Syllabus for Math 2250-004 Differential Equations and Linear Algebra

Spring 2018

Instructor Professor Nick Korevaar
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office LCB 204, 801.581.7318
office hours M 2:00-3:00 p.m. LCB 204, T 4:30-6:00 location TBA, and by appointment. Also available after class (briefly).
Lecture MTWF 10:45-11:35 a.m. MWF in WEB L105, T in JWB 335, don't forget!

Laboratory sections with Dihan Dai, dai@math.utah.edu
2250-005 H 10:45-11:35 a.m. LCB 219
2250-006 H 9:40-10:30 a.m. AEB 310
with Jose Yanez, yanez@math.utah.edu
2250-015 H 10:45-11:35 a.m. JWB 308
2250-016 H 9:40-10:30 a.m. JWB 308

Course websites
Daily lecture notes and weekly homework assignments will be posted on our public home page.
http://www.math.utah.edu/∼korevaar/2250spring18
There are blank spaces in the notes where we will work out examples and fill in details together. Research has shown that class attendance with active participation - including problem solving and writing notes by hand - are effective ways to learn class material, for almost everyone. Passively watching a lecture is not usually effective. Class notes will be posted at least several days before we use them, so that you have ample time to print them out. Printing for math classes is free in the Math Department Rushing Student Center, in the basement of LCB. There will often be additional class discussion related to homework and lab problems.
Grades and exam material will be posted on our CANVAS course page; access via Campus Information Systems.
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Textbook *Linear Algebra and Differential Equations: with Introductory Partial Differential Equations*, by Edwards, Penney, and Haberman, a custom-printed textbook for the University of Utah: (ISBN: 13-9781269425575)
This text is a hybrid of the three texts: Differential Equations and Linear Algebra 3rd Edition, by Edwards and Penney; Applied Partial Differential Equations with Fourier Series and Boundary Value Problems, 5th edition, by Haberman; Elementary Linear Algebra, by Edwards and Penney. You should buy this version of the text if you plan to take the 4th semester in the new engineering math sequence, Math 3140, or the PDE course Math 3150, soon.
If your math courses will terminate with Math 2250, then the 3rd or 4th editions of the Differential Equations and Linear Algebra text by Edwards-Penney will suffice:
Differential Equations and Linear Algebra (4th Edition) (978-0134497181) brand new this year;
Differential Equations and Linear Algebra (3rd Edition) (978-0136054252) will go out of print at some point.

Final Exam logistics: ~~Thursday~~ Friday April 27, 10:30 a.m.-12:30 p.m., in our MWF classroom WEB L105. This is the University scheduled time and location.

Catalog description for Math 2250: This is a hybrid course which teaches the allied subjects of linear algebra and differential equations. These topics underpin the mathematics required for most students in the Colleges of Science, Engineering, Mines & Earth Science.

Prerequisites: Math 1210-1220 or 1310-1320 (or 1250-1260 or 1311-1321, i.e. single-variable calculus.) You are expected to have learned about vectors and parametric curves in one of these courses, or in Math 2210 or or Physics 2210 or 3210. Practically speaking, you are better prepared for this course if you've had elements of multivariable calculus in courses such as 1320, 1321, or 2210 and if your grades in the prerequisite courses were above the "C" level.

Grading – no worse than 90/80/70 scale

Math 2250-004 is graded on a curve. **note:** In order to receive a grade of at least "C" in the course you must earn a grade of at least "C" on the final exam. To see historical distributions of grades in my Math 2250 classes you can look at my old course home pages, on the exam pages.

Details about the content of each assignment type, and how much they count towards your final grade are as follows:

- Homework (10%): Homework from roughly three textbook sections is due every Wednesday at the beginning of class, based on lecture sections covered through the preceding Tuesday. All assignments will be posted on our public page, the Wednesday before they are due. Several problems per section will be randomly selected for grading. Two of a student's lowest homework scores will be dropped. Only hard-copy assignments will be accepted—no digital copies—and no late homework will be accepted.

- Quizzes (10%): At the end of most Wednesday classes, a short 1-2 problem quiz will be given, taking roughly 10 minutes to do. The quiz will cover relevant topics from the week's lectures, homework, and lab section work. Two of a student's lowest quiz scores will be dropped. There are no makeup quizzes. You will be allowed and encouraged to work together on these quizzes.

- Midterm exams (30%): Two class-length midterm exams will be given, On Friday February 16 and Friday March 30. Review for the exams will occur either in lecture or in the lab sections, in the days before the exams. No midterm scores are dropped.

- Final exam (30%): A two-hour comprehensive exam will be given at the end of the semester. As with the midterms, a practice final will be posted. Please check the final exam time, which is the official University scheduled time. It is your responsibility to make yourself available for that time, so make any arrangements (e.g., with your employer) as early as possible.

- Lab (20%): Every week a Teaching Assistant (TA) -directed lab section will be held. In lab, the TA will hand out problem worksheets and will facilitate student-led group work. The worksheet problems will provide guided practice with both basic methods, as well as longer in-depth problems with physical and engineering applications. Completed lab assignments will be due at the start of the following week's lab class. Credit will be given for both lab attendance and completed worksheets. Students should expect that worksheets will take additional time outside of lab to finish completely. None of the lab worksheet grades will be dropped. The TA's will be available for additional office hours.

Strategies for success

- Attend class and lab regularly.

- Read or skim the relevant text book sections and lecture note outlines *before* you attend class.

- Ask questions and become involved.

- Plan to do homework daily; try homework on the same day that the material is covered in lecture; do not wait until just before homework and lab reports are due to begin serious work.

- Form study groups with other students.

Students with disabilities

The University of Utah seeks to provide equal access to its programs, services and activities for people with disabilities. If you will need accommodations in the class, reasonable prior notice needs to be given to the Center for Disability Services, 162 Olpin Union Building, 581-5020 (V/TDD). CDS will work with you and the instructor to make arrangements for accommodations. All information in this course can be made available in alternative format with prior notification to the Center for Disability Services.

Learning Objectives for 2250

The goal of Math 2250 is to master the basic tools and problem solving techniques important in differential equations and linear algebra. These basic tools and problem solving skills are described below.

The essential topics

Be able to model dynamical systems that arise in science and engineering, by using general principles to derive the governing differential equations or systems of differential equations. These principles include linearization, compartmental analysis, Newton's laws, conservation of energy and Kirchoff's law.

Learn solution techniques for first order separable and linear differential equations. Solve initial value problems in these cases, with applications to problems in science and engineering. Understand how to approximate solutions even when exact formulas do not exist. Visualize solution graphs and numerical approximations to initial value problems via slope fields. Understand phase diagram analysis for autonomous first order differential equations.

Become fluent in matrix algebra techniques, in order to be able to compute the solution space to linear systems and understand its structure; by hand for small problems and with technology for large problems.

Be able to use the basic concepts of linear algebra such as linear combinations, span, independence, basis and dimension, to understand the solution space to linear equations, linear differential equations, and linear systems of differential equations.

Understand the natural initial value problems for first order systems of differential equations, and how they encompass the natural initial value problems for higher order differential equations and general systems of differential equations.

Learn how to solve constant coefficient linear differential equations via superposition, particular solutions, and homogeneous solutions found via characteristic equation analysis. Apply these techniques to understand the solutions to the basic unforced and forced mechanical and electrical oscillation problems.

Learn how to use Laplace transform techniques to solve linear differential equations, with an emphasis on the initial value problems of mechanical systems, electrical circuits, and related problems.

Be able to find eigenvalues and eigenvectors for square matrices. Apply these matrix algebra concepts to find the general solution space to first and second order constant coefficient homogeneous linear systems of differential equations, especially those arising from compartmental analysis and mechanical systems.

Understand and be able to use linearization as a technique to understand the behavior of nonlinear dynamical systems near equilibrium solutions. Apply these techniques to non-linear mechanical oscillation problems. (Additional material, subject to time availability: Apply linearization to autonomous systems of two first order differential equations, including interacting populations. Relate the phase portraits of non-linear systems near equilibria to the linearized data, in particular to understand stability.)

Develop your ability to communicate modeling and mathematical explanations and solutions, using technology and software such as Maple, Matlab or internet-based tools as appropriate.

Problem solving fluency

Students will be able to read and understand problem descriptions, then be able to formulate equations modeling the problem usually by applying geometric or physical principles. Solving a problem often requires specific solution methods listed above. Students will be able to select the appropriate operations, execute them accurately, and interpret the results using numerical and graphical computational aids.

Students will also gain experience with problem solving in groups. Students should be able to effectively transform problem objectives into appropriate problem solving methods through collaborative discussion. Students will also learn how to articulate questions effectively with both the instructor and TA, and be able to effectively convey how problem solutions meet the problem objectives.

Week-by-Week Topics Plan

Topic schedule is subject to slight modifications as the course progresses, but exam dates are fixed.

Week 1: 1.1-1.4; differential equations, mathematical models, integral as general and particular solutions, slope fields, separable differential equations.

Week 2: 1.4-1.5, EP 3.7, 2.1-2.2; separable equations cont., linear differential equations, circuits, mixture models, population models,equilibrium solutions and stability.

Week 3: 2.2-2.4; equilibrium solutions and stability cont., acceleration-velocity models, numerical solutions.

Week 4: 2.5-2.6, 3.1; numerical solutions cont., linear systems.

Week 5: 3.1-3.4; linear systems, matrices, Gaussian elimination, reduced row echelon form, matrix operations.

Week 6: 3.5-3.6; matrix inverses, determinants, review; **Midterm exam 1 on Friday February 16** covering material from weeks 1-6.

Week 7: 4.1-4.3; vector spaces, linear combinations in $\mathbb{R}^n$, span and independence, subspaces, bases and dimension.

Week 8: 4.4, 5.1-5.3; second-order linear DEs, general solutions, superposition, homogeneity and constant coefficients.

Week 9: 5.3-5.5; mechanical vibrations, pendulum model, particular solutions to non-homogeneous problems.

Week 10: 5.5-5.6, EP 3.7; forced oscillations and associated physical phenomena.practical resonance Laplace transforms, solving IVPs with transforms, partial fractions and translations.

Week 11: 10.1-3; Laplace transforms, solving IVPs with transforms, partial fractions and translations. **Midterm exam 2 on Friday March 30** covering material from weeks 7-11.

Week 12: 10.4-10.5, EP 7.6, 6.1-6.2 Unit steps, convolutions, impulse function forcing; eigenvalues, eigenvectors and diagonalizability.

Week 13: 6.1-6.2 continued; 7.1-7.3; first order systems of differential equations; framework for differential equations in which every DE is equivalent to a first order system of DE's. Matrix systems of DEs

Week 14: 7.3-7.4; solution algorithms and applications for first and second order systems of differential equations; input-output modeling and mechanical systems.

Week 15: 7.3-7.4 continued, and review. **Final exam Friday April 27, 10:30 a.m. - 12:30 p.m. in classroom WEB L105. This is the University scheduled time.**

• Pick up syllabus & copy of 1^{st} week outline
• I expect our public page to be up some time this evening
• " " " CANVAS " " " later this week.

Math 2250-004 Week 1 notes

We will not necessarily finish the material from a given day's notes on that day. Or on an amazing day we may get farther than I've predicted. We may also add or subtract some material as the week progresses, but these notes represent an outline of what we will cover. These week 1 notes are for sections 1.1-1.3, with hints of 1.4.

Monday January 8:

- Course Introduction ✓
- 1.1: differential equations and initial value problems

also touching on

- 1.3: slope fields for first order differential equations

• Go over course information on syllabus and course homepage:

http://www.math.utah.edu/~korevaar/2250spring18

• Note that there is a quiz this Wednesday on the material we cover today and tomorrow, and that your first lab meeting is this Thursday. Your first homework assignment will be due next Wednesday, January 17.

Then, let's begin!

• What is an n^{th} order differential equation (DE)?

any equation involving a function $y = y(x)$ and its derivatives, for which the highest order *derivative appearing in the equation is the n^{th} one, $y^{(n)}(x)$; i.e. any equation which can be written as*

$$F\left(x, y(x), y'(x), y''(x), ...y^{(n)}(x)\right) = 0.$$

Exercise 1: Which of the following are differential equations? For each DE determine the order.

a) For $y = y(x)$, $(y''(x))^2 + \sin(y(x)) = 0$ yes! 2^{nd} order

b) For $x = x(t)$, $x'(t) = 3\,x(t)(10 - x(t))$. yes! 1^{st} order

means same

c) For $x = x(t)$, $x' = 3\,x\,(10 - x)$. yes!

d) For $z = z(r)$, $z'''(r) + 4\,z(r)$. no! not an equation (no "=" sign)

e) For $y = y(x)$, $y' = y^2$. yes! 1^{st} order

short for $y'(x) = (y(x))^2$

f) $y' = y^{(2)}$ for $y(x)$ yes! 2^{nd} order!!

Definition: *A* <u>*solution*</u> *function* $y(x)$ *to a first order differential equation* $F(x, y, y') = 0$ *on the interval* I *is any function* $y(x)$ *which, when substituted into the differential equation, yields a true identity.*

Definition: *If the solution function also satisifies* $y(x_0) = y_0$ *for a specified* $x_0 \in I$ *and* $y_0 \in \mathbb{R}$*, then* $y(x)$ *is called a* <u>*solution to the initial value problem*</u> *(IVP)*

$$\text{IVP} \begin{cases} F(x, y, y') = 0 \\ y(x_0) = y_0 \end{cases} .$$

<u>Exercise 2</u>: Consider the differential equation $\frac{dy}{dx} = y^2$ from (1e).

DE $y'(x) = (y(x))^2$

<u>2a</u>) Show that functions $y(x) = \frac{1}{C - x}$ solve the DE (on any interval not containing the constant C).

plug $y(x)$ into the DE
does it yield a true identity.

$$\underset{\text{LHS}}{y'(x)} = \underset{\text{RHS}}{(y(x))^2}$$

we got lucky!
$y(x) = x^2$
LHS $y'(x) = 2x$
RHS $y^2(x) = x^4$
LHS $\neq$ RHS not a soltn.

if $y(x) = \frac{1}{C-x}$

$\Rightarrow$ LHS $y'(x) = (C-x)^{-2} = \frac{1}{(C-x)^2}$

$\Rightarrow$ RHS $= y(x)^2 = \frac{1}{(C-x)^2}$

[$y(x) = (C-x)^{-1}$
$\Rightarrow y'(x) = -1 \cdot (C-x)^{-2}(-1)$

Since LHS = RHS, $y(x)$ solves the DE.

<u>2b</u>) Find the appropriate value of C in the collection of solution functions $y(x) = \frac{1}{C - x}$ in order to solve the initial value problem

$$\text{IVP} \begin{cases} y' = y^2 \\ y(1) = 2 . \end{cases}$$

if $y(x) = \frac{1}{C-x}$

$y(1) = \frac{1}{C-1} \overset{\text{want}}{=} 2 \Rightarrow C = 1.5$

$\left(\frac{1}{C-1} = \frac{1}{.5} = 2 \checkmark \right)$

soltn to IVP is

$y(x) = \frac{1}{1.5 - x}$

$$\begin{cases} y' = y^2 \\ y(1) = 2 \end{cases}$$

2c) What is the largest interval on which your solution to (b) is defined as a differentiable function? Why?

$y(1)=2$

$(-\infty, 1.5)$

$-\infty < x < 1.5$

2d) Do you expect that there are any other solutions to the IVP in 2b? Hint: The graph of the IVP solution function we found is superimposed onto a "slope field" below: The line segments at points (x, y) have values y^2, because solutions graphs to the differential equation

$$y' = y^2$$

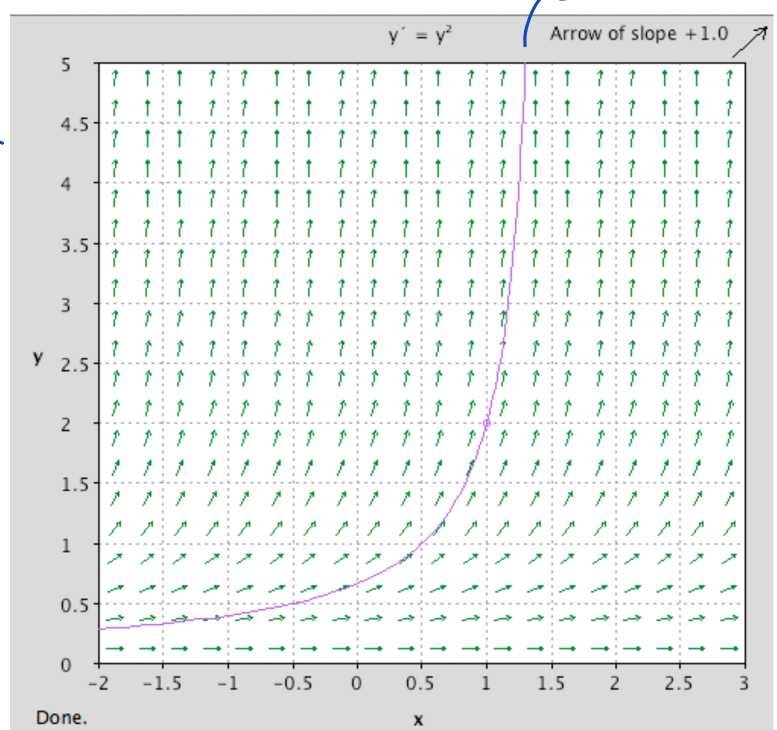
will have slopes given by the derivatives of the solutions $y(x)$. This might give you some intuition about whether you expect more than one solution to the IVP.

$y = \frac{1}{1.5 - x}$

$y' = y^2$

at any point on soln graph (x, y)

the slope of graph (y') is y^2



HW text

1.1 : 4, 6, 15, 19

2e) Where did the solution functions $y(x) = \frac{1}{C - x}$ for the differential equation

$$y' = y^2$$

come from? Many of you learned how to find solutions to "separable"' differential equations in a previous Calculus class. (Section 1.4 of the text is a review of separable differential equations.) Let's see if we can use the method of substitution from integration theory to see where those solution functions came from.

Math 2250-004
Tues Jan 9

- Course introduction, continued with an example
- 1.2 Differential equations that can be solved by direct antidifferentiation

Announcements:
- Office Hours on T. usually 4:30-6:00 are 5:00-6:00 today in JWB 240
- Web page is up (Canvas later)
- Quiz tomorrow :)

Warm-up Exercise: Solve this initial value problem for position function $x(t)$

IVP $\begin{cases} x'(t) = 4\sin(2t) \\ x(0) = 0 \end{cases}$

e.g. t = time sec
x = position in meters

hint: $x(t) = \int 4\sin(2t)\,dt = -2\cos 2t + C$

('til 10:50)

$\left(\frac{d}{dt} -2\cos 2t = -2(-\sin(2t))\cdot 2 = 4\sin(2t) \checkmark \right)$

$x(0) = 0 = -2 + C$

Answer $x(t) = -2\cos 2t + 2$ ✓

(Question: What if DE was $x'(t) = 4\sin(2x)$ $\quad x = x(t)$
cannot just antidiff w.r.t. t
because RHS depends on $x(t)$, which you don't know yet.)

• **important course goals:** understand some of the key differential equations which arise in modeling real-world dynamical systems from science, mathematics, engineering; how to find the solutions to these differential equations if possible; how to understand properties of the solution functions (sometimes even without formulas for the solutions) in order to effectively model or to test models for dynamical systems.
In fact, you've encountered differential equations in previous mathematics and/or physics classes:

• 1st order differential equations: rate of change of function depends in some way on the function value, the variable value, and nothing else. For example, you've studied the population growth/decay differential equation for $P = P(t)$, and k a constant, given ~~by~~

$$P'(t) = k\,P(t)$$

and having applications in biology, physics, finance.

how fast P(t) changes

is proportional to P(t).

• 2nd order DE's: Newton's second law (change in momentum equals net forces) often leads to second order differential equations for particle position functions $x = x(t)$ in physics.

$m\,x''(t) =$ net forces

$= \underbrace{f(t)}_{\text{e.g.}} - \underbrace{k\,x(t)}_{\text{Hooke's Law}} - c\,x'(t)$ ← damped mass spring

drag coeff.

external forcing function

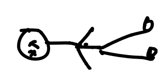
The following exercise is an illustration of how modeling and differential equations tie together...

Exercise 1) Newton's law of cooling is a model for how objects are heated or cooled by the temperature of an ambient medium surrounding them. In this model, the body temperature $T = T(t)$ changes at a rate $T'(t)$

proportional to to the difference between it (T) and the ambient temperature $A(t)$. In the simplest models A is constant.

a) Convert the description above into the differential equation for how the temperature of a heating or cooling object changes:

$$\frac{dT}{dt} = -k(T - A)\,.$$



T(t) temperature @ time t

ambient medium temperature A(t)

$T'(t) = k_1 \cdot (T(t) - A(t))$

how fast temp changes

if $T < A$
object is heating
so $T'(t) > 0$. $T - A < 0$

$T'(t) = k_1(T - A)$ so $k_1 < 0$

b) Would the differential equation have been correct if we wrote $\frac{dT}{dt} = k_1(T - A)$ instead?

Yes. (In this case, the k_1 is neg.) so let $k = -k_1$, $k > 0$

c) Find all solution functions to the Newton's law of cooling differential equation

$$\frac{dT}{dt} = -k(T - A).$$

using separation of variables or the method of substitution. const.

$$\frac{T'(t)}{T(t) - A} = -k$$

$$\int \frac{T'(t)}{T(t)-A}\,dt = \int -k\,dt$$

$$= -kt + C_1$$

let $u(t) = T(t) - A$

$du = T'(t)\,dt$

$$\int \frac{du}{u} = \ln|u| + C_2$$

so $\ln|T(t) - A| + C_2 = -kt + C_1$

$\ln|T(t) - A| = -kt + C_3$

e e

$|T(t) - A| = e^{-kt} e^{C_3}$

$|T(t) - A| = C_4 e^{-kt}$

$T(t) - A = C_5 e^{-kt}$ $C_5 = \pm C_4$

$$\boxed{T(t) = A + Ce^{-kt}}$$

makes sense as $t \to \infty$.

d) Use the Newton's law of cooling model to partially solve a murder mystery: At 3:00 p.m. a deceased body is found. Its temperature is 70 ° F. An hour later the body temperature has decreased to 60 °. It's been a winter inversion in SLC, with constant ambient temperature 30 °. Assuming the Newton's law model, estimate the time of death.

$T = \cancel{A} + Ce^{-kt}$

$T = 30 + Ce^{-kt}$

$t = 0$ is when you found body

$T(1) = 60°$

$T(0) = 70°$

$A = 30$

get out of quiz free
bring soln

or get 5 more pts on quiz

Section 1.2: differential equations equivalent to ones of the form

$$y'(x) = f(x)$$

which we solve by direct antidifferentiation

$$y(x) = \int f(x)\, dx = F(x) + C.$$

Exercise 2   Solve the initial value problem

$$\frac{dy}{dx} = x\sqrt{x^2 + 4}$$

$$y(0) = 0$$

(1.1 like warmup)

HW   rest of §1.1 : 27, 30, 33, 34.

§1.2   w/ 1.1 (antidiff)

2, 6, 7, 9, 16, 18

(maybe 24, 26, 33)

after Wed class, all of 1.1 & 1.2 homework is within your grasp, although some of the problems may be challenging.

An important class of such problems arises in physics usually as velocity/acceleration problems via Newton's second law. Recall that if a particle is moving along a number line and if $x(t)$ is the particle **position** function at time t, then the rate of change of $x(t)$ (with respect to t) namely $x'(t)$, is the **velocity** function. If we write $x'(t) = v(t)$ then the rate of change of velocity $v(t)$, namely $v'(t)$, is called the **acceleration** function $a(t)$, i.e.

$$x''(t) = v'(t) = a(t) .$$

Thus if $a(t)$ is known, e.g. from Newton's second law that force equals mass times acceleration, then one can antidifferentiate once to find velocity, and one more time to find position.

$$m x''(t) = \underbrace{\text{net forces}}_{\text{often just a fcn of time}}$$

Exercise 3:

a) If the units for position are meters m and the units for time are seconds s, what are the units for velocity and acceleration? (These are *mks* units.)

veloc. m/s $\quad v(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$ $\quad \frac{m}{s}$

accel. m/s^2 $\quad a(t) = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t}$ $\quad \frac{m/s}{s}$ $\quad \frac{m}{s^2}$

b) Same question, if we use the English system in which length is measure in feet and time in seconds. Could you convert between *mks* units and English units?

v: f/s $\qquad$ google

a: f/s^2 $\qquad$ mars probe crashes bad units

Exercise 4:
Suppose the acceleration function is a negative constant $-a$,

$$\boxed{x''(t) = -a}$$

(This could happen for vertical motion, e.g. near the earth's surface with $a = g \approx 9.8 \frac{m}{s^2} \approx 32 \frac{ft}{s^2}$, as well as in other situations such as a car breaking with constant deceleration.)
Write $x(0) = x_0$, $v(0) = v_0$ for the initial position and velocity. Find formulas for $v(t)$ and $x(t)$.

2 init conds.
anti diff $x'' = \underbrace{-a}_{\text{const.}}$ twice to get $v(t)$; then $x(t)$.

$v(t) = x'(t) = \int x''(t)\,dt = \int -a\,dt = -at + C$

$v(0) = v_0 = 0 + C \Rightarrow \boxed{v(t) = -at + v_0}$

$x(t) = \int v(t)\,dt = \int -at + v_0\,dt$

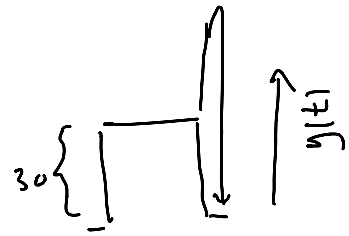
$x(t) = -\frac{1}{2}at^2 + v_0 t + C$ $\qquad x(0) = x_0 = 0 + 0 + C \Rightarrow C = x_0$

$$\boxed{x(t) = -\frac{1}{2}at^2 + v_0 t + x_0}$$

Exercise 5: A projectile with ~~very low air~~ resistance is fired almost straight up from the roof of a building 30 meters high, with initial velocity 50 m/s. Assume that the only acceleration of the object is from the force of gravity. If its initial horizontal velocity is near zero, but large enough so that the object lands on the ground rather than the roof.

- a) Neglecting friction, how high will the object get above ground?

b) When does the object land ?



$v_0 = 50$

$x_0 = 30$ (if $x=0$ is ground level).

Newtons Law $\cancel{m}y''(t) = -\cancel{m}g$ $g \approx 9.8\ m/s^2$

$y''(t) = -g = -9.8$

$$\Rightarrow \boxed{\begin{aligned} y'(t) &= -gt + 50 = -9.8t + 50 \\ y(t) &= -4.9t^2 + 50t + 30 \end{aligned}}$$

a) set $v(t) = 0$, then solve for t, plug into y.

$-9.8t + 50 = 0$ $t = 5.1$ sec.

$y(5.1) = 157.6$ m.

b) lands when $y(t) = 0$ (because we set ground level to be $y=0$).

$-4.9t^2 + 50t + 30 = 0$

$t = 10.77$ sec.

Homework: all of § 1.1 & 1.2 should be worth trying now

Math 2250-004
Wed Jan 10

- 1.3 slope fields and differential equations
- Quiz today at end of class, on section 1.1-1.2 material

Announcements:

- quiz today (hand in dead body prob. with quiz)
- Labs tomorrow!
- Finish Tuesday's notes first

Warm-up Exercise:

Solve the IVP

until ~~10:48~~ 10:50 :)

$$\begin{cases} \frac{dy}{dx} = x\sqrt{x^2+4} \\ y(1) = 2 \end{cases}$$

§1.2 $\quad y = \int x\sqrt{x^2+4}\,dx$

$u = x^2+4$
$du = 2x\,dx \qquad x\,dx = \frac{1}{2}du$

$= \int u^{1/2}\,\frac{1}{2}du$

$= \frac{1}{\cancel{2}}\frac{\cancel{2}}{3}u^{3/2} + C$

$= \frac{1}{3}(x^2+4)^{3/2} + C$

$y(1) = 2 = \frac{1}{3}(5)^{3/2} + C$

$2 - \frac{1}{3}5^{3/2} = C$

$$y(x) = \frac{(x^2+4)^{3/2}}{3} + C$$

$$C = 2 - \frac{5^{3/2}}{3} \leftarrow$$

Section 1.3: slope fields and graphs of differential equation solutions: Consider the first order DE IVP for a function $y(x)$:

$$y' = f(x, y) \ \ , y(x_0) = y_0 \, .$$

If $y(x)$ is a solution to this IVP and if we consider its graph $y = y(x)$, then the IC means the graph must pass through the point (x_0, y_0). The DE means that at every point (x, y) on the graph the slope of the graph must be $f(x, y)$. (So we often call $f(x, y)$ the "slope function" for the differential equation.) This gives a way of understanding the graph of the solution $y(x)$ even without ever actually finding a formula for $y(x)$! Consider a **slope field** near the point (x_0, y_0): at each nearby point (x, y), assign the slope given by $f(x, y)$. You can represent a slope field in a picture by using small line segments placed at representative points (x, y), with the line segments having slopes $f(x, y)$.

Exercise 1: Consider the differential equation $\boxed{\dfrac{dy}{dx} = x - 3}$ and then the IVP with $y(1) = 2$.

a) Fill in (by hand) segments with representative slopes, to get a picture of the slope field for this DE, in the rectangle $0 \le x \le 5, 0 \le y \le 6$. Notice that in this example the value of the slope field only depends on x, so that all the slopes will be the same on any vertical line (having the same x-coordinate). (In general, curves on which the slope field is constant are called **isoclines**, since "iso" means "the same" and "cline" means inclination.) Since the slopes are all zero on the vertical line for which $x = 3$, I've drawn a bunch of horizontal segments on that line in order to get started, see below.

b) Use the slope field to create a qualitatively accurate sketch for the graph of the solution to the IVP above, without resorting to a formula for the solution function $y(x)$.

c) This is a DE and IVP we can solve via antidifferentiation. Find the formula for $y(x)$ and compare its graph to your sketch in (b).

a) draw the field of slopes

"isoclines"	slope values
$x=3$	0
$x=4$	1
$x=5$	2
$x=2$	-1
$x=1$	-2
$x=0$	-3

slope fcn is $f(x,y) = x-3$

$x-3=1 \Rightarrow x=4$

$x-3=-1 \Rightarrow x=2$

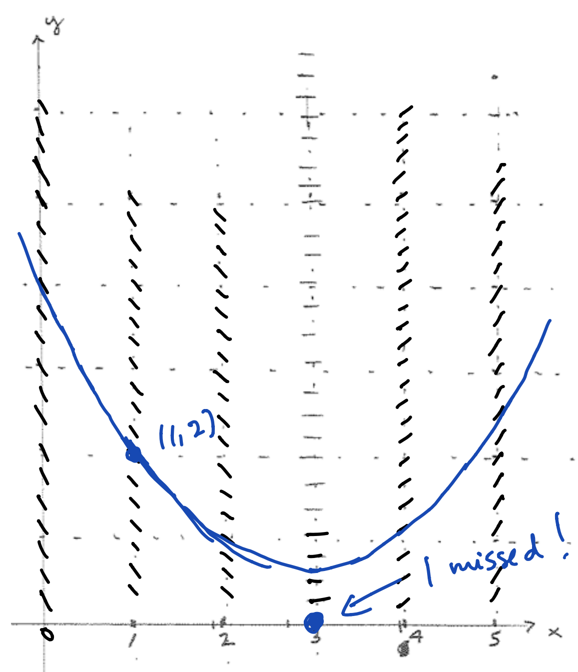
$x-3=-2 \Rightarrow x=1$

$x-3=-3 \Rightarrow x=0$

b) Sketch graph for IVP soltn.

$y(1)=2$

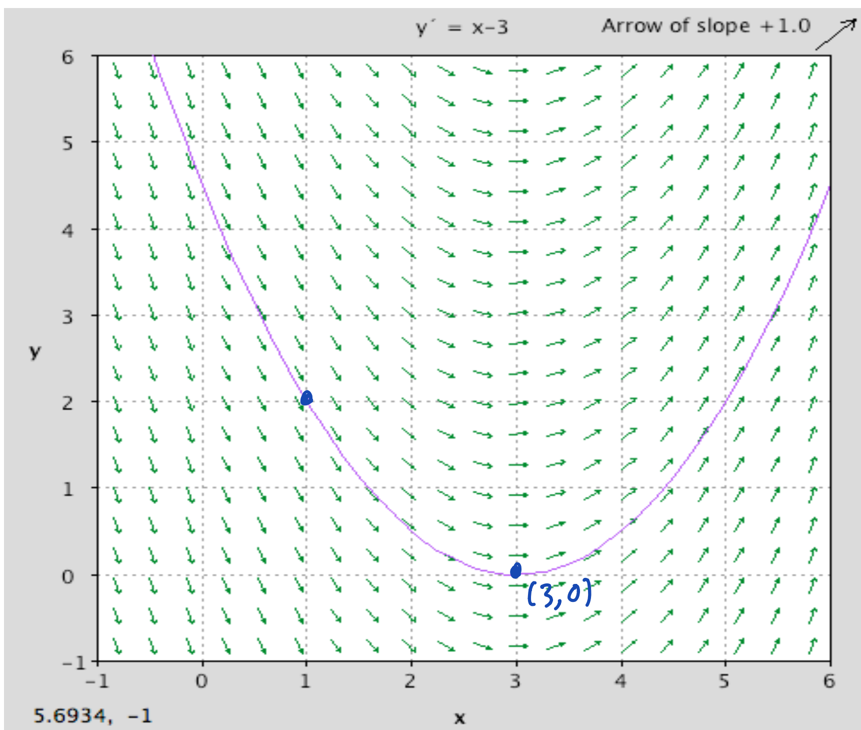
$\Rightarrow (1,2)$ is on graph



The procedure of drawing the slope field $f(x, y)$ associated to the differential equation $y'(x) = f(x, y)$ can be automated. And, by treating the slope field as essentially constant on small scales, i.e. using

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx} = f(x, y)$$

one can make discrete steps in x and y, starting from the initial point (x_0, y_0). In this way one can approximate solution functions to IVPs, and their graphs. You can find an applet to do this by googling "dfield" (stands for "direction field", which is a synonym for slope field). Here's a picture like the one we sketched by hand on the previous page. The solution graph was approximated using numerical ideas as above, and this numerical technique works for much more complicated differential equations, e.g. when solutions exist but don't have closed form formulas. The program "dfield" was originally written for Matlab, and you can download a version to run inside that package. Or, you can download stand-alone java code.



LHS ↓ RHS ↓

<u>Exercise 2</u>: Consider the IVP

$$\frac{dy}{dx} = y - x$$
$$y(0) = 0$$

a) Check that $y(x) = x + 1 + C\,e^x$ gives a family of solutions to the DE (C=const). Notice that we haven't yet discussed a method to derive these solutions, but we can certainly check whether they work or not.

This was your quiz!

for $y(x) = x + 1 + Ce^x$

LHS: $y'(x) = 1 + Ce^x$

RHS: $y(x) - x = x + 1 + Ce^x - x = 1 + Ce^x$

LHS = RHS, so fcns make DE true, so are sol'ns

b) Solve the IVP by choosing appropriate C.

$y(x) = x + 1 + Ce^x$

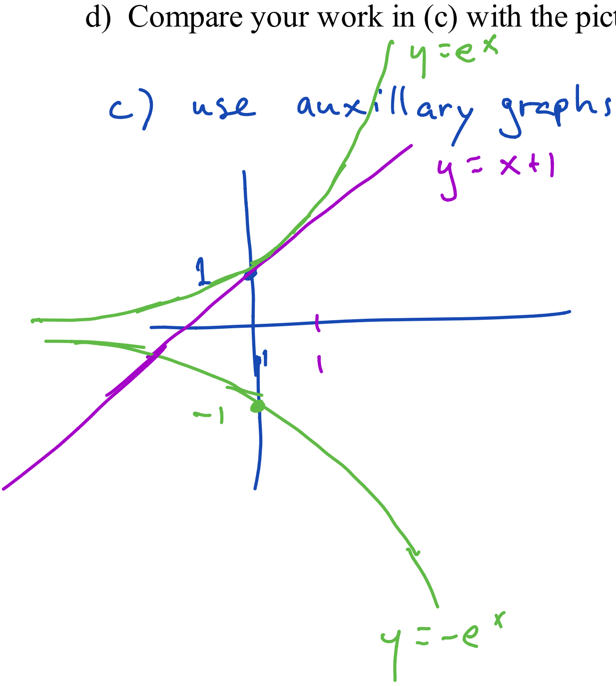
$y(0) = 0 = 1 + C \implies C = -1$

$\boxed{y(x) = x + 1 - e^x}$

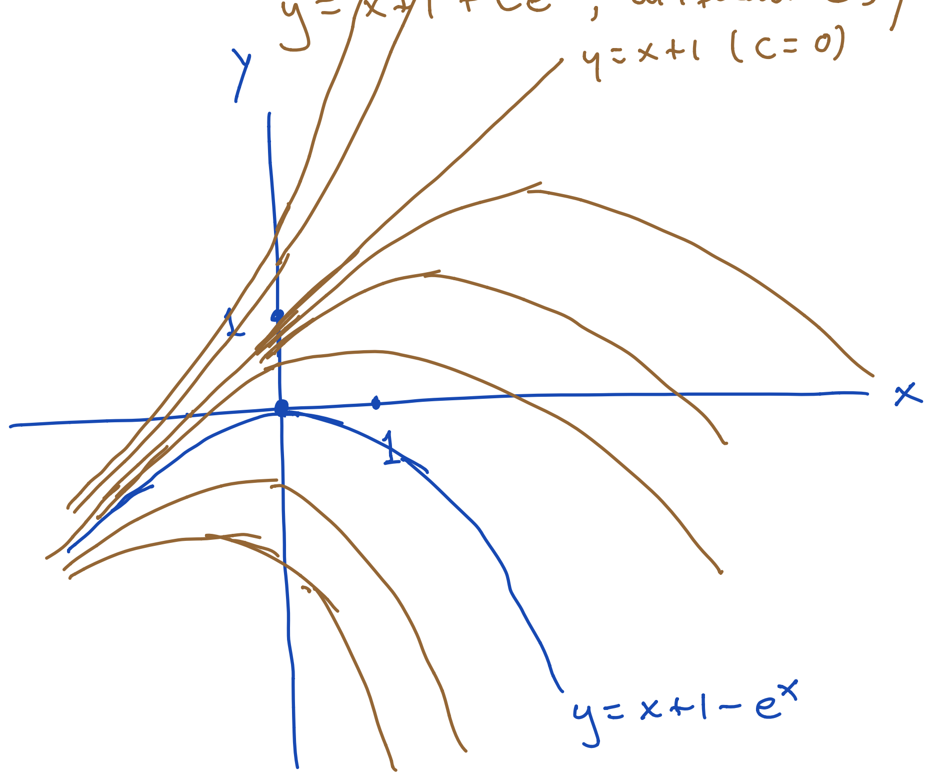
c) Sketch the solution by hand, for the rectangle $-3 \le x \le 3, -3 \le y \le 3$. Also sketch typical solutions for several different C-values. Notice that this gives you an idea of what the slope field looks like. How would you attempt to sketch the slope field by hand, if you didn't know the general solutions to the DE? What are the isoclines in this case?

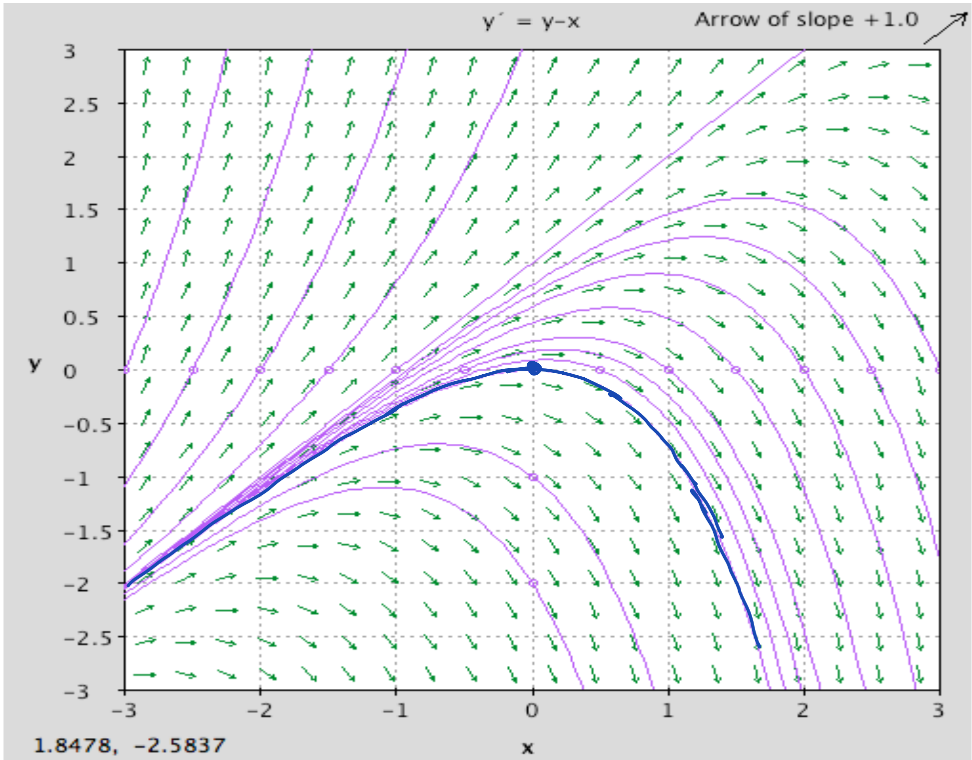
d) Compare your work in (c) with the picture created by dfield on the next page.

c) use auxillary graphs



to get graph of $y = x + 1 - e^x$ add the vertical displacements from x-axis of the component graphs (and do the same for $y = x + 1 + Ce^x$, different C's)




y´ = y-x
Arrow of slope +1.0
y
x
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−3
−3
−2
−1
0
1
2
3
1.8478, −2.5837

Math 2250-004
Fri Jan 12

- 1.3 slope fields and the existence-uniqueness theorem for initial value problems
- 1.4 separable differential equations

Announcements:
- How did labs go?
- Monday is MLK day
- next week notes posted over weekend & I'll bring copies on T.
- to get in to class: request permission code from www.math.utah.edu
  they send ticket #
  email korevaar@math.utah.edu #
- first do Wed notes...

Warm-up Exercise: a) Solve the IVP

$$\begin{cases} y'(x) = x-3 \\ y(1) = 2 \end{cases}$$

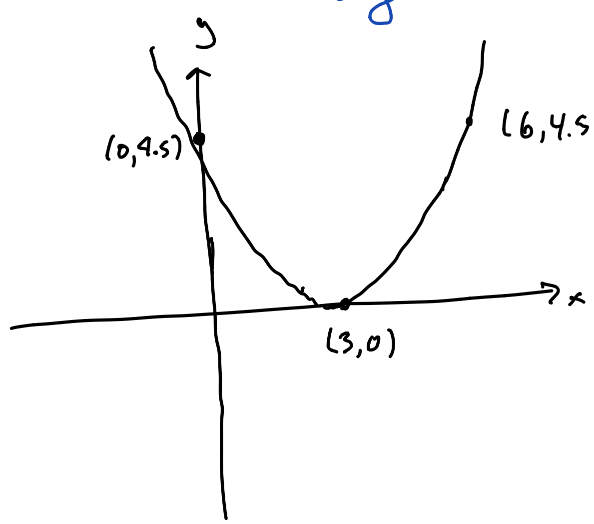
this should be straightforward.

b) sketch the solution graph using Calc or pre-calc. ($y = y(x)$)
you might be a little rusty on this

1a) Soltn $y(x) = \frac{1}{2}x^2 - 3x + 4.5$

1b) $y'(x) = x-3 = 0$ @ $x=3$
$y(3) = \frac{1}{2}9 - 9 + \frac{9}{2} = 0.$
vertex @ $(3,0)$
y-intercept (@ $x=0$) is 4.5

or $y(x) = \frac{1}{2}(x^2 - 6x + 9)$
$y = \frac{1}{2}(x-3)^2$



1.3-1.4: slope fields; existence and uniqueness for solutions to IVPs; examples we can check with separation of variables.

$y'(x) - y(x)^2 = 1$?

$\int y'(x) - y(x)^2\, dx = \int 1\, dx$

Exercise 1: Consider the differential equation

$$\frac{dy}{dx} = 1 + y^2 .$$

a) Use separation of variables to find solutions to this DE...the "magic" algorithm that we talked about at the start of the week, but didn't explain the reasoning for. It is de-mystified on the next page of today's notes.

legal

$y'(x) = 1 + y(x)^2$

$\frac{y'(x)}{1 + y(x)^2} = 1$

$\int \frac{y'(x)}{1+y(x)^2}\, dx = \int 1\, dx = x$

$u = y(x)$

$du = y'(x)\, dx$

$\int \frac{du}{1+u^2}$

$= \arctan(u) = \arctan(y(x))$

$\Rightarrow \arctan(y) = x + C$

$\tan \Rightarrow \boxed{y = \tan(x+c)}$

magic way

$\frac{dy}{dx} = 1 + y^2$

$\frac{dy}{1+y^2} = 1 \cdot dx$ magic

$\int \frac{dy}{1+y^2} = \int 1 \cdot dx$ "

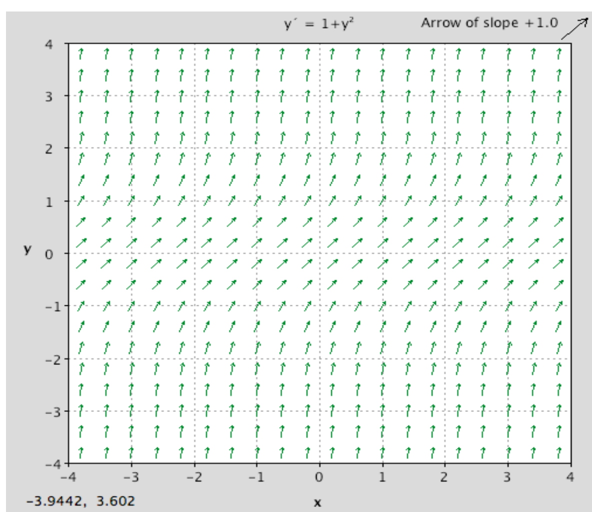
$\arctan(y) = x + C$

b) Use the slope field below to sketch some solution graphs. Are your graphs consistent with the formulas from a? (You can sketch by hand, I'll use "dfield" on my browser.)
c) Explain why each IVP has a solution, but this solution does not exist for all *x*.

You can download the java applet "dfield" from the URL

http://math.rice.edu/~dfield/dfpp.html

(You also have to download a toolkit, following the directions there.)



1.4 Separable differential applications: Important applications, as well as a lot of the examples we study in slope field discussions of section 1.3 are separable DE's. So let's discuss precisely what they are, and why the separation of variables algorithm works.

Definition: A separable first order DE for a function $y = y(x)$ is one that can be written in the form:

$$\frac{dy}{dx} = f(x)\varphi(y) .$$

It's more convenient to rewrite this DE as

$$\frac{1}{\varphi(y)}\frac{dy}{dx} = f(x), \qquad \text{(as long as } \varphi(y) \neq 0\text{)} .$$

Writing $g(y) = \dfrac{1}{\varphi(y)}$ the differential equation reads

$$g(y)\frac{dy}{dx} = f(x) .$$

Solution (math justified): The left side of the modified differential equation is short for $g(y(x))\dfrac{dy}{dx}$.

Even though we don't know the solution functions $y(x)$ yet, once we find them it will be true that they make this antidifferentiated identity true:

$$\int g(y(x))y'(x)dx = \int f(x)\, dx.$$

And if $G(y)$ is any antiderivative of $g(y)$, then this identity can be rewritten as

$$\int G'(y(x))y'(x)dx = \int f(x)\, dx.$$

By the chain rule (read backwards), the integrand on the left is nothing more than

$$\frac{d}{dx}G(y(x)).$$

So we can antidifferentiate both sides of the integral identity to get

$$G(y(x)) = \int f(x)\, dx = F(x) + C.$$

where $F(x)$ is any antiderivative of $f(x)$. Thus solutions $y(x)$ to the original differential equation satisify

$$G(y) = F(x) + C .$$

This expresses solutions $y(x)$ implicitly as functions of x. You may be able to use algebra to solve this equation explicitly for $y = y(x)$, and (working the computation backwards) $y(x)$ will be a solution to the DE. (Even if you can't algebraically solve for $y(x)$, this still yields implicitly defined solutions.)

Solution (differential magic): Treat $\frac{dy}{dx}$ as a quotient of differentials dy, dx, and multiply and divide the DE to "separate" the variables:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$
$$g(y)dy = f(x)dx\,.$$

Antidifferentiate each side with respect to its variable (?!)

$$\int g(y)dy = \int f(x)dx\,,\ \text{i.e.}$$

$G(y) + C_1 = F(x) + C_2 \Rightarrow G(y) = F(x) + C$. Agrees with "math-justified" implicit solutions.

This is the same differential magic that you used for the "method of substitution" in antidifferentiation, which was essentially the "chain rule in reverse" for integration techniques.

Exercise 2a) Use separation of variables to solve the IVP

$$\frac{dy}{dx} = y^{\left(\frac{2}{3}\right)}$$
$$y(0) = 0$$

2b) But there are actually a lot more solutions to this IVP! (Solutions which don't arise from the separation of variables algorithm are called singular solutions.) Once we find these solutions, we can figure out why separation of variables missed them.

2c) Sketch some of these singular solutions onto the slope field below.

2a) $\frac{dy}{dx} = y^{2/3}$

$y \neq 0$: $\frac{dy}{y^{2/3}} = 1 \cdot dx$

$\int \frac{dy}{y^{2/3}} = \int dx$

$\int y^{-2/3}\,dy$

$3y^{1/3} = x + C$

$y^{1/3} = \frac{x+C}{3}$

$y = \left(\frac{x+C}{3}\right)^3$

$y(0) = 0 = \left(\frac{C}{3}\right)^3 \Rightarrow C = 0$

$y(x) = \left(\frac{x}{3}\right)^3 = \frac{x^3}{27}$

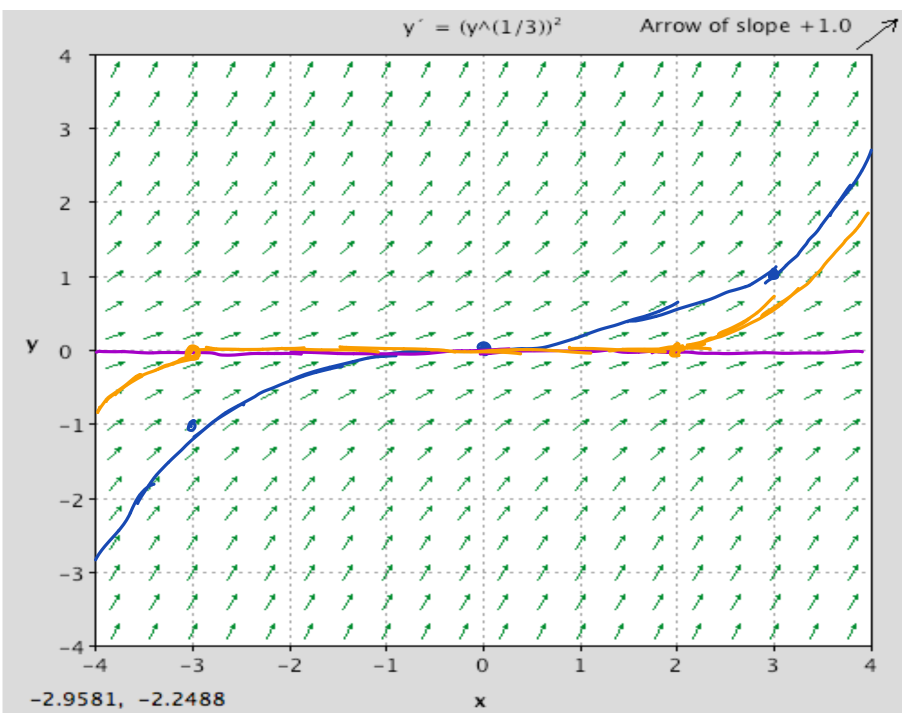
$y(3) = 1$

2b) $y(x) \equiv 0$

LHS $y' = 0$

RHS $y^{2/3} = 0$

2c) ∞'ly many soltns!!

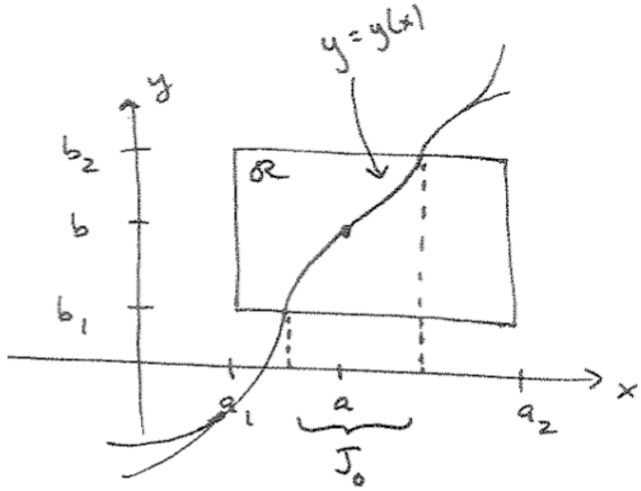


Here's what's going on (stated in 1.3 page 24 of text; partly proven in Appendix A.)
<u>Existence - uniqueness theorem for the initial value problem</u>
Consider the IVP

$$\frac{dy}{dx} = f(x, y)$$
$$y(a) = b$$

• Let the point (a, b) be interior to a coordinate rectangle $\mathcal{R} : a_1 \leq x \leq a_2, b_1 \leq y \leq b_2$ in the $x-y$ plane.
• <u>Existence:</u> If $f(x, y)$ is <u>continuous</u> in $\mathcal{R}$ (i.e. if two points in $\mathcal{R}$ are close enough, then the values of f at those two points are as close as we want). Then there exists a solution to the IVP, defined on some subinterval $J \subseteq [a_1, a_2]$.
• <u>Uniqueness:</u> If the partial derivative function $\frac{\partial}{\partial y} f(x, y)$ is also continuous in $\mathcal{R}$, then for any subinterval $a \in J_0 \subseteq J$ of x values for which the graph $y = y(x)$ lies in the rectangle, the solution is unique!

See figure below. The intuition for existence is that if the slope field $f(x, y)$ is continuous, one can follow it from the initial point to reconstruct the graph. The condition on the y-partial derivative of $f(x, y)$ turns out to prevent multiple graphs from being able to peel off.



<u>Exercise 3</u>: Discuss how the existence-uniqueness theorem is consistent with our work in Wednesday's Exercises 1-2, and in today's Exercises 1-2 where we were able to find explicit solution formulas because the differential equations were actually separable.

Wed #1 $\begin{cases} \dfrac{dy}{dx} = x + 3 \\ y(1) = 2 \end{cases}$

Wed #2 $\begin{cases} \dfrac{dy}{dx} = y - x \\ y(0) = 0 \end{cases}$

Fri #1 $\begin{cases} \dfrac{dy}{dx} = 1 + y^2 \\ y(0) = 0 \end{cases}$

Fri #2 $\begin{cases} \dfrac{dy}{dx} = y^{2/3} \\ y(0) = 0 \end{cases}$