Math 2250-004 Week 3 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes include material from 1.5, EP3.7, 2.1-2.2 and an introduction to 2.3.

Mon Jan 22
1.5, EP3.7 linear DEs, and applications.

Announcements:

Warm-up Exercise:

Input-out models often lead to the linear (and separable) differential equation IVP for functions $x(t)$

$$x'(t) + a\,x(t) = b$$
$$x(0) = x_0$$

where a, b are constants. As our warm-up exercise on Friday we found that the solution to this IVP is

$$x(t) = \frac{b}{a} + \left(x_o - \frac{b}{a}\right)e^{-a\,t}.$$

<u>Exercise 1:</u> Use the result above to solve a pollution problem IVP and answer the following question (p. 55-56 text): Lake Huron typically has a constant concentration for a certain pollutant. Due to an industrial accident, Lake Erie has suddenly obtained a concentration five times as large. Lake Erie has a volume of 480 km^3 , and water flows into and out of Lake Erie at a rate of 350 km^3 per year. Essentially all of the in-flow is from Lake Huron and the outflow goes to Lake Ontario (see map). We expect that as time goes by, the water from Lake Huron will flush out Lake Erie. Assuming that the pollutant concentration is roughly the same everywhere in Lake Erie, about how long will it be until this concentration is only twice the background concentration from Lake Huron?

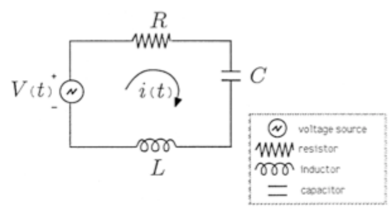
<u>a)</u> Set up the initial value problem. Maybe use symbols
c for the background concentration (in Huron),
$V = 480\ km^3$
$r = 350\ \frac{km^3}{y}$

http://www.enchantedlearning.com/usa/statesbw/greatlakesbw.GIF

<u>b)</u> Solve the IVP, and then answer the question posed in the exercise.

EP 3.7 This is a supplementary section. I've posted a .pdf on our homework page.

Often the same DE can arise in completely different-looking situations. For example, first order linear DE's also arise (as special cases of second order linear DE's) in simple *RLC* circuit modeling.



circuit element	voltage drop	units
inductor	$L\,I'(t)$	L Henries (H)
resistor	$R\,I(t)$	R Ohms (Ω)
capacitor	$\frac{1}{C}Q(t)$	C Farads (F)

http://cnx.org/content/m21475/latest/pic012.png

Charge $Q(t)$ coulombs accumulates on the capacitor, at a rate $I(t)$ ($i(t)$ in the diagram above) amperes (coulombs/sec), i.e $Q'(t) = I(t)$.

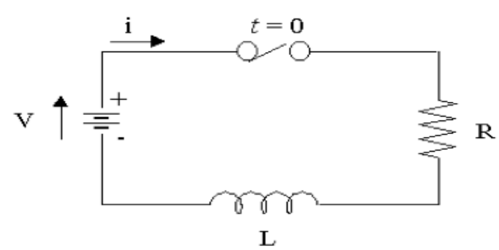
Kirchoff's Law: The sum of the voltage drops around any closed circuit loop equals the applied voltage $V(t)$ (volts). The units of voltage are energy units - Kirchoff's Law says that a test particle traversing any closed loop returns with the same potential energy level it started with:

For $Q(t)$: $$L\,Q''(t) + R\,Q'(t) + \frac{1}{C}Q(t) = V(t)$$

For $I(t)$: $$L\,I''(t) + R\,I'(t) + \frac{1}{C}I(t) = V'(t)$$

if no inductor, or if no capacitor, then Kirchoff's Law yields 1^{st} order linear DE's, as below:

Exercise 2: Consider the $R - L$ circuit below, in which a switch is thrown at time $t = 0$. Assume the voltage V is constant, and $I(0) = 0$. Find $I(t)$. Interpret your results.



http://www.intmath.com/differential-equations/5-rl-circuits.php

Tues Jan 23
2.1 Improved population models

Announcements:

Warm-up Exercise:

2.1: Let $P(t)$ be a population at time t. Let's call them "people", although they could be other biological organisms, decaying radioactive elements, accumulating dollars, or even molecules of solute dissolved in a liquid at time t (2.1.23). Consider:

$B(t)$, birth rate (e.g. $\frac{people}{year}$);

$$\beta(t) := \frac{B(t)}{P(t)}$$, fertility rate ($\frac{people}{year}$ per $person$)

$\mathrm{D}(t)$, death rate (e.g. $\frac{people}{year}$);

$$\delta(t) := \frac{\mathrm{D}(t)}{P(t)}$$, mortality rate ($\frac{people}{year}$ per $person$)

Then in a closed system (i.e. no migration in or out) we can write the governing DE two equivalent ways:

$$P'(t) = B(t) - \mathrm{D}(t)$$
$$P'(t) = \big(\beta(t) - \delta(t)\big)P(t) .$$

<u>Model 1:</u> constant fertility and mortality rates, $\beta(t) \equiv \beta_0 \geq 0,\ \delta(t) \equiv \delta_0 \geq 0$, constants.

$$\Rightarrow P' = \left(\beta_0 - \delta_0\right)P = k\,P .$$

This is our familiar exponential growth/decay model, depending on whether $k > 0$ or $k < 0$.

<u>Model 2:</u> population fertility and mortality rates only depend on population P, but they are not constant:

$$\beta = \beta_0 + \beta_1 P$$
$$\delta = \delta_0 + \delta_1 P$$

with $\beta_0, \beta_1, \delta_0, \delta_1$ constants. This implies

$$P' = (\beta - \delta)P = \left(\left(\beta_0 + \beta_1 P\right) - \left(\delta_0 + \delta_1 P\right)\right)P$$
$$= \left(\left(\beta_0 - \delta_0\right) + \left(\beta_1 - \delta_1\right)P\right)P .$$

For viable populations, $\beta_0 > \delta_0$. For a sophisticated (e.g. human) population we might also expect $\beta_1 < 0$, and resource limitations might imply $\delta_1 > 0$. With these assumptions, and writing $\beta_1 - \delta_1 = -a < 0$, $\beta_0 - \delta_0 = b > 0$ one obtains the <u>logistic differential equation:</u>

$$P' = (b - a\,P)P$$
$$P' = b\,P - a\,P^2 \text{, or equivalently}$$
$$P' = a\,P\left(\frac{b}{a} - P\right) = k\,P(M - P).$$

$k = a > 0$, $M = \frac{b}{a} > 0$. (One can consider other cases as well.)

Exercise 1a): Discuss qualitative features of the slope field for the logistic differential equation for $P = P(t)$. Notice that the "isoclines" (curves where the slope function is constant) are horizontal lines

$$P' = k\,P(M - P)$$

Also note that there are two constant ("equilibrium") solutions. What are they?

b) Sketch the slope field and apparent solutions graphs in a qualitatively accurate way. We'll also include the 1-dimensional "phase portrait" associated to these slope fields.

c) When discussing the logistic equation, the value M is called the "carrying capacity" of the (ecological or other) system. Discuss why this is a good way to describe M. Hint: if $P(0) = P_0 > 0$, and $P(t)$ solves the logistic equation, what is the apparent value of $\lim_{t \to \infty} P(t)$? Note that by the existence-uniqueness theorem, different solution graphs may never touch each other, so the time-varying solution graphs never touch the horizontal graph asymptotes.

Exercise 2: Solve the logistic DE IVP

$$P' = k\,P(M - P)$$
$$P(0) = P_0$$

via separation of variables. Verify that the solution formula is consistent with the slope field and phase diagram discussion from exercise 1. Hint: You should find that

$$P(t) = \frac{MP_0}{(M - P_0)e^{-Mkt} + P_0} .$$

Solution (we will work this out step by step in class, using the fact that the logistic DE is separable. It is not linear!!):

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-Mkt}} .$$

Notice that because $\lim_{t \to \infty} e^{-Mkt} = 0$,

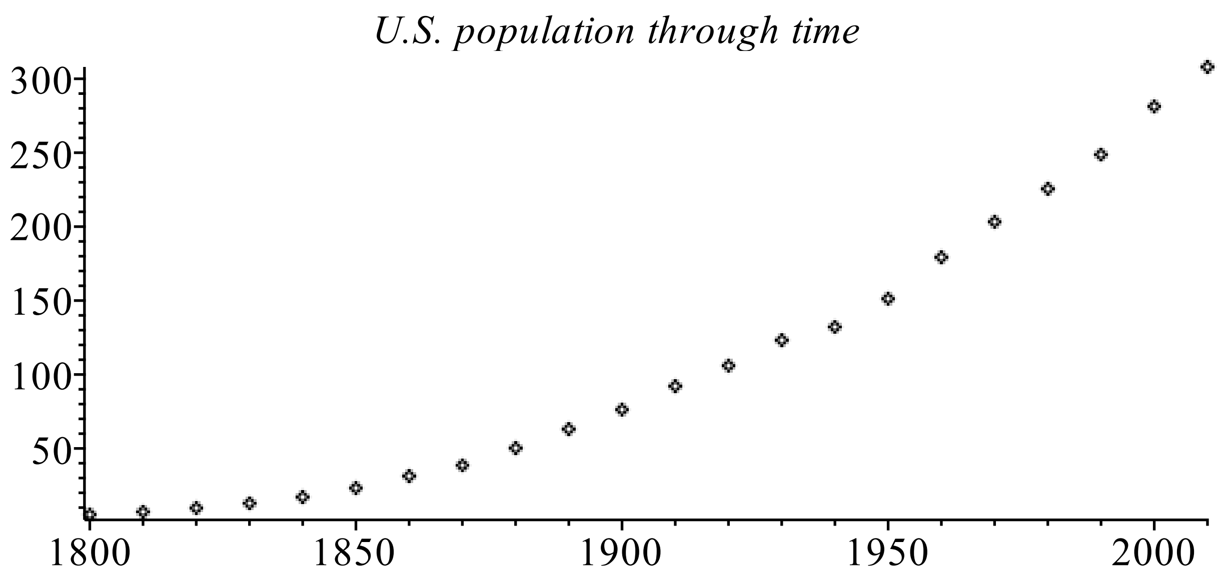
$$\lim_{t \to \infty} P(t) = \frac{MP_0}{P_0} = M \text{ as expected.}$$

Note: If $P_0 > 0$ the denominator stays positive for $t \geq 0$, so we know that the formula for $P(t)$ is a differentiable function for all $t > 0$. (If the denominator became zero, the function would blow up at the corresponding vertical asymptote.) To check that the denominator stays positive check that (i) if $P_0 < M$ then the denominator is a sum of two positive terms; if $P_0 = M$ the separation algorithm actually fails because you divided by 0 to get started but the formula actually recovers the constant equilibrium solution $P(t) \equiv M$; and if $P_0 > M$ then $|M - P_0| < P_0$ so the second term in the denominator can never be negative enough to cancel out the positive P_0, for $t > 0$.)

Application!

The Belgian demographer P.F. Verhulst introduced the logistic model around 1840, as a tool for studying human population growth. Our text demonstrates its superiority to the simple exponential growth model, and also illustrates why mathematical modelers must always exercise care, by comparing the two models to actual U.S. population data.

```
> restart : # clear memory
  Digits := 5 : #work with 5 significant digits
> pops := [[1800, 5.3], [1810, 7.2], [1820, 9.6], [1830, 12.9],
      [1840, 17.1], [1850, 23.2], [1860, 31.4], [1870, 38.6],
      [1880, 50.2], [1890, 63.0], [1900, 76.2], [1910, 92.2],
      [1920, 106.0], [1930, 123.2], [1940, 132.2], [1950, 151.3],
      [1960, 179.3], [1970, 203.3], [1980, 225.6], [1990, 248.7],
      [2000, 281.4], [2010, 308.]] : #I added 2010 - between 306-313
      # I used shift-enter to enter more than one line of information
      # before executing the command.
> with(plots) : # plotting library of commands
  pointplot(pops, title = `U.S. population through time`);
```




```
>
```

Unlike Verhulst, the book uses data from 1800, 1850 and 1900 to get constants in our two models. We let t=0 correspond to 1800.

Exponential Model: For the exponential growth model $P(t) = P_0\, e^{r\,t}$ we use the 1800 and 1900 data to get values for P_0 and r :

```
> P0 := 5.308;
  solve(P0·exp(r·100) = 76.212, r);
```

$$P0 := 5.308$$

$$0.026643 \qquad \textbf{(1)}$$

```
> P1 := t→5.308· exp(.02664· t);#exponential model -eqtn (9) page 83
```

$$P1 := t \rightarrow 5.308\, e^{0.02664\, t} \qquad \textbf{(2)}$$

```
>
```

Logistic Model: We get P_0 from 1800, and use the 1850 and 1900 data to find k and M :

```
> P2 := t→M·P0/(P0 + (M−P0)·exp(−M·k·t)); # logistic solution we worked out
```

$$P2 := t \rightarrow \frac{M\, P0}{P0 + (M - P0)\, e^{-M\,k\,t}} \qquad \textbf{(3)}$$

```
> solve({P2(50) = 23.192, P2(100) = 76.212}, {M, k});
```

$$\{M = 188.12,\ k = 0.00016772\} \qquad \textbf{(4)}$$

```
> M := 188.12;
   k := .16772e-3;
  P2(t); #should be our logistic model function,
     #equation (11) page 84.
```

$$M := 188.12$$

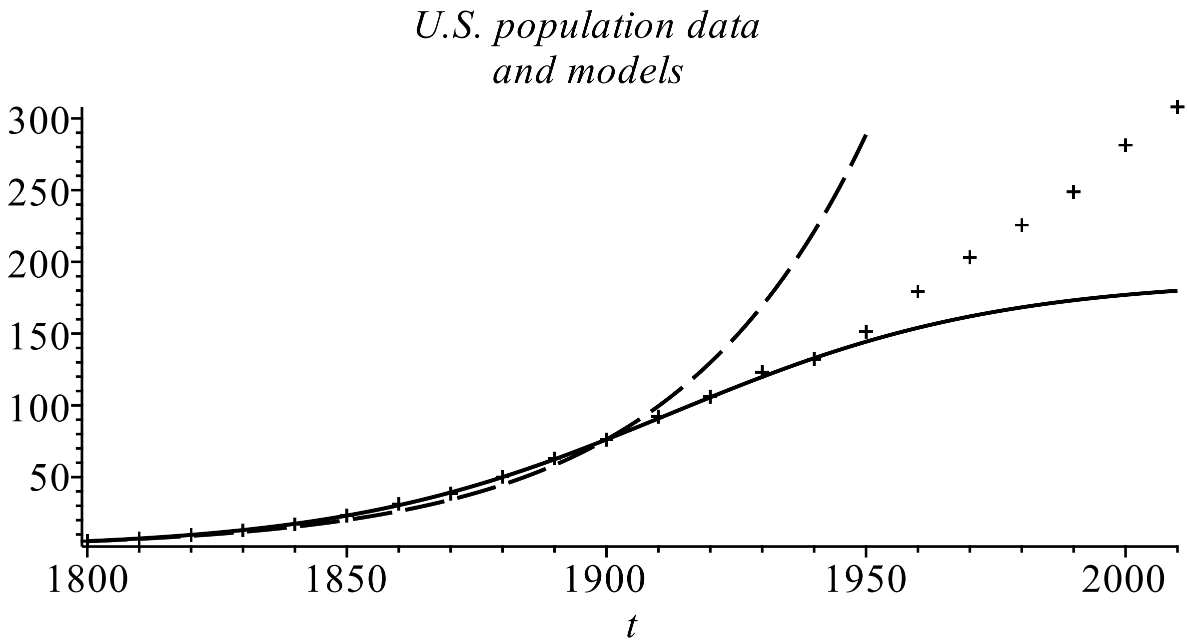
$$k := 0.00016772$$

$$\frac{998.54}{5.308 + 182.81\, e^{-0.031551\, t}} \qquad \textbf{(5)}$$

```
>
```

Now compare the two models with the real data, and discuss. The exponential model takes no account of the fact that the U.S. has only finite resources.

```
> plot1 := plot(P1(t-1800), t = 1800..1950, color = black, linestyle = 3) :
  #this linestyle gives dashes for the exponential curve
  plot2 := plot(P2(t-1800), t = 1800..2010, color = black) :
  plot3 := pointplot(pops, symbol = cross) :
  display({plot1, plot2, plot3}, title = `U.S. population data
  and models`);
```




```
>
```

Any ideas on why the logistic model begins to fail (with our parameters) around 1950?

Wednesday Jan 24
2.2: Autonomous Differential Equations.

Announcements:

Warm-up Exercise:

2.2: Recall, that a first order DE for $x = x(t)$ is written as

$$x' = f(t, x) \ ,$$

which is shorthand for

$$x'(t) = f(t, x(t)).$$

Definition: If the slope function f only depends on the value of $x(t)$, and not on t itself, then we call the first order differential equation *autonomous*:

$$x' = f(x) \ .$$

Example: The logistic DE, $P' = k\,P(M - P)$ is an autonomous differential equation for $P(t)$, because how fast the population is changing only depends on the value of the population.

Definition: Constant solutions $x(t) \equiv c$ to autonomous differential equations $x' = f(x)$ are called *equilibrium solutions*. Since the derivative of a constant function $x(t) \equiv c$ is zero, the values c of equilibrium solutions are exactly the roots c to $f(c) = 0$.

Example: The functions $P(t) \equiv 0$ and $P(t) \equiv M$ are the equilibrium solutions for the logistic DE.

Exercise 1: Find the equilibrium solutions of

1a) $x'(t) = 3\,x - x^2$

1b) $x'(t) = x^3 + 2\,x^2 + x$

1c) $x'(t) = \sin(x)$.

<u>Def:</u> Let $x(t) \equiv c$ be an equilibrium solution for an autonomous DE. Then

· c is a *stable* equilibrium solution if solutions with initial values close enough to c stay close to c.

There is a precise way to say this, but it requires quantifiers: For every $\varepsilon > 0$ there exists a $\delta > 0$ so that for solutions with $|x(0) - c| < \delta$, we have $|x(t) - c| < \varepsilon$ for all $t > 0$.

· c is an *unstable* equilibrium if it is not stable.

· c is an *asymptotically stable* equilibrium solution if it's stable and in addition, if $x(0)$ is close enough to c , then $\lim_{t \to \infty} x(t) = c$.

Precisely there exists a $\delta > 0$ so that if $|x(0) - c| < \delta$ then $\lim_{t \to \infty} x(t) = c$.

Notice that if c is asymptotically stable, then the horizontal line $x = c$ will be an *asymptote* to nearby solution graphs $x = x(t)$)

<u>Exercise 2:</u> Use phase diagram analysis to guess the stability of the equilibrium solutions in Exercise 1. For (a) you've worked out a solution formula already, so you'll know you're right. For (b), (c), use the Theorem on the next page to justify your answers.

2a) $x'(t) = 3x - x^2$

2b) $x'(t) = x^3 + 2x^2 + x$

2c) $x'(t) = \sin(x)$.

Theorem: Consider the autonomous differential equation

$$x'(t) = f(x)$$

with $f(x)$ and $\frac{\partial}{\partial x} f(x)$ continuous (so local existence and uniqueness theorems hold). Let $f(c) = 0$, i.e. $x(t) \equiv c$ is an equilibrium solution.

Suppose c is an *isolated zero* of f, i.e. there is an open interval containing c so that c is the only zero of f in that interval. The the stability of the equilibrium solution c can is completely determined by the local phase diagrams:

$$sign(f): \ ----0+++ \quad \Rightarrow \quad \leftarrow\leftarrow\leftarrow c \rightarrow\rightarrow\rightarrow \quad \Rightarrow \quad c \text{ is unstable}$$

$$sign(f): \ +++0---- \quad \Rightarrow \quad \rightarrow\rightarrow\rightarrow c \leftarrow\leftarrow\leftarrow \quad \Rightarrow \quad c \text{ is asymptotically stable}$$

$$sign(f): \ +++0+++ \quad \Rightarrow \quad \rightarrow\rightarrow\rightarrow c \rightarrow\rightarrow\rightarrow \quad \Rightarrow \quad c \text{ is unstable (half stable)}$$

$$sign(f): ----0---- \quad \Rightarrow \quad \leftarrow\leftarrow\leftarrow c \leftarrow\leftarrow\leftarrow \quad \Rightarrow \quad c \text{ is unstable (half stable)}$$

We can actually prove this Theorem with calculus!! (want to try?)

Constructing many solutions from one ....

Exercise 3) Use the chain rule to check that if $x(t)$ solves the autonomous DE

$$x'(t) = f(x)$$

Then $X(t) := x(t - c)$ solves the same DE. What does this say about the geometry of representative solution graphs to autonomous DEs? Have we already noticed this?

Exercise 4) Use the chain rule to check that if $x(t)$ solves the autonomous DE

$$x'(t) = f(x)$$

Then $X(t) := x(-t)$ solves

$$X'(t) = -f(X)$$

Application of Exercise 4: Understanding the "doomsday-extinction" model via the logistic model. With different hypotheses about fertility and mortality rates, one can arrive at a population model which looks like logistic, except the right hand side is the opposite of what it was in that case:

Logistic:	$P'(t) = -a P^2 + b P$
Doomsday-extinction:	$Q'(t) = a Q^2 - b Q$

For example, suppose that the chances of procreation are proportional to population density (think alligators or insect plagues), i.e. the fertility rate $\beta = a\,Q(t)$, where $Q(t)$ is the population at time t. Suppose the morbidity rate is constant, $\delta = b$. With these assumptions the birth and death rates are $a\,Q^2$ and $-b\,Q$ .... which yields the DE above. In this case factor the right side:

$$Q'(t) = a\,Q\left(Q - \frac{b}{a}\right) = k\,Q(Q - M).$$

Exercise 5) Construct the phase diagram for the general doomsday-extinction model and discuss the stability of the equilbrium solutions.

If $P(t)$ solves the logistic differential equation

$$P'(t) = k\,P(M - P)$$

we deduce from exercise 4 that $Q(t) := P(-t)$ solves the doomsday-extinction differential equation

$$Q'(t) = k\,Q(Q - M)\,.$$

We can use this fact to recover a formula for solutions to doomsday-extinction IVPs. What does this say about how representative solution graphs are related, for the logistic and the doomsday-extinction models? The solution to the logistic IVP is

$$P(t) = \frac{MP_0}{\left(M - P_0\right)e^{-Mkt} + P_0}\,.$$

So for the doomsday-extinction IVP

$$Q'(t) = k\,Q(Q - M)$$
$$Q(0) = Q_0$$

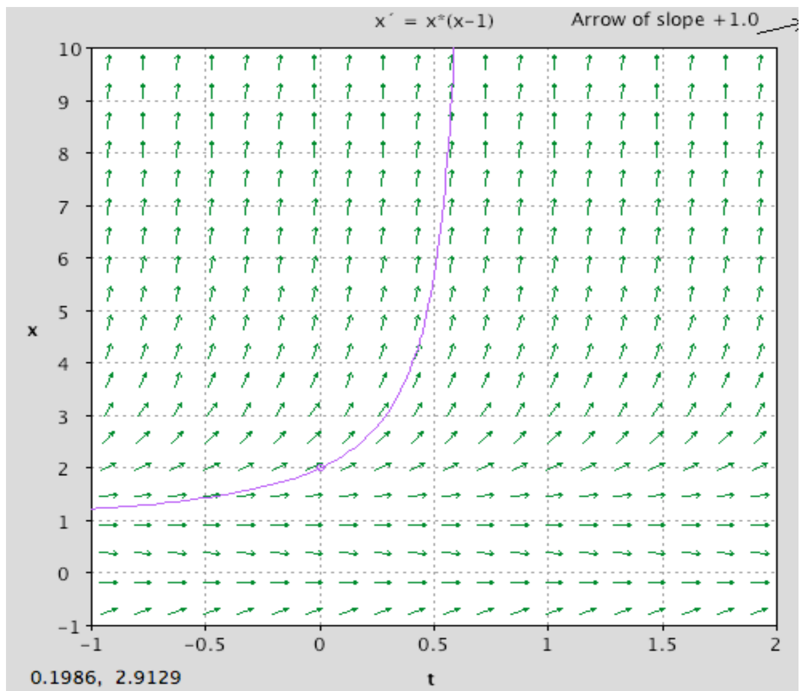
the solution is

$$Q(t) = P(-t) = \frac{MQ_0}{\left(M - Q_0\right)e^{Mkt} + Q_0}$$

Example: We can use the formula on the previous page or work from scratch using partial fractions, to write down the solution to the doomsday-extinction IVP

$$x'(t) = x(x-1)$$
$$x(0) = 2 .$$

Does the solution exist for all $t > 0$? (Hint: no, there is a very bad doomsday at $t = \ln 2$.)



Math 2250-004
Friday Jan 26
2.2 - 2.3

- harvesting a logistic population (2.2)
- improved velocity models (2.3)

Announcements:

Warm-up Exercise:

2.2: <u>Further application of phase-portrait analysis:</u> harvesting a logistic population...text p.97 (or, why do fisheries sometimes seem to die out "suddenly"?) Consider the DE

$$P'(t) = a\,P - b\,P^2 - h\,.$$

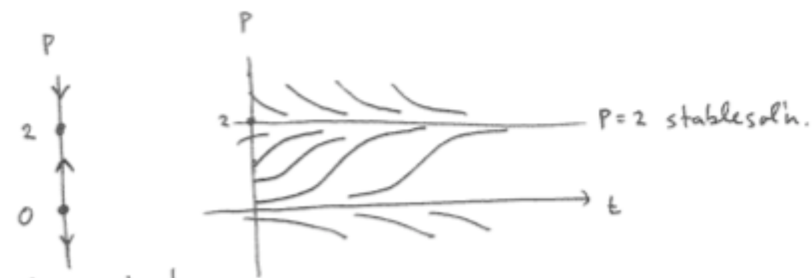
Notice that the first two terms represent a logistic rate of change, but we are now harvesting the population at a rate of h units per time. For simplicity we'll assume we're harvesting fish per year (or thousands of fish per year etc.) One could model different situations, e.g. constant "effort" harvesting, in which the effect on how fast the population was changing could be $h\,P$ instead of P.

For computational ease we will assume $a = 2$, $b = 1$. (One could actually change units of population and time to reduce to this case.)

for computational simplicity take $a=2$, $b=1$

<u>Case 0</u> no harvesting

$$P'(t) = 2P - P^2 = P(2-P)$$



phase portrait.

if $P_0>0$, $P(t) \to 2$.
(units could be millions of fish, e.g.)

with harvesting:

$$P'(t) = 2P - P^2 - h = -(P^2 - 2P + h) = -(P-P_1)(P-P_2)$$

$$P_1, P_2 = \frac{2 \pm \sqrt{4-4h}}{2} = 1 \pm \sqrt{1-h}$$

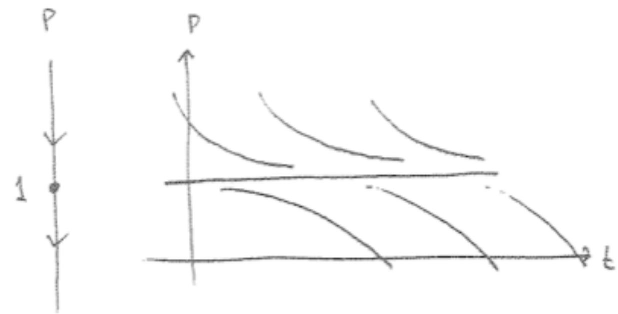
<u>Case 1</u>: subcritical harvesting

$0 < h < 1$

Case 2: Critical harvesting

$h = 1$

$P'(t) = -(P-1)^2$



Case 3 Over harvesting

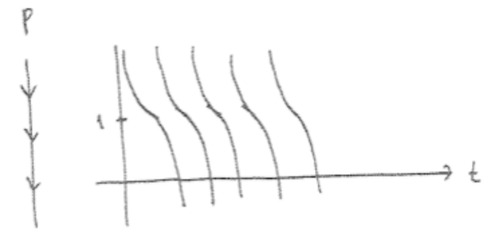
$h > 1$

complex roots.

$P'(t) = -(P^2 - 2P + h)$

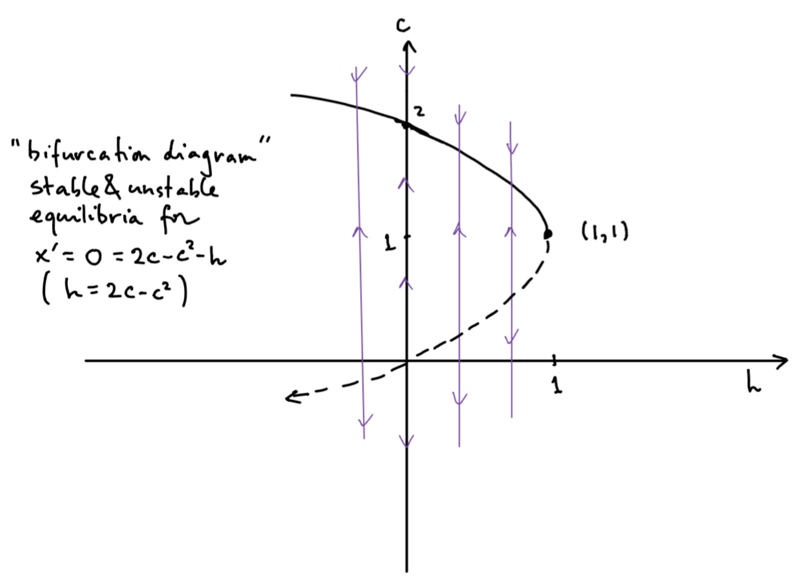
$= -[(P-1)^2 + (h-1)]$

$< 0.$



This model gives a plausible explanation for why many fisheries have "unexpectedly" collapsed in modern history. If $h < 1$ but near 1 and something perturbs the system a little bit (a bad winter, or a slight increase in fishing pressure), then the population and/or model could suddenly shift so that $P(t) \to 0$ very quickly.

Here's one picture that summarizes all the cases - you can think of it as collection of the phase diagrams for different fishing pressures h. The upper half of the parabola represents the stable equilibria, and the lower half represents the unstable equilibria. Diagrams like this are called "bifurcation diagrams". In the sketch below, the point on the h- axis should be labeled $h = 1$, not h. What's shown is the parabola of equilibrium solutions, $c = 1 \pm \sqrt{1-h}$, i.e. $2c - c^2 - h = 0$, i.e. $h = c(2-c)$.



2.3 Improved velocity models: velocity-dependent drag forces

For particle motion along a line, with

position $x(t)$ (or $y(t)$) ,
velocity $x'(t) = v(t)$, and
acceleration $x''(t) = v'(t) = a(t)$

We have Newton's 2^{nd} law

$$m\, v'(t) = F$$

where F is the net force.

- We're very familiar with constant force $F = m\,\alpha$, where α is a constant:

$$v'(t) = \alpha$$
$$v(t) = \alpha\, t + v_0$$
$$x(t) = \frac{1}{2}\alpha\, t^2 + v_0\, t + x_0 \, .$$

Examples we've seen a lot of:

- $\alpha = -g$ near the surface of the earth, if up is the positive direction, or $\alpha = g$ if down is the positive direction.
- boats or cars or "particles" subject to constant acceleration or deceleration.

New today !!! Combine a constant force with a velocity-dependent drag force, at the same time. The text calls this a "resistance" force:

$$m\, v'(t) = m\,\alpha + F_R$$

Empirically/mathematically the resistance forces F_R depend on velocity, in such a way that their magnitude is

$$|F_R| \approx k\,|v|^p \; , 1 \le p \le 2 \, .$$

- $p = 1$ (linear model, drag proportional to velocity):

$$m\,v'(t) = m\,\alpha\ - k\,v$$

This linear model makes sense for "slow" velocities, as a linearization of the frictional force function, assuming that the force function is differentiable with respect to velocity...recall Taylor series for how the velocity resistance force might depend on velocity:

$$F_R(v) = F_R(0) + F_R{}'(0)\,v + \frac{1}{2!}F_R{}''(0)v^2 + \ldots$$

$F_R(0) = 0$ and for small enough v the higher order terms might be negligable compared to the linear term, so

$$F_R(v) \approx F_R{}'(0)\,v \approx -k\,v\ .$$

We write $-k\,v$ with $k > 0$, since the frictional force opposes the direction of motion, so sign opposite of the velocity's.

http://en.wikipedia.org/wiki/Drag_(physics)#Very_low_Reynolds_numbers:_Stokes.27_drag

Exercise 1: Rewrite the linear drag model as

$$v'(t) = \alpha - \rho\,v$$

where the $\rho = \frac{k}{m}$. Construct the phase diagram for v . Notice that $v(t)$ has exactly one constant (equilibrium) solution, and find it. Its value is called the *terminal velocity*. Explain why *terminal velocity* is an appropriate term of art, based on your phase diagram for velocity.

(This is a constant coefficient linear DE, so it's easy to solve. We'll do that in detail on Monday, and then we'll antidifferentiate $v(t)$ to find the position function $x(t)$. The text also works this out in detail.)

• $p = 2$, for the power in the resistance force. This can be an appropriate model for velocities which are not "near" zero....described in terms of "Reynolds number". Accounting for the fact that the resistance opposes direction of motion we get

$$m\,v'(t) = m\,\alpha - k\,v^2 \quad \text{if } v > 0$$
$$m\,v'(t) = m\,\alpha + k\,v^2 \quad \text{if } v < 0.$$

http://en.wikipedia.org/wiki/Drag_(physics)#Drag_at_high_velocity

Exercise 2) Once again letting $\rho = \frac{k}{m}$ we can rewrite the DE's as

$$v'(t) = \alpha - \rho\,v^2 \quad \text{if } v > 0$$
$$v'(t) = \alpha + \rho\,v^2 \quad \text{if } v < 0.$$

Consider the case in which $\alpha = -g$, so we are considering vertical motion, with up being the positive direction. Draw the phase diagrams. Note that each diagram contains a half line of v-values. Make conclusions about velocity behavior in case $v_0 > 0$ and $v_0 \leq 0$. Is there a terminal velocity?

(We'll get two different nonlinear but separable DE's when we try to find the velocity in these two cases. They're messier than the linear drag case, and then it's also messier to antidifferentiate the velocity functions to recover the position functions.)