatives, e.g. for $y = y(x)$, $f(x, y, y', \ldots y^{(n)}) = 0$.

1^{st} order IVP:
$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

1b) How do you check whether a function solves a differential equation? An initial value problem?

substitute the function into the DE & see whether a true identity results. For IVP see if the function satisfies the initial condition(s)

1c) What is the connection between a first order differential equation and a slope field for that differential equation? The connection between an IVP and the slope field?

the graphs of solutions $y(x)$ to the DE $y' = f(x,y)$ will have slopes at (x,y) equal to $f(x,y)$, so the graphs are tangent to the slope fields. Graphs of solutions to IVP's go through the initial points.

1d) Do you expect solutions to IVP's to exist, at least for values of the input variable close to its initial value? Why? Do you expect uniqueness? What can cause solutions to not exist beyond a certain input variable value?

Expect soltns, expect uniqueness, although this can fail if the conditions of the existence-uniqueness theorem aren't satisfied

1e) What is Euler's numerical method for approximating solutions to first order IVP's, and how does it relate to slope fields?

for $y' = f(x,y)$:
$$x_{i+1} = x_i + \Delta x$$
$$y_{i+1} = y_i + f(x_i, y_i)\Delta x$$

You're using the value of the slope function at (x_i, y_i) as a constant slope to estimate y_{i+1}.

1f) What's an autonomous differential equation? What's an equilibrium solution to an autonomous differential equation? What is a phase diagram for an autonomous first order DE, and how do you construct one? How does a phase diagram help you understand stability questions for equilibria? What does the phase diagram for an autonomous first order DE have to do with the slope field?

autonomous: for $y(x)$: $y' = f(y)$ (so $x(t)$, $x' = f(x)$, etc.)

phase diagram contains equilibrium points (constant solutions), and arrows on the intervals between the equilibrium points, to indicate whether solutions are increasing or decreasing.

You can decide whether constant solutions are stable or unstable (or asymptotically stable or semi-stable) using phase diagram. You can think of the phase diagram as a projection of the slope field onto the vertical axis

1g) Can you recognize the first order differential equations for which we've studied solution algorithms, even if the DE is not automatically given to you pre-set up for that algorithm? Do you know the algorithms for solving these particular first order DE's?

separable for $y(x)$, $y' = f(x)g(y)$ or equivalent
($\text{for } x(t),\ x' = f(t)g(x)$, etc.).

linear: for $y(x)$, $y' + P(x)y = Q(x)$
($\text{for } x(t),\ x' + P(t)x = Q(t)$, etc.)

know how to recognize & solve!

2) Can you convert a description of a dynamical system in terms of rates of change, or a geometric configuration in terms of slopes, into a differential equation? What are the models we've studied carefully in Chapters 1-2? What sorts of DE's and IVP's arise? Can you solve these basic application DE's, once you've set up the model as a differential equation and/or IVP?

- recognize whether DE is separable, linear, (or something you can't solve)
  & be able to solve

applications!
- population models ... $P'(t) = aP^2 + bP + c$
- improved velocity models (mainly linear drag).
- input-output modeling "tanks"
- Newton's law of cooling, exponential growth & decay.

be able to set up and/or interpret the model, and then be able to solve the resulting DE's & IVP's, and interpret the results