Math 2250-004 Week 5 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes include material from 3.1-3.5

Mon Feb 5
- 3.1-3.3 systems of linear equations and how to solve them via Gaussian elimination and the reduced row echelon form of augmented matrices.

Announcements:

Warm-up Exercise: The set of solutions $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to a single linear equation

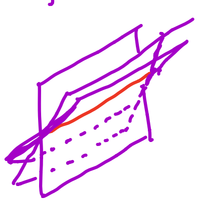
'til 10:47

$$a_1 x + a_2 y + a_3 z = b$$

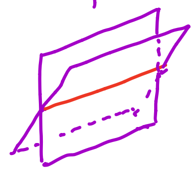
is a plane in $\mathbb{R}^3$. (Unless $a_1 = a_2 = a_3 = b = 0$, in which case it is all of $\mathbb{R}^3$.) Using your geometric intuition, can you list all possible types of solution sets for systems of linear equations in three unknowns?

5 kinds!

- all of $\mathbb{R}^3$ if each eqtn is $0x + 0y + 0z = 0$.
- plane in $\mathbb{R}^3$ if soltn set for each eqtn is same plane: $x + 2y - 3z = 6$, $2x + 4y - 6z = 12$
- line in $\mathbb{R}^3$

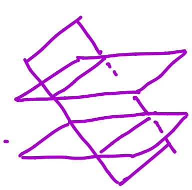


- point (typical for 3 eqtns in 3 unknowns)



3rd plane cuts thru red line at a point

- no solution, i.e. $\emptyset$: e.g.



or

(3 vertical planes)

The last thing we did on Friday was to solve a system of 3 linear equations in 3 unknowns. One interpretation of this system is that we were looking for the intersection of three planes in $\mathbb{R}^3$. We used Gaussian elimination:

From Friday's notes

Solutions to linear equations in 3 unknowns:

What is the geometric question you're answering?

Exercise 4) Consider the system

$$\begin{aligned} x + 2y + z &= 4 \\ 3x + 8y + 7z &= 20 \\ 2x + 7y + 9z &= 23\,. \end{aligned}$$

Use elementary equation operations (or if you prefer, elementary row operations in the synthetic version) to find the solution set to this system. There's a systematic way to do this, which we'll talk about. It's called Gaussian elimination.

Hint: The solution set is a single point, $[x, y, z] = [5, -2, 3]$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right]$$

$$\begin{array}{l} \\ -3R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 7 & 15 \end{array}\right]$$

$$\begin{array}{l} \\ \frac{R_2}{2} \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 7 & 15 \end{array}\right]$$

$$\begin{array}{l} \\ \\ -3R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

row echelon form
from here you can "backsolve" to find solns

$z = 3 \Rightarrow y + 6 = 4 \Rightarrow y = -2$
$\Rightarrow x - 4 + 3 = 4$
$\Rightarrow x = 5$

$$\begin{array}{l} -R_3 + R_1 \to R_1 \\ -2R_3 + R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{l} -2R_2 + R_1 \to R_1 \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$x = 5$
$y = -2$
$z = 3$

Summary of the systematic method known as Gaussian elimination:

We write the linear system (LS) of m equations for the vector $\underline{\boldsymbol{x}} = \left[x_1, x_2, \ldots x_n\right]^T$ of the n unknowns as

$$
\begin{array}{c}
a_{11}\,x_1 + a_{12}\,x_2 + \ldots + a_{1\,n}\,x_n = b_1 \\
a_{21}\,x_1 + a_{22}\,x_2 + \ldots + a_{2\,n}\,x_n = b_2 \\
\vdots \qquad\qquad\qquad\qquad \vdots \quad \vdots \\
a_{m1}\,x_1 + a_{m2}\,x_2 + \ldots + a_{m\,n}\,x_n = b_m
\end{array}
$$

The matrix that we get by adjoining (augmenting) the right-side $\underline{\boldsymbol{b}}$-vector to the coefficient matrix $A = \left[a_{i\,j}\right]$ is called the *augmented matrix* $\langle A|\underline{\boldsymbol{b}}\rangle$:

$$
\left[\begin{array}{ccccc|c}
a_{11} & a_{12} & a_{13} & \ldots & a_{1\,n} & b_1 \\
a_{21} & a_{22} & a_{23} & \ldots & a_{2\,n} & b_2 \\
\vdots & & & & \vdots & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \ldots & a_{m\,n} & b_m
\end{array}\right]
$$

Our goal is to find all the solution vectors $\underline{\boldsymbol{x}}$ to the system - i.e. the *solution set* .

There are three types of *elementary equation operations* that don't change the solution set to the linear system. They are

- interchange two of equations
- multiply one of the equations by a non-zero constant
- replace an equation by its sum with a multiple of a different equation.

When working with the augmented matrix $\langle A|\underline{\boldsymbol{b}}\rangle$ these correspond to the three types of *elementary row operations:*

- interchange ("swap") two rows
- multiply one of the rows by a non-zero constant
- replace a row by its sum with a multiple of a different row.

Gaussian elimination: Use elementary row operations and work column by column (from left to right) and row by row (from top to bottom) to first get the augmented matrix for an equivalent system of equations which is in

row-echelon form:

(1) All "zero" rows (having all entries $= 0$) lie beneath the non-zero rows.

(2) The leading (first) non-zero entry in each non-zero row lies strictly to the right of the one above it. The locations of these leading non-zero entries are sometimes called *pivot positions*, and the entries are called *pivots*.

(At this stage you could "backsolve" to find all solutions.)

$$\left[\begin{array}{cccc|c} 2 & * & * & * & * \\ 0 & 0 & 3 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Next, continue but by working from bottom to top and from right to left instead, so that you end with an augmented matrix that is in

reduced row echelon form: (1),(2), together with

(3) Each leading non-zero row entry has value 1 . These pivot entries are called "*leading 1's* " in our text.

(4) Each column that has pivot leading 1, has $0's$ in all the other entries.

Finally, read off how to explicitly specify the solution set, by "backsolving" from the reduced row echelon form.

$$\left[\begin{array}{cccc|c} 1 & * & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Note: There are lots of row-echelon forms for a matrix, but only one reduced row-echelon form. All mathematical software will have a command to find the reduced row echelon form of a matrix.

Exercise 1 As a follow-up to the exercise we worked on Friday, in the two systems below we keep all of the coeffients the same except for a_{33}, and we change the right side in the third equation, for 1a. Work out what happens in each case.

1a)

$$\begin{aligned} x + 2y + z &= 4 \\ 3x + 8y + 7z &= 20 \\ 2x + 7y + 8z &= 20\,. \end{aligned}$$

1b)

$$\begin{aligned} x + 2y + z &= 4 \\ 3x + 8y + 7z &= 20 \\ 2x + 7y + 8z &= 23\,. \end{aligned}$$

1c) What are the possible solution sets (and geometric configurations) for 1, 2, 3, 4,... equations in 3 unknowns?

$$\begin{array}{ccc|c|c} & & & 1a) & 1b) \\ 1 & 2 & 1 & 4 & 4 \\ 3 & 8 & 7 & 20 & 20 \\ 2 & 7 & 8 & 20 & 23 \end{array}$$

$-3R_1 + R_2 \to R_2$
$-2R_1 + R_3 \to R_3$

$$\begin{array}{ccc|c|c} 1 & 2 & 1 & 4 & 4 \\ 0 & 2 & 4 & 8 & 8 \\ 0 & 3 & 6 & 12 & 15 \end{array}$$

$R_2/2 \to R_2$

$$\begin{array}{ccc|c|c} 1 & 2 & 1 & 4 & 4 \\ 0 & 1 & 2 & 4 & 4 \\ 0 & 3 & 6 & 12 & 15 \end{array}$$

$-3R_2 + R_3 \to R_3$

$$\begin{array}{ccc|c|c} 1 & 2 & 1 & 4 & 4 \\ 0 & 1 & 2 & 4 & 4 \\ 0 & 0 & 0 & 0 & 3 \end{array}$$

row echelon form

for system 1b 3rd eqtn says
$0x + 0y + 0z = 3$
so system 1b) has no solutions
"inconsistent system"

$$\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}$$

$-2R_2 + R_1 \to R_1$

$$\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}$$

reduced row echelon form.

$x \quad -3z = -4.$
$y + 2z = 4$

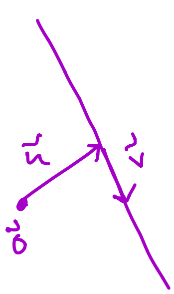
$x = -4 + 3z$
$y = 4 - 2z$
$z = \text{free}$

or

$x = -4 + 3t$
$y = 4 - 2t$
$z = t \in \mathbb{R}$

line in $\mathbb{R}^3$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 3t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$



$\vec{u} + t\vec{v},\ t \in \mathbb{R}$

position vector for a line thru $(-4, 4, 0)$ with velocity vector $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = 3\hat{i} - 2\hat{j} + \hat{k}$

This exercise illustrates Gaussian elimination in a larger example:

Exercise 2 Find all solutions to the system of 3 linear equations in 5 unknowns

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 &= 10 \\ 2x_1 - 4x_2 + 8x_3 + 3x_4 + 10x_5 &= 7 \\ 3x_1 - 6x_2 + 10x_3 + 6x_4 + 5x_5 &= 27\,. \end{aligned}$$

Here's the augmented matrix:

$$\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{array}\right] \end{array}$$

Find the reduced row echelon form of this augmented matrix and then backsolve to explicitly parameterize the solution set. (Hint: it's a two-dimensional plane in $\mathbb{R}^5$, if that helps. :-))

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{array}\right]$$

$$\begin{array}{l} \\ -2R_1 + R_2 \to R_2 \\ -3R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 2 & -1 & 8 & -13 \\ 0 & 0 & 1 & 0 & 2 & -3 \end{array}\right]$$

$$\begin{array}{l} \\ R_3 \to R_2 \\ R_2 \to R_3 \end{array} \left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 2 & -1 & 8 & -13 \end{array}\right]$$

$$\begin{array}{l} \\ \\ -2R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & -1 & 4 & -7 \end{array}\right]$$

$$\begin{array}{l} \\ \\ -R_3 \to R_3 \end{array} \left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{array}\right]$$

row echelon form

$$\begin{array}{l} -2R_3 + R_1 \\ \\ \\ \end{array} \left[\begin{array}{ccccc|c} 1 & -2 & 3 & 0 & 9 & -4 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{array}\right]$$

$$\begin{array}{l} -3R_2 + R_1 \\ \\ \\ \end{array} \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{array}\right]$$

$$\boxed{\begin{aligned} x_1 &= 5 + 2p - 3t \\ x_2 &= p \in \mathbb{R} \\ x_3 &= -3 - 2t \qquad \text{2nd eqtn.} \\ x_4 &= 7 + 4t \\ x_5 &= \text{free} = t \in \mathbb{R} \end{aligned}}$$

$$x_4 - 4x_5 = 7$$
$$x_4 = 7 + 4x_5 = 7 + 4t$$

about free variable choices:

$$\begin{aligned} x_1 - 2x_2 \qquad\qquad + 3x_5 &= 5 \\ x_3 \qquad + 2x_5 &= -3 \\ x_4 - 4x_5 &= 7 \end{aligned}$$

easier to let x_2 be free.

then solve for x_3 in terms of x_5

I could let x_5 or x_4 be free

but because of rref shape, it's easier to let x_5 be free:

- x_5 shows up in other eqtns. x_4 does not
- and, pivot coeff for x_4 is 1 so easier to solve for x_4 in terms of x_5.

Tuesday discussion about free variables

$$\begin{aligned} x_1 &= 5 + 2t_2 - 3t_5 \\ x_2 &= t_2 \quad (\text{free}) \\ x_3 &= -3 - 2t_5 \\ x_4 &= 7 + 4t_5 \\ x_5 &= t_5 \quad (\text{free}) \end{aligned}$$

Maple says:

```
> with(LinearAlgebra) :   # matrix and linear algebra library
> A := Matrix(3, 5, [1, -2, 3, 2, 1,
                     2, -4, 8, 3, 10,
                     3, -6, 10, 6, 5]) :
  b := Vector([10, 7, 27]) :
  <A|b>;     # the mathematical augmented matrix doesn't actually have
             # a vertical line between the end of A and the start of b
  ReducedRowEchelonForm(<A|b>);
```

$$\begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{bmatrix} \quad \textbf{(1)}$$

```
> LinearSolve(A, b);
      #  this command will actually write down the general solution, using
      # Maple's way of writing free parameters, which actually makes
      # some sense.  Generally when there  are free parameters involved,
      #  there will be equivalent ways to express the solution that may
      # look different.  But usually Maple's version will look like yours,
      # because it's using the same algorithm and choosing the free parameters
      # the same way too.
```

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 + 2\_t_2 - 3\_t_5 \\ \_t_2 \\ -3 - 2\_t_5 \\ 7 + 4\_t_5 \\ \_t_5 \end{bmatrix} \quad \textbf{(2)}$$

```
>
```

Matlab says, so watch out...

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reduce.m

```
A=[1,-2,3,2,1;
   2,-4,8,3,10;
   3,-6,10,6,5]
b=[10;7;27]
Aaugb=[A,b]
rref(Aaugb)
linsolve(A,b)
```

Command Window

New to MATLAB? See resources for Getting Started.

```
     1    -2     0     0     3     5
     0     0     1     0     2    -3
     0     0     0     1    -4     7


ans =

         0
   -5.1250
    0.5000
         0
   -1.7500
```

oh oh.

Math 2250-004

Tues Feb 6

- 3.3 The structure of the solution sets to systems of linear equations, based on reduced row echelon form properties.

Announcements: continue talking about solutions to systems of linear equations, using reduced row echelon form

— start with Monday's notes, talking about why we chose the non-pivot column variables to be free, and solve for the pivot variables in terms of them.

Warm-up Exercise: Exercise 1 on next page! which matrices are in r.r.e.f.?

Recall that the four conditions for a matrix to be in reduced row echelon form are :

<u>Row echelon form:</u>

- (1) All "zero" rows (having all entries $= 0$) lie beneath the non-zero rows.
- (2) The leading (first) non-zero entry in each non-zero row lies strictly to the right of the one above it.

These entries are called *pivots* and their locations are called *pivot positions*.

(At this stage you could "backsolve" to find all solutions.)

plus:

<u>reduced row echelon form:</u> (1),(2), together with

- (3) Each pivot has value 1 . These pivot entries are called "*leading 1's* " in our text.
- (4) Each column that has pivot, has 0's in all the other entries.

$$\begin{bmatrix} 1 & 0 & * & 0 & * & * \\ 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Tuesday warm-up exercise 'til 10:47

<u>Exercise 1</u> Are the following matrices in reduced row echelon form or not? Explain. If they aren't in reduced row echelon form, what is the reduced row echelon form?

a)

no. (2) fails

$$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{R_2 \to R_1 \\ R_1 \to R_2}} \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b)

yes

$$\begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

c)

no

(4) fails

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$-2R_3 + R_1 \to R_1$

$R_3 + R_2 \to R_2$

$$\begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

d)

yes!

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

e)

yes

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

Exercise 2 Coefficient matrix taken from problem #19, section 3.3, page 174, together with its reduced row echelon form:

$$A := \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 2 & 7 & -10 & -19 & 13 \\ 1 & 3 & -4 & -8 & 6 \\ 1 & 0 & 2 & 1 & 3 \end{bmatrix} \end{array} \qquad rref(A) = \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let's consider three different linear systems for which A is the coefficient matrix. In the first one, the right hand sides are all zero (what we call the "homogeneous" problem), and I have carefully picked the other two right hand sides. The three right hand sides are separated by the dividing line below, and you might want to add a similar line in the reduced matrix:

$$C = \left[\begin{array}{ccccc|c:c:c} 2 & 7 & -10 & -19 & 13 & 0 & 7 & 7 \\ 1 & 3 & -4 & -8 & 6 & 0 & 0 & 3 \\ 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \end{array}\right] \qquad rref(C) = \left[\begin{array}{ccccc:c:c:c} 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

(1) (2) (3)

2a) Find the solution sets for each of the three systems, using the reduced row echelon form of C.

(1)
$x_1 = -2t_3 - t_4 - 3t_5$
$x_2 = 2t_3 + 3t_4 - t_5$ (Eqtn$_2$ says $x_2 - 2t_3 - 3t_4 + t_5 = 0$)
$x_3 = t_3 \in \mathbb{R}$
$x_4 = t_4 \in \mathbb{R}$
$x_5 = t_5 \in \mathbb{R}$

in vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2t_3 \\ 2t_3 \\ t_3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -t_4 \\ 3t_4 \\ 0 \\ t_4 \\ 0 \end{bmatrix} + \begin{bmatrix} -3t_5 \\ -t_5 \\ 0 \\ 0 \\ t_5 \end{bmatrix} = t_3 \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_5 \begin{bmatrix} -3 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(2) 3^{rd} eqtn says $0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 1$ impossible
no solutions. This system is inconsistent!

(3)
$x_1 = -2t_3 - t_4 - 3t_5$
$x_2 = 1 + 2t_3 + 3t_4 - t_5$
$x_3 = t_3$
$x_4 = t_4$
$x_5 = t_5$

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \bigcirc$$

general soln to problem 1

particular solution

$$rref(A) = \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad rref(C) = \begin{array}{c} \\ \left[\begin{array}{ccccc|ccc} \textcircled{1} & \textcircled{2} & \textcircled{3} \end{array}\right. \end{array}$$

$$rref(C) = \left[\begin{array}{ccccc|ccc} 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Important conceptual questions:

2b) Which of these three solutions could you have written down just from the reduced row echelon form of A, i.e. without using the augmented matrix and the reduced row echelon form of the augmented matrix? Why?

$$\left[\begin{array}{ccccc|c} 2 & 7 & -10 & -19 & 13 & 0 \\ 1 & 3 & -4 & -8 & 6 & 0 \\ 1 & 0 & 2 & 1 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 3 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

for problem 1, only need rref(A), because augmented column stays all zero during reduction

2c) Linear systems in which right hand side vectors equal zero are called homogeneous linear systems. Otherwise they are called inhomogeneous or nonhomogeneous. Notice that the general solution to the consistent inhomogeneous system is the sum of a particular solution to it, together with the general solution to the homogeneous system!!! It's related to an important general concept which will come up later in the course.

2d) Can you tell how many free parameters the solutions to a matrix system $A\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$ will have, based on the reduced row echelon form of A alone (assuming the system is consistent, i.e. has at least one solution)?

three: # of non-pivot variables (i.e. # of columns without pivots)

The previous exercise shows that the reduced row echelon form of just the matrix A can tell us a lot about the possible solution sets to linear systems with augmented matrices $\langle A|\underline{\boldsymbol{b}}\rangle$, independently of what the value for the vector $\underline{\boldsymbol{b}}$ is.

Before we continue that discussion, let's introduce notation that will let us abbreviate how we write systems of linear equations:

$$\begin{array}{ccccccccc} a_{11}\,x_1 & + & a_{12}\,x_2 & + \ldots + & a_{1\,n}\,x_n & = & b_1 \\ a_{21}\,x_1 & + & a_{22}\,x_2 & + \ldots + & a_{2\,n}\,x_n & = & b_2 \\ \vdots & & & & \vdots & & \vdots \\ a_{m1}\,x_1 & + & a_{m2}\,x_2 & + \ldots + & a_{m\,n}\,x_n & = & b_m \end{array}$$

can be written more efficiently using the rule we use to multiply a matrix times a vector,

$A_{m\times n}\,\vec{x}_{n\times 1}$

$$\boldsymbol{A}\,\underline{\boldsymbol{x}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1\,n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2\,n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{m\,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1\,n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2\,n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{m\,n}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

so that they system above can be abbreviated by $\boxed{A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}}$.

Exercise 3) To make sure you understand the rule for multiplying a matrix times a vector, compute the one of these two expressions that makes sense:

3a) $\begin{bmatrix} 1 & -2 & 2 \\ 3 & 2 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \end{bmatrix} =$ oops. — I'd need 3 entries in the vector.

3b) $\begin{bmatrix} 1 & 3 \\ -2 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 1\cdot 6 + 3\cdot 2 \\ -2\cdot 6 + 2\cdot 2 \\ 2\cdot 6 + 0\cdot 2 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \\ 12 \end{bmatrix}$

Exercise 4)

Then consider the matrix A below, and answer all questions:

$$B := \left[\begin{array}{ccccc|c|c} 2 & 7 & -10 & -19 & 13 & 0 & b_1 \\ 1 & 3 & -4 & -8 & 6 & 0 & b_2 \end{array}\right] \qquad rref(B) = \left[\begin{array}{ccccc|c|c} 1 & 0 & 2 & 1 & 3 & 0 & c_1 \\ 0 & 1 & -2 & -3 & 1 & 0 & c_2 \end{array}\right]$$

4a) Is the homogeneous problem $\underbrace{B\,\mathbf{x}=\mathbf{0}}$ always solvable for $\mathbf{x}$?

always consistent

$\vec{x} = \vec{0}$ is always a soln

$B\vec{0} = \vec{0}$

4b) Is the inhomogeneous problem $B\,\mathbf{x}=\mathbf{b}$ solvable for $\mathbf{x}$ no matter the choice of $\mathbf{b}$?

Wed.

$$2x_1 + 7x_2 - 10x_3 - 19x_4 + 13x_5 = b_1$$
$$x_1 + 3x_2 - 4x_3 - 8x_4 + 6x_5 = b_2$$

consistent. : x_3, x_4, x_5 free. then solve for x_2, x_1 in terms of them.

done for today !!

$$\rightarrow \quad x_1 \quad + 2x_3 + x_4 + 3x_5 = c_1$$
$$x_2 - 2x_3 - 3x_4 + x_5 = c_2$$

∞'ly many   3

4c) How many solutions are there? How many free parameters are there in the solution? How does this number relate to the reduced row echelon form of A, the number of pivots and the number of columns?

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 & 1 \end{bmatrix}$$

3 = # columns without pivot
= # free parameters.

Exercise 5) Now consider the matrix E and similar questions:

$$E := \left[\begin{array}{cc|c|c} 1 & 2 & 0 & b_1 \\ -1 & 3 & 0 & b_2 \\ 4 & 2 & 0 & b_3 \end{array}\right] \qquad rref(E) = \left[\begin{array}{cc|c|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & c_3 \end{array}\right]$$

5a) How many solutions to the homogeneous problem $\boxed{E\,\underline{x} = \underline{\mathbf{0}}}$?

$$E\vec{x} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

actual system:

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ -x_1 + 3x_2 &= 0 \\ 4x_1 + 2x_2 &= 0 \end{aligned}$$

soltn: from rref $x_1 = 0$, $x_2 = 0$

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

a single solution.

5b) Is the inhomogeneous problem $E\,\underline{x} = \underline{\boldsymbol{b}}$ solvable for every right side vector $\underline{\boldsymbol{b}}$?

$$\begin{bmatrix} x_1 + 2x_2 \\ -x_1 + 3x_2 \\ 4x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \longrightarrow \begin{aligned} x_1 &= c_1 \\ x_2 &= c_2 \\ 0 &= c_3 \end{aligned}$$

(a combo of the b_1, b_2, b_3)

won't be solvable if the c_3 turned ~~&~~ out not be zero. And that happens for a lot, because I would pick any $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ with $c_3 \neq 0$, and do row operations in reverse, to find $\vec{b}$'s giving such $\vec{c}$'s.

5c) When the inhomogeneous problem is solvable, how many solutions does it have?

this is the case when c_3 turned out to be zero

one solution: $x_1 = c_1$, $x_2 = c_2$

Math 2250-004

Wed Feb 7

- 3.3 The structure of the solution sets to systems of linear equations, based on reduced row echelon form computations.
- 3.4 Matrix algebra

Announcements: quiz: use Gaussian elimination

finish 3.3 discussion

start 3.4

stop in time for the quiz; 11:22

Warm-up Exercise: Compute: 'til 10:47

a)
$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 10 \end{bmatrix} \begin{array}{l} \leftarrow 1\cdot 2 + 2\cdot 1 \\ \leftarrow -1\cdot 2 + 3\cdot 1 \\ \leftarrow 4\cdot 2 + 2\cdot 1 \end{array}$$

b)
$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ -x_1 + 3x_2 \\ 4x_1 + 2x_2 \end{bmatrix}$$

c) Rewrite the system

$$x_1 + 2x_2 - x_3 = 4$$
$$2x_1 - 3x_2 + x_3 = 5$$

in the form
$$A\underset{\sim}{x} = \underset{\sim}{b}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Let's continue our discussion from Tuesday about the implications for solving a matrix equation based on the reduced row echelon form of just the coefficient matrix.

Exercise 1) Square matrices (i.e number of rows equals number of columns) with 1's down the diagonal which runs from the upper left to lower right corner are special. They are called identity matrices, I (because $I\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}$ is always true (as long as the vector $\underline{\boldsymbol{x}}$ is the right size)). Here is an example where the matrix A reduces to the identity matrix:

$$C := \left[\begin{array}{cccc|cc} 1 & 0 & -1 & 1 & 0 & b_1 \\ 22 & -1 & 3 & 5 & 0 & b_2 \\ 7 & 4 & 6 & 2 & 0 & b_3 \\ 3 & 5 & 7 & 13 & 0 & b_4 \end{array}\right] \qquad rref(C) = \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & c_1 \\ 0 & 1 & 0 & 0 & 0 & c_2 \\ 0 & 0 & 1 & 0 & 0 & c_3 \\ 0 & 0 & 0 & 1 & 0 & c_4 \end{array}\right]$$

"the diagonal"

RHS $= \vec{0}$

1a) How many solutions to the homogeneous problem $C\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$?

from reduced augmented matrix

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \\ x_4 &= 0 \end{aligned} \qquad \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

one solution

1b) Is the inhomogeneous problem $C\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ solvable for every choice of $\underline{\boldsymbol{b}}$?

$$\begin{aligned} x_1 &= c_1 \\ x_2 &= c_2 \\ x_3 &= c_3 \\ x_4 &= c_4 \end{aligned}$$

c_1, c_2, c_3, c_4 are the result of the elementary rows on b_1, b_2, b_3, b_4

1c) How many solutions? one solution! $\vec{x} = \vec{c}$.

Exercise 2: **What are the general conclusions we can draw from today's and yesterday's examples and reasoning?**

$A_{m \times n}$ matrix : m rows, n columns.

2a) What conditions on the reduced row echelon form of just the matrix A guarantee that the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions?

there are free variables

i.e. there are columns without pivots.

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

only one solution

$$\text{rref}(A) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

∞'ly many!

2b) What conditions on the dimensions of A (i.e. number of rows and number of columns) always force infinitely many solutions to the homogeneous problem?

pivots $\leq m$ & n always

so if # columns ("n") > # rows ("m")

there will at least $n-m$ non-pivot columns

at least $n-m$ free variables.

2c) What conditions on just the reduced row echelon form of A guarantee that solutions $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$ are always unique (if they exist)?

no free parameters

i.e. each column has a pivot.

2d) If A is a square matrix ($m=n$), what can you say about the solution set to $A\mathbf{x} = \mathbf{b}$ when

* The reduced row echelon form of A is the identity matrix?

every problem has a unique sol'n.

* The reduced row echelon form of A is not the identity matrix?

$< n$ pivots

- bottom row is zero $\rightarrow$ $A\vec{x} = \vec{b}$ can be inconsistent
- not all columns have pivots so if system is consistent, ∞'ly many sol'ns

3.4 Matrix algebra

Matrix vector algebra that we've already touched on, but that we want to record carefully:

Vector addition and scalar multiplication:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ \vdots \\ x_n + y_n \end{bmatrix} ; \qquad c \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} c\,x_1 \\ c\,x_2 \\ c\,x_3 \\ \vdots \\ c\,x_n \end{bmatrix}$$

Vector dot product , which yields a scalar (i.e. number) output (regardless of whether vectors are column vectors or row vectors):

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} := x_1 y_1 + x_2 y_2 + \ldots + x_n y_n .$$

Matrix times vector: If A is an $m \times n$ matrix and $\underline{\boldsymbol{x}}$ is an n column vector, then

$$A\underline{\boldsymbol{x}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} Row_1(A) \cdot \underline{\boldsymbol{x}} \\ Row_2(A) \cdot \underline{\boldsymbol{x}} \\ \vdots \\ Row_m(A) \cdot \underline{\boldsymbol{x}} \end{bmatrix}$$

Compact way to write our usual linear system:

$$\underbrace{A\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}}_{} .$$

Exercise 1a) Compute done!

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 2 & 1 & -2 \\ -5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}.$$

# of columns in A ("n"), = # rows in B ("n")

Matrix times matrix: Let $\boxed{A_{m \times n}, B_{n \times p}}$ be two matrices such that the number of columns of A equals the number of rows of B. ~~Then the product~~ AB is an $m \times p$ matrix, with

- $$col_j(AB) = A\, col_j(B).$$

In other words, you just compute matrix times vector, for each column of B, to get the corresponding column of the product AB. So, the resulting matrix will have as many columns as B and as many rows as A.

Exercise 1b) Compute did in warm-up!

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 2 & 1 & -2 \\ -5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -4 & 1 \\ 0 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ -8 & -3 \\ -7 & 2 \end{bmatrix}$$

* In warmup Friday, we just computed

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 2 & 1 & -2 \\ -5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -7 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 2 & 1 & -2 \\ -5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 2 \end{bmatrix}$$

Summary of different ways to think of the matrix product $A\,B$:

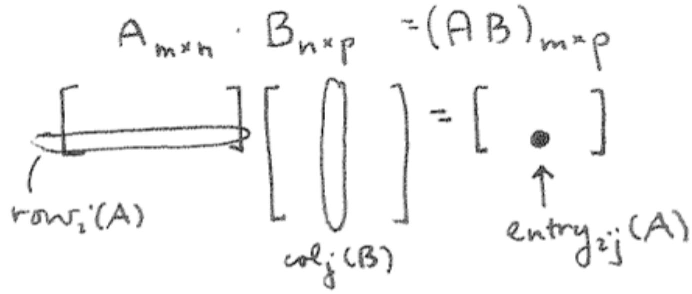
- The j^{th} column of AB is given by A times the j^{th} column of B

$$col_j(AB) = A\ col_j(B)$$

- The i^{th} entry in the j^{th} column of $A\,B$, i.e. $entry_{i\,j}(AB)$ is the dot product of the i^{th} row of A with the j^{th} column of B:

$$entry_{i\,j}(AB) := row_i(A) \bullet col_j(B) = \sum_{k=1}^{n} a_{i\,k} b_{k\,j}\,.$$

This stencil might help:



More matrix operations:

- addition and scalar multiplication: Let $A_{m \times n}$, $B_{m \times n}$ be two matrices of the same dimensions (m rows and n columns). Let $entry_{ij}(A) = a_{ij}$, $entry_{ij}(B) = b_{ij}$. (In this case we write $A = [a_{ij}]$, $B = [b_{ij}]$.) Let c be a scalar. Then

$$entry_{ij}(A + B) := a_{ij} + b_{ij}.$$
$$entry_{ij}(c\,A) := c\,a_{ij}.$$

In other words, addition and scalar multiplication are defined analogously as for vectors. In fact, for these two operations you can just think of matrices as vectors written in a rectangular rather than row or column format.

Exercise 3) Let $A := \begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 0 & 3 \end{bmatrix}$ and $B := \begin{bmatrix} 0 & 27 \\ 5 & -1 \\ -1 & 1 \end{bmatrix}$. Compute $4\,A - B$.

3×2 matrices.

$$4A = \begin{bmatrix} 4 & -8 \\ 12 & -4 \\ 0 & 12 \end{bmatrix} \qquad -B = \begin{bmatrix} 0 & -27 \\ -5 & 1 \\ 1 & -1 \end{bmatrix}$$

$$4A - B = 4A + (-B) = \begin{bmatrix} 4 & -35 \\ 7 & -3 \\ 1 & 11 \end{bmatrix}$$

try to get so that you can write down $4A-B$ entry by entry, in one step.

Properties for the algebra of matrix addition and multiplication :

Multiplication is not commutative in general (AB usually does not equal BA , even if you're multiplying square matrices so that at least the product matrices are the same size).

size: $A_{m \times n} B_{n \times m} = (AB)_{m \times m}$

$B_{n \times m} A_{m \times n} = (BA)_{n \times n}$

not even same size, unless $m = n$ (i.e. square matrices), even for square matrices, almost never true that $AB = BA$

$$\begin{bmatrix} 2 & 6 \\ 1 & 7 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 0 & 0 \end{bmatrix}$$

But other properties you're used to do hold:

- $+$ is commutative $\quad A + B = B + A$

  ij entry: $a_{ij} + b_{ij} = b_{ij} + a_{ij}$

- $+$ is associative $\quad (A + B) + C = A + (B + C)$

  ij entry on both sides $a_{ij} + b_{ij} + c_{ij}$

- scalar multiplication distributes over $+$ $\quad c(A + B) = cA + cB$.

  ij entry on both sides: $c(a_{ij} + b_{ij}) = ca_{ij} + cb_{ij}$

$$c\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = c\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} c4 \\ c6 \end{bmatrix}$$

$$c\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} c \\ 2c \end{bmatrix} + \begin{bmatrix} 3c \\ 4c \end{bmatrix} = \begin{bmatrix} 4c \\ 6c \end{bmatrix}$$

- multiplication is associative $\quad (AB)C = A(BC)$.

  * magic.

  you can do this by brute force also, see HW for example.

- matrix multiplication distributes over $+$ $\quad A(B + C) = AB + AC$; $\quad (A + B)C = AC + BC$

e.g.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\left(\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_1 + c_1 \\ b_2 + c_2 \end{bmatrix} = \begin{bmatrix} a_{11}(b_1 + c_1) + a_{12}(b_2 + c_2) \\ a_{21}(b_1 + c_1) + a_{22}(b_2 + c_2) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} a_{11}b_1 + a_{12}b_2 \\ a_{21}b_1 + a_{22}b_2 \end{bmatrix}}_{b \text{ terms}} + \underbrace{\begin{bmatrix} a_{11}c_1 + a_{12}c_2 \\ a_{21}c_1 + a_{22}c_2 \end{bmatrix}}_{c \text{ terms}} = A\vec{b} + A\vec{c}$$

- If A is an $m \times n$ matrix, and we use the letter I for identity matrices, then $I_{m \times m} A_{m \times n} = A$ and $A_{m \times n} I_{n \times n} = A$.

$$I_{2\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ldots$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 2 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 2 & 0 \\ 1 & -1 \end{bmatrix}; \quad \begin{bmatrix} 3 & 3 \\ 2 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 2 & 0 \\ 1 & -1 \end{bmatrix}$$

Math 2250-004

Fri Feb 9

- 3.4 Matrix algebra
- 3.5 Matrix inverses

Announcements: • finish Wed notes on matrix algebra

• start 3.5 on matrix inverses

... to be continued on Monday

exam thru §3.5

* what's it all good for?

Warm-up Exercise: Compute

'til 10:47

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 2 & 1 & -2 \\ -5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -7 \end{bmatrix}$$

$3\cdot 2 + 2\cdot(-4) + 1\cdot 0 - 2\cdot 3$
$= 6 - 8 - 6 = -8$

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 2 & 1 & -2 \\ -5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 2 \end{bmatrix}$$

$-5\cdot 0 + 0\cdot 1 + 0\cdot(-1) + 1\cdot 2$
$= 2$

so!

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 2 & 1 & -2 \\ -5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -4 & 1 \\ 0 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ -8 & -3 \\ -7 & 2 \end{bmatrix}$$

$$col_j(AB) = A\, col_j(B)$$

$$A_{3\times 4} B_{4\times 2} = (AB)_{3\times 2}$$

this is how we define matrix multiplication, column by column.

We've been talking about matrix algebra: addition, scalar multiplication, multiplication, and how these operations combine. If necessary, finish those notes.

But I haven't told you what all that algebra is good for. Today we'll start to find out. By way of comparison, think of a scalar linear equation with known numbers a, b, c, d and an single unknown number x,

$$a\,x + b = c\,x + d$$

We know how to solve it by collecting terms and doing scalar algebra:

$$a\,x - c\,x = d - b$$

$$(a - c)\,x = d - b \qquad *$$

$$x = \frac{d - b}{a - c}\,.$$

How would you solve such an equation if A, B, C, D were square matrices, and X was a vector (or matrix) ? Well, you could use the matrix algebra properties we've been discussing to get to the $*$ step. And then if X was a vector you could solve the system * with Gaussian elimination. In fact, if X was a matrix, you could solve for each column of X (and do it all at once) with Gaussian elimination.

But you couldn't proceed as with scalars and do the final step after the * because it is not possible to divide by a matrix. Today we'll talk about a potential shortcut for that last step that is an analog of of dividing, in order to solve for X . It involves the concept of *inverse matrices.*

$$AX + B = CX + D$$

$$-CX \quad = -CX$$

$$-B \qquad -B$$

$$AX - CX = D - B$$

$$(A - C)X = D - B.$$

can't divide by a matrix

$$\cancel{X = \frac{D-B}{A-C}}$$

Matrix inverses: A square matrix $A_{n \times n}$ is invertible if there is a matrix $B_{n \times n}$ so that

$$AB = BA = I,$$

where I is the $n \times n$ identity matrix. In this case we call B the inverse of A, and write $B = A^{-1}$.

Remark: A matrix A can have at most one inverse, because if we have two candidates B, C with

$$AB = BA = I \quad \text{and also} \quad AC = CA = I$$

then

$$(BA)C = IC = C$$
$$B(AC) = BI = B$$

so since the associative property $(BA)C = B(AC)$ is true, it must be that

$$B = C.$$

Exercise 1a) Verify that for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ the inverse matrix is $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$BA = I$ too!

Inverse matrices are very useful in solving algebra problems. For example

Theorem: If A^{-1} exists then the only solution to $\boxed{A\underline{\mathbf{x}} = \underline{\mathbf{b}}}$ is $\underline{\mathbf{x}} = A^{-1}\underline{\mathbf{b}}$. $\Rightarrow$

$$A\vec{x} = \vec{b}$$
$$A^{-1}(A\vec{x}) = A^{-1}\vec{b}$$
$$(A^{-1}A)\vec{x} = A^{-1}\vec{b}$$
$$I\vec{x} = A^{-1}\vec{b}$$
$$\boxed{\vec{x} = A^{-1}\vec{b}}$$

Exercise 1b) Use the theorem and A^{-1} in 1a, to write down the solution to the system

$$x + 2y = 5$$
$$3x + 4y = 6$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 9/2 \end{bmatrix}.$$

check $x + 2y = -4 + 9 = 5$ ✓
$3x + 4y = -12 + 18 = 6$ ✓

Exercise 2a) Use matrix algebra to verify why the Theorem on the previous page is true. Notice that the correct formula is $\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$ and not $\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}\,A^{-1}$ (this second product can't even be computed because the dimensions don't match up!).

2b) Assuming A is a square matrix with an inverse A^{-1}, and that the matrices in the equation below have dimensions which make for meaningful equation, solve for X in terms of the other matrices:

$$XA + C = B$$

But where did that formula for A^{-1} come from?

<u>Answer:</u> Consider A^{-1} as an unknown matrix, $A^{-1} = X$. We want

$$A\,X = I\,.$$

We can break this matrix equation down by the columns of *X*. In the two by two case we get:

$$A\left[\begin{array}{c|c} col_1(X) & col_2(X) \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array}\right].$$

In other words, the two columns of the inverse matrix *X* should satisfy

$$A\,\left(col_1(X)\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\quad A\,\left(col_2(X)\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We can solve for both of these mystery columns at once, as we've done before when we had different right hand sides:

<u>Exercise 3:</u> Reduce the double augmented matrix

$$\left[\begin{array}{cc|c|c} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

to find the two columns of A^{-1} for the previous example.

Exercise 4: Will this always work? Can you find A^{-1} for

$$A := \begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix} ?$$

Exercise 5) Will this always work? Try to find B^{-1} for $B := \begin{bmatrix} 1 & 5 & 5 \\ 2 & 5 & 0 \\ 2 & 7 & 4 \end{bmatrix}$.

Hint: We'll discover that it's impossible for B to have an inverse.