Math 2250-004
Week 14 April 16-20: sections 7.1-7.3 first order systems of linear differential equations; 7.4 mass-spring systems.

Mon Apr 16
7.1-7.2 Systems of differential equations (7.1), and the vector Calculus we need to study them (7.2).
Every differential equation or system of differential equations can be converted into a first order system of differential equations (7.1).

Announcements:

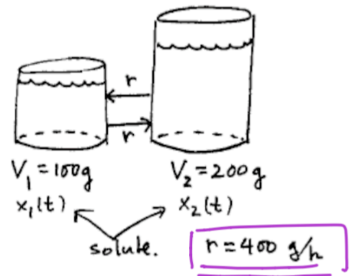
Warm-up Exercise:

Summary and continuation of Friday overview/introduction to Chapter 7, which is about systems of differential equations. We began with a specific two-tank input-output model, with the goal of tracking the vector

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

of solute amounts in each tank. The initial value problem for this tank system was of the form

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0$$



g = gallons
h = hours

Exercise 1) Find differential equations for solute amounts $x_1(t), x_2(t)$ above, using input-output modeling. Assume solute concentration is uniform in each tank. If $x_1(0) = b_1, x_2(0) = b_2$, write down the initial value problem that you expect would have a unique solution.

$$x_1'(t) = r \cdot \frac{x_2}{200} - r\frac{x_1}{100} = 400\frac{x_2}{200} - 400\frac{x_1}{100} = -4x_1 + 2x_2$$

$$x_2' = 400\frac{x_1}{100} - 400\frac{x_2}{200} = 4x_1 - 2x_2$$

IVP
$$\begin{cases} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \end{cases}$$

answer (in matrix-vector form):

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

The more general first order system of differential equations, and associated initial value problem is,

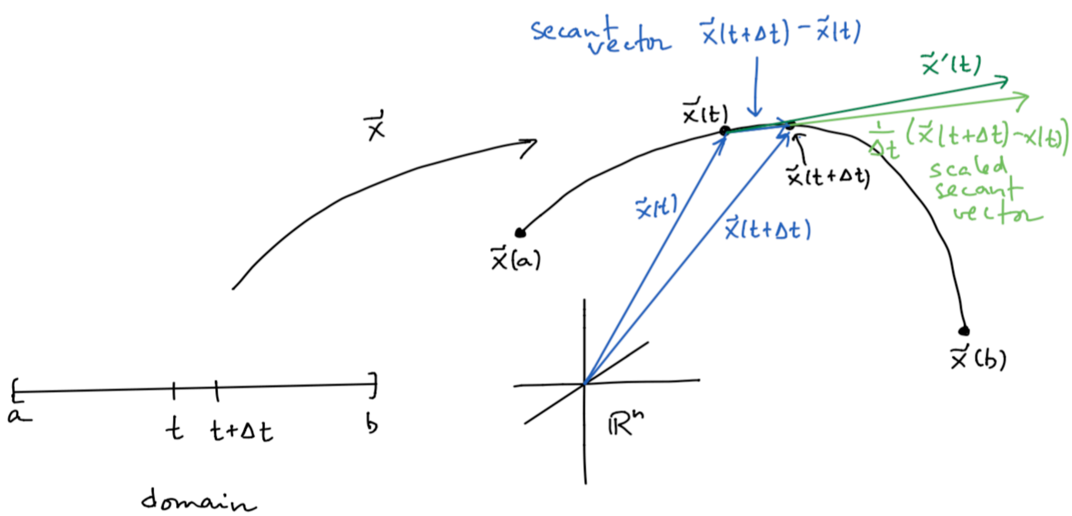
$$\underline{\boldsymbol{x}}'(t) = \underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t))$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

Existence and Uniqueness for solutions to IVP's: The above IVP is a vectorized version of the scalar first order DE IVP that we considered in Chapter 1. In Chapter 1 we understood why (with the right conditions on the right hand side), these IVP's have unique solutions. There is an analogous existence-uniqueness theorem for the vectorized version we study in Chapter 7, and it's believable for the same reasons the Chapter 1 theorem seemed reasonable. We just have to remember the geometric meaning of the *tangent* vector $\underline{\boldsymbol{x}}'(t)$ to a parametric curve in $\mathbb{R}^n$ (which is also called the *velocity* vector in physics, when you study particle motion):

Algebra:

$$\underline{\boldsymbol{x}}'(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\begin{bmatrix} x_1(t+\Delta t) \\ x_2(t+\Delta t) \\ : \\ x_n(t+\Delta t) \end{bmatrix} - \begin{bmatrix} x_1(t) \\ x_2(t) \\ : \\ x_n(t) \end{bmatrix}\right) = \lim_{\Delta t \to 0} \begin{bmatrix} \frac{1}{\Delta t}\left(x_1(t+\Delta t) - x_1(t)\right) \\ \frac{1}{\Delta t}\left(x_2(t+\Delta t) - x_2(t)\right) \\ : \\ \frac{1}{\Delta t}\left(x_n(t+\Delta t) - x_n(t)\right) \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ : \\ x_n'(t) \end{bmatrix}$$

Geometric interpretation in terms of displacement vectors along a parametric curve:



So the existence-uniquess theorem for first order systems of DE's is true because if you know where you start at time t_0, namely $\underline{\boldsymbol{x}}_0$; and if you know your tangent vector $\underline{\boldsymbol{x}}'(t)$ at every later time -in terms of your location $\underline{\boldsymbol{x}}(t)$ and what time t it is, as specified by the vector function $\underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t))$; then there should only be one way the parametric curve $\underline{\boldsymbol{x}}(t)$ can develop. This is analogous to our reasoning in Chapter 1 that there should only be one way to follow a slope field, given the initial point one starts at.

For the two-tank example, we used output from *pplane*, the sister program to *dfield*, to illustrate how solutions follow tangent vector fields, and tracked the solution to

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -4\,x_1(t) + 2\,x_2(t) \\ 4\,x_1(t) - 2\,x_2(t) \end{bmatrix} = \left(4\,x_1(t) - 2\,x_2(t)\right) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

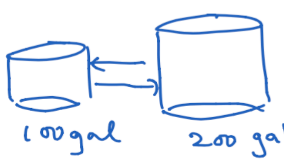
$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

We noticed that the limiting solute amounts appeared to be $\begin{bmatrix} x_1(\infty) \\ x_2(\infty) \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, which makes complete sense.

Note: This system of DE's is *autonomous*, since the formula $\underline{\boldsymbol{x}}'(t)$ only depends on the value of $\underline{\boldsymbol{x}}$ and not on the value of *t*, so the tangent field is not changing in time; the pplane *phase portrait* is analgous to the *phase diagram* lines we drew for autonomous first order differential equations in Chapter 2.

$$\lim_{t\to\infty} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

2c)

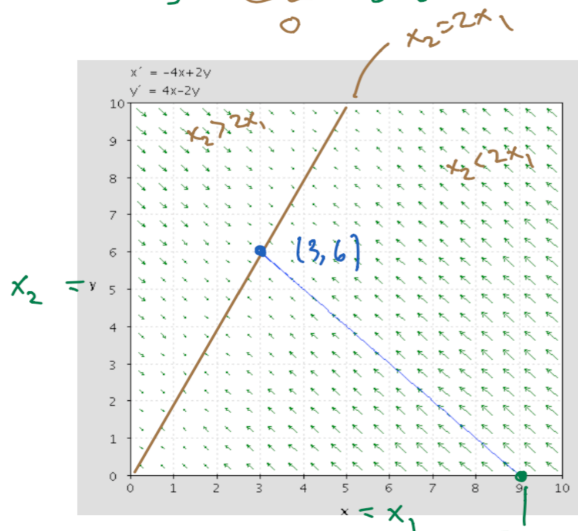


expect same concentrations in each tank as $t \to \infty$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

& expect total of 9 lbs & twice as much in 2nd tank because twice the vol.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{(2x_1 - x_2)}_{0} \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$



$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

On Friday, we began to solve the two-tank IVP example analytically, using eigenvalues and eigenvectors from the matrix A in that problem (!). That is typical of what we will do in section 7.3, to solve the first order system of DE's

$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}}$$
$$\underline{\mathbf{x}}(0) = \underline{\mathbf{x}}_0 \,.$$

But before we finish that computation, it's a better idea to review and extend some differentiation rules you probably learned in multivariable Calculus, when you studied the calculus of parametric curves. This is related to material in section 7.2 of the text.

1) If $\underline{\mathbf{x}}(t) = \underline{\boldsymbol{b}}$ is a constant vector, then $\underline{\mathbf{x}}'(t) = \underline{\mathbf{0}}$ for all t, and vise-verse. (Because all of the entries in the vector $\underline{\boldsymbol{b}}$ are constants, and their derivatives are zero. And if the derivatives of all entries of a vector are identically zero, then the entries are constants.)

2) Sum rule for differentiation:

$$\frac{d}{dt}\left(\underline{\mathbf{x}}(t) + \underline{\mathbf{y}}(t)\right) = \underline{\mathbf{x}}'(t) + \underline{\mathbf{y}}'(t): \qquad \text{Both sides simplify to} \begin{bmatrix} x_1'(t) + y_1'(t) \\ x_2'(t) + y_2'(t) \\ \vdots \\ x_n'(t) + y_n'(t) \end{bmatrix}$$

3) Constant multiple rule for differentiation:

$$\frac{d}{dt}\left(c\,\underline{\mathbf{x}}(t)\right) = c\,\underline{\mathbf{x}}'(t): \qquad \text{Both sides simplify to} \quad c \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}.$$

4) Matrix-valued functions sometimes show up and need to be differentiated. This is done with the limit definition, and amounts to differentiating each entry of the matrix. For example, if $A(t)$ is a 2 × 2 matrix, then

$$\frac{d}{dt}\begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\begin{bmatrix} a_{11}(t+\Delta t) & a_{12}(t+\Delta t) \\ a_{21}(t+\Delta t) & a_{22}(t+\Delta t) \end{bmatrix} - \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}\right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\begin{bmatrix} a_{11}(t+\Delta t) - a_{11}(t) & a_{12}(t+\Delta t) - a_{12}(t) \\ a_{21}(t+\Delta t) - a_{21}(t) & a_{22}(t+\Delta t) - a_{22}(t) \end{bmatrix}$$

$$= \lim_{\Delta t \to 0}\begin{bmatrix} \frac{a_{11}(t+\Delta t) - a_{11}(t)}{\Delta t} & \frac{a_{12}(t+\Delta t) - a_{12}(t)}{\Delta t} \\ \frac{a_{21}(t+\Delta t) - a_{21}(t)}{\Delta t} & \frac{a_{22}(t+\Delta t) - a_{22}(t)}{\Delta t} \end{bmatrix} = \begin{bmatrix} a_{11}'(t) & a_{12}'(t) \\ a_{21}'(t) & a_{22}'(t) \end{bmatrix}.$$

5) The constant rule (1), sum rule (2), and constant multiple rule (3) also hold for matrix derivatives.

Universal product rule: Shortcut to take the derivatives of

$f(t)\underline{\mathbf{x}}(t)$ (scalar function times vector function),
$f(t)A(t)$ (scalar function times matrix function),
$A(t)\underline{\mathbf{x}}(t)$ (matrix function times vector function),
$\underline{\mathbf{x}}(t) \cdot \underline{\mathbf{y}}(t)$ (vector function dot product with vector function),
$\underline{\mathbf{x}}(t) \times \underline{\mathbf{y}}(t)$ (cross product of two vector functions),
$A(t)B(t)$ (matrix function times matrix function).

As long as the "product" operation distributes over addition, and scalars times the product equal the products where the scalar is paired with either one of the terms, there is a product rule. Since the product operation is not assumed to be commutative you need to be careful about the order in which you write down the terms in the product rule, though.

Theorem. Let $A(t)$, $B(t)$ be differentiable scalar, matrix or vector-valued functions of t, and let $*$ be a product operation as above. Then

$$\frac{d}{dt}(A(t)*B(t)) = A'(t)*B(t) + A(t)*B'(t).$$

The explanation just rewrites the limit definition explanation for the scalar function product rule that you learned in Calculus, and assumes the product distributes over sums and that scalars can pass through the product to either one of the terms, as is true for all the examples above. It also uses the fact that differentiable functions are continuous, that you learned in Calculus. Here is one explanation that proves all of those product rules at once:

$$\frac{d}{dt}(A(t)*B(t)) := \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(A(t+\Delta t)*B(t+\Delta t) - A(t)*B(t)\right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(A(t+\Delta t)*B(t+\Delta t) - A(t+\Delta t)*B(t) + A(t+\Delta t)*B(t) - A(t)*B(t)\right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(A(t+\Delta t)*B(t+\Delta t) - A(t+\Delta t)*B(t)\right) + \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(A(t+\Delta t)*B(t) - A(t)*B(t)\right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Big(A(t+\Delta t)*\left(B(t+\Delta t) - B(t)\right) + \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(A(t+\Delta t) - A(t)\right)*B(t)$$

$$= \lim_{\Delta t \to 0} \left(A(t+\Delta t)*\left(\frac{1}{\Delta t}\left(B(t+\Delta t) - B(t)\right)\right)\right) + \lim_{\Delta t \to 0}\left(\frac{1}{\Delta t}\left(A(t+\Delta t) - A(t)\right)\right)*B(t)$$

$$= A(t)*B'(t) + A'(t)*B(t).$$

It is always the case that an initial value problem for single differential equation, or for a system of differential equations is equivalent to an initial value problem for a larger system of first order differential equations, as in the previous example. (See examples and homework problems in section 7.1) This gives us a new perspective on e.g. homogeneous differential equations from Chapter 5.

For example, consider this overdamped problem from Chapter 5:

$$x''(t) + 7x'(t) + 6x(t) = 0$$
$$x(0) = 1$$
$$x'(0) = 4.$$

Exercise 3a) Solve the IVP above, using Chapter 5 and characteristic polynomial.

3b) Show that if $x(t)$ solves the IVP above, then $[x(t), x'(t)]^T$ solves the first order system of DE's IVP

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

Use your work to write down the solution to the IVP in 3b.

3c) Show that if $[x_1(t), x_2(t)]^T$ solves the IVP in 3b then the first entry $x_1(t)$ solves the original second order DE IVP. So converting a second order DE to a first order system is a reversible procedure.

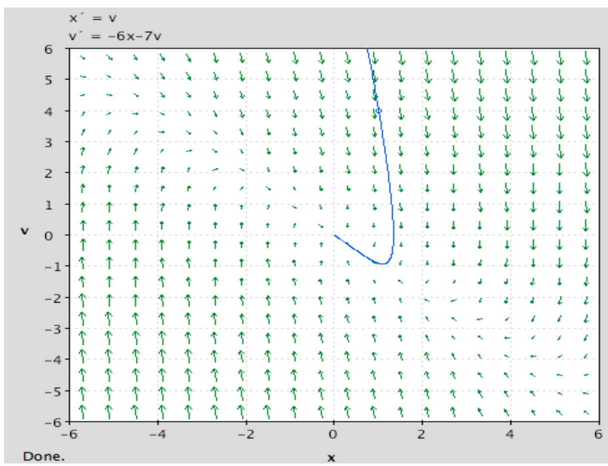
3d) Compare the characteristic polynomial for the homogeneous DE in 3a, to the one for the matrix in the first order system in 3b. It's a mystery (for now)!

$$x''(t) + 7x'(t) + 6x(t) = 0$$

$$\begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix}$$

Pictures of the phase portrait for the system in 3b, which is tracking position and velocity of the solution to 3a.

From pplane, for the system:



From Wolfram alpha, for the underdamped second order DE in 3a.

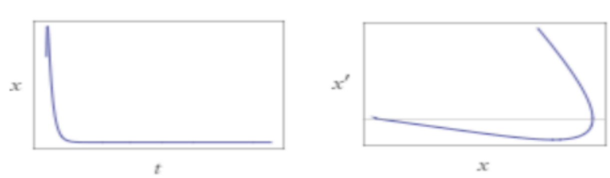
Input:

$\{x''(t) + 7x'(t) + 6x(t) = 0,\ x(0) = 1,\ x'(0) = 4\}$

Differential equation solution:

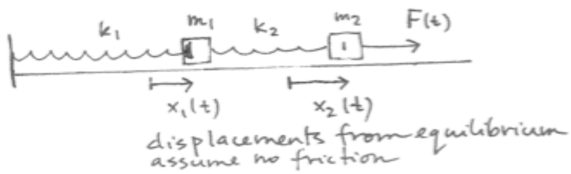
$x(t) = e^{-6t}(2e^{5t} - 1)$

Plots of the solution:



A larger example of converting higher order DE's and systems of DE's into first-order ones....

Example:
Consider this configuration of two coupled masses and springs:



Exercise 4) Use Newton's second law to derive a system of two second order differential equations for $x_1(t), x_2(t)$, the displacements of the respective masses from the equilibrium configuration. What initial value problem do you expect yields unique solutions in this case? (See homework in section 7.1)

Exercise 5) Consider the IVP from Exercise 4, with the special values
$m_1 = 2$, $m_2 = 1$; $k_1 = 4$, $k_2 = 2$; $F(t) = 40\sin(3t)$:

$$\begin{aligned}
&x_1'' = -3x_1 + x_2\\
&x_2'' = 2x_1 - 2x_2 + 40\sin(3t)\\
&x_1(0) = b_1, x_1'(0) = b_2\\
&x_2(0) = c_1, x_2'(0) = c_2 .
\end{aligned}$$

5a) Show that if $x_1(t), x_2(t)$ solve the IVP above, and if we define

$$\begin{aligned}
v_1(t) &:= x_1'(t)\\
v_2(t) &:= x_2'(t)
\end{aligned}$$

then $x_1(t), x_2(t), v_1(t), v_2(t)$ solve the first order system IVP

$$\begin{aligned}
&x_1' = v_1\\
&x_2' = v_2\\
&v_1' = -3x_1 + x_2\\
&v_2' = 2x_1 - 2x_2 + 40\sin(3t)\\
&x_1(0) = b_1\\
&v_1(0) = b_2\\
&x_2(0) = c_1\\
&v_2(0) = c_2 .
\end{aligned}$$

5b) Conversely, show that if $x_1(t), x_2(t), v_1(t), v_2(t)$ solve the IVP of four first order DE's, then $x_1(t), x_2(t)$ solve the original IVP for two second order DE's.

Tues Apr 17

7.2-7.3 Linear systems of differential equations (7.2); Solving homogeneous linear systems $\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}(t)$ by finding basis solutions of the form $e^{\lambda t}\underline{\boldsymbol{v}}$. (7.3).

Announcements:

Warm-up Exercise:

The focus of sections 7.2-7.3 is linear systems of first order differential equations, and their associated initial value problems:

$$\underline{\mathbf{x}}'(t) = A(t)\underline{\mathbf{x}}(t) + \underline{\boldsymbol{f}}(t)$$
$$\underline{\mathbf{x}}(t_0) = \underline{\mathbf{x}}_0$$

If the matrix $A(t)$ and the vector function $\underline{\boldsymbol{f}}(t)$ are continuous on an open interval I containing t_0 then a solution $\underline{\mathbf{x}}(t)$ exists and is unique, on the entire interval. We had a similar fact for the scalar version of this system, in Chapter 1. There, we had an integrating factor technique to find the solutions. Not so, here. When we want to emphasize the *linear* nature of this sort of system we may re-write it as

$$\underline{\mathbf{x}}'(t) - A(t)\underline{\mathbf{x}}(t) = \underline{f}(t)$$

$$\underline{\mathbf{x}}(t_0) = \underline{\mathbf{x}}_0$$

We checked on Friday that the operator on the left, namely

$$L(\underline{\mathbf{x}}(t)) := \underline{x}'(t) - A(t)\underline{\mathbf{x}}(t)$$

is linear:

$$L(\underline{\mathbf{x}}(t) + \underline{z}(t)) = L(\underline{\mathbf{x}}(t)) + L(\underline{z}(t))$$

$$L(c\,\underline{\mathbf{x}}(t)) = c\,L(\underline{\mathbf{x}}(t)).$$

This is how the check went (we suppress the t):

$$L(\underline{\mathbf{x}}(t) + \underline{z}(t)) = L(\underline{\mathbf{x}} + \underline{z}) := (\underline{\mathbf{x}} + \underline{z})' - A(\underline{\mathbf{x}} + \underline{z})$$

$$= \underline{\mathbf{x}}' + \underline{z}' - A\,\underline{\mathbf{x}} - A\,\underline{z} = (\underline{\mathbf{x}}' - A\,\underline{\mathbf{x}}) + (\underline{z}' - A\,\underline{z}) = L(\underline{\mathbf{x}}) + L(\underline{z}).$$

$$L(c\,\underline{\mathbf{x}}(t)) = L(c\,\underline{\mathbf{x}}) = (c\,\underline{\mathbf{x}})' - A\,(c\,\underline{\mathbf{x}}) = c\,\underline{\mathbf{x}}' - c\,A\,\underline{\mathbf{x}} = c(\underline{\mathbf{x}}' - A\,\underline{\mathbf{x}}) = c\,L(\underline{\mathbf{x}}).$$

Thus, from vector space theory,

1a) The solution space to the homogeneous linear system of DE's

$$\underline{\boldsymbol{x}}'(t) - A(t)\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{0}}.$$

is a subspace.

1b) If $\underline{\boldsymbol{x}}(t)$ is an n-vector and A is an $n \times n$ matrix, then the initial value problems

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t)$$

$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

have n free parameters (i.e. the n entries of the intial vector $\underline{\boldsymbol{x}}_0$), so it will take a set of n linearly independent solutions $\{\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), ...\underline{\boldsymbol{X}}_n(t)\}$ to uniquely solve each IVP with a linear combination

$$\underline{\boldsymbol{x}}(t) = c_1\underline{\boldsymbol{X}}_1(t) + c_2\underline{\boldsymbol{X}}_2(t) + ... c_n\underline{\boldsymbol{X}}_n(t).$$

In other words, the solution space has dimension n.

By the way, the matrix that has n solutions in its columns is called the *Wronskian matrix* of the solutions, and its determinant is called the *Wronskian determinant.*

2) The general solution to the inhomogeneous system

$$\underline{\boldsymbol{x}}'(t) - A(t)\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{f}}(t)$$

which we often write in the form

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{f}}(t)$$

is

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{x}}_p(t) + \underline{\boldsymbol{x}}_H(t)$$

where $\underline{\boldsymbol{x}}_p(t)$ is any single particular solution and $\underline{\boldsymbol{x}}_H(t)$ is the general solution to the homogeneous problem

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t).$$

Just like in Chapter 5!

Section 7.3 How to find a basis for the solution space to homogeneous first order systems of differential equations

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$

when the matrix A is constant and diagonalizable.

Here's how!! If $\underline{\boldsymbol{v}} \neq \underline{\boldsymbol{0}}$ is an eigenvector of A, i.e.

$$A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$$

then

$$\underline{\boldsymbol{X}}(t) = e^{\lambda t}\underline{\boldsymbol{v}}$$

satisifies

$$\underline{\boldsymbol{X}}'(t) = \lambda\, e^{\lambda t}\,\underline{\boldsymbol{v}}$$

and

$$A\,\underline{\boldsymbol{X}}(t) = A\left(e^{\lambda t}\underline{\boldsymbol{v}}\right) = e^{\lambda t} A\,\underline{\boldsymbol{v}} = e^{\lambda t}\lambda\,\underline{\boldsymbol{v}}\ .$$

So $\underline{\boldsymbol{X}}(t) = e^{\lambda t}\underline{\boldsymbol{v}}$ is a solution to the system of differential equations

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}.$$

If A is diagonalizable then there is a basis of $\mathbb{R}^n$ made out of its eigenvectors, and we will have a corresponding basis for the solution space to the first order system:

$$\left\{ e^{\lambda_1 t}\underline{\boldsymbol{v}}_1,\ e^{\lambda_2 t}\underline{\boldsymbol{v}}_2\,, \ldots\ e^{\lambda_n t}\underline{\boldsymbol{v}}_n \right\}.$$

In other words, we can solve each initial value problem

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0$$

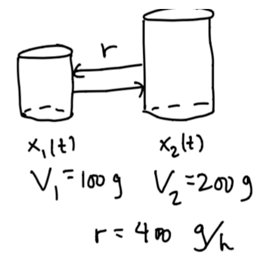
with a unique linear combination of the basis solutions,

$$\underline{\boldsymbol{x}}(t) = c_1 e^{\lambda_1 t}\underline{\boldsymbol{v}}_1 + c_2\, e^{\lambda_2 t}\underline{\boldsymbol{v}}_2 + c_n e^{\lambda_n t}\underline{\boldsymbol{v}}_n\ .$$

Exercise 1 For the tank problem last Friday,

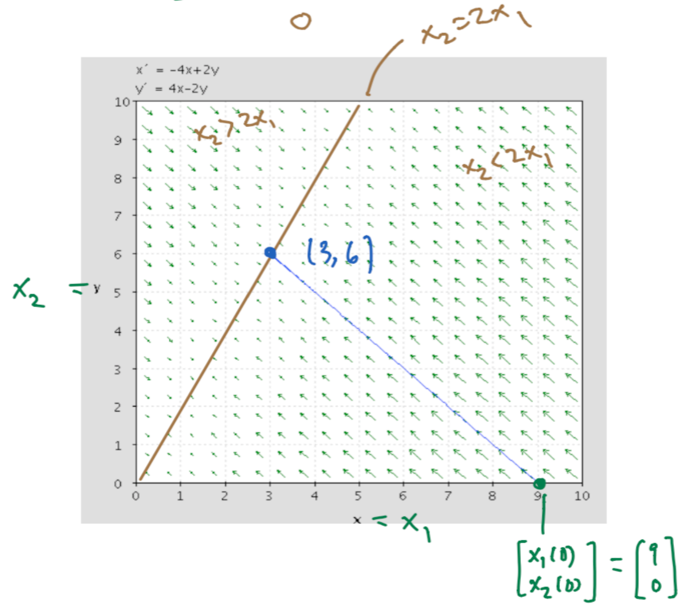
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$



We found the eigendata for the matrix *A*. But it's been a while, so let's recompute. Then write down the general solution to the first order system and solve the initial value problem. Compare the solution to the pplane analysis we did on Friday, on the next page.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{(2x_1 - x_2)}_{0} \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$



Exercise 2) For the overdamped mass-spring IVP on Monday

$$x''(t) + 7x'(t) + 6x(t) = 0$$
$$x(0) = 1$$
$$x'(0) = 4$$

we found the solution

$$x(t) = 2e^{-t} - e^{-6t}.$$

We noted that the equivalent first order system IVP for $[x(t), x'(t)]^T$ is

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

2a) Solve this IVP using the algorithm for first order systems of DE's, and verify that $x_1(t)$ is the solution to the original second order DE IVP.

2b) Compare the Chapter 5 "Wronskian matrix" for the second order DE, to the Chapter 7 "Wronskian matrix" for the system. (We already noted that the characteristic polynomials were the same.)

$$x''(t) + 7x'(t) + 6x(t) = 0$$

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Wed Apr 18
7.3 continued: solving $\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$ when the eigenvalues and eigenvectors are complex. More examples.

Announcements:

Warm-up Exercise:

Consider this second order underdamped IVP for $x(t)$:

$$x'' + 2x' + 5x = 0$$
$$x(0) = 4$$
$$x'(0) = -4 \ .$$

Exercise 1)

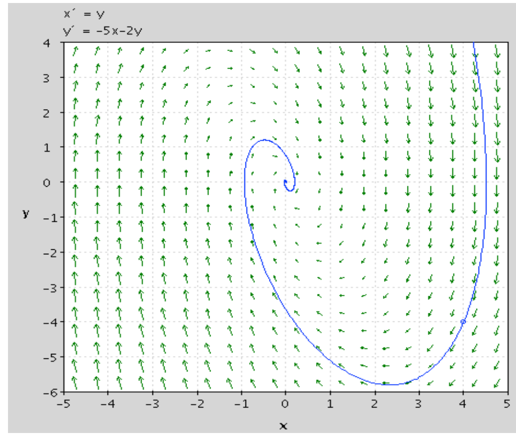
1a) Convert this single second order IVP into an equivalent first order system IVP for $x_1(t) := x(t)$ and $x_2(t) := x'(t)$.

1b) Solve the second order IVP in order to deduce a solution to the first order IVP. Use Chapter 5 methods.

1c) How does the Chapter 5 "characteristic polynomial" in 3b compare with the Chapter 6 (eigenvalue) "characteristic polynomial" for the first order system matrix in 3a? hmmm.

In your homework assignment for next week, as practice in using complex eigendata, you will solve the first order system IVP in 3a and verify that the first component of the solution, $x_1(t)$ is the solution $x(t)$ to the second order DE.

phase portrait for the underdamped oscillator and and associated first order system, via pplane and Wolfram alpha.



ODE classification:

second-order linear ordinary differential equation

Alternate form:

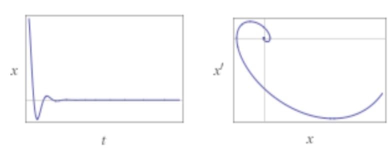
$\{x''(t) = -2\,x'(t) - 5\,x(t),\ x(0) = 4,\ x'(0) = -4\}$

Differential equation solution:

$x(t) = 4\,e^{-t}\cos(2\,t)$

Need a step by step solution for this problem? >>

Plots of the solution:



So far we've not considered the possibility of complex eigenvalues and eigenvectors. Linear algebra theory works the same with complex number scalars and vectors - one can talk about complex vector spaces, linear combinations, span, linear independence, reduced row echelon form, determinant, dimension, basis, etc. Then the model space is $\mathbb{C}^n$ rather than $\mathbb{R}^n$.

Definition: $\underline{\mathbf{v}} \in \mathbb{C}^n$ ($\underline{\mathbf{v}} \neq \underline{\mathbf{0}}$) is a complex eigenvector of the matrix A, with eigenvalue $\lambda \in \mathbb{C}$ if $A\,\underline{\mathbf{v}} = \lambda\,\underline{\mathbf{v}}$.

Just as before, you find the possibly complex eigenvalues by finding the roots of the characteristic polynomial $|A - \lambda I|$. Then find the eigenspace bases by reducing the corresponding matrix (using complex scalars in the elementary row operations).

The best way to see how to proceed in the case of complex eigenvalues/eigenvectors is to work an example. There is a general discussion on the page after this example that we will refer to along the way:

Glucose-insulin model (adapted from a discussion on page 339 of the text "Linear Algebra with Applications," by Otto Bretscher)

Let $G(t)$ be the excess glucose concentration (mg of G per 100 ml of blood, say) in someone's blood, at time t hours. Excess means we are keeping track of the difference between current and equilibrium ("fasting") concentrations. Similarly, Let $H(t)$ be the excess insulin concentration at time t hours. When blood levels of glucose rise, say as food is digested, the pancreas reacts by secreting insulin in order to utilize the glucose. Researchers have developed mathematical models for the glucose regulatory system. Here is a simplified (linearized) version of one such model, with particular representative matrix coefficients. It would be meant to apply between meals, when no additional glucose is being added to the system:

$$\begin{bmatrix} G'(t) \\ H'(t) \end{bmatrix} = \begin{bmatrix} -0.1 & -0.4 \\ 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix}$$

Exercise 2a) Understand why the signs of the matrix entries are reasonable.

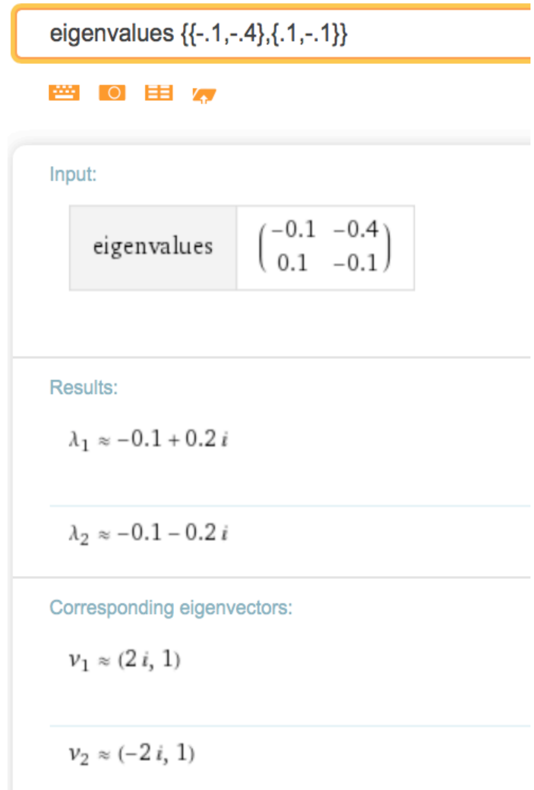
Now let's solve the initial value problem, say right after a big meal, when

$$\begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} G'(t) \\ H'(t) \end{bmatrix} = \begin{bmatrix} -0.1 & -0.4 \\ 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix}$$

$$\begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

<u>2b)</u> The first step is to get the eigendata of the matrix. Do this, and compare with the Wolfram check on the next page.



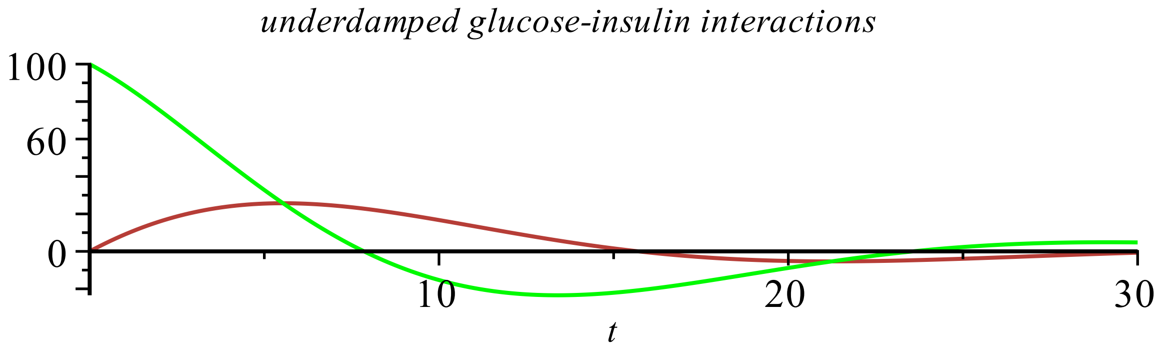
2c) Extract a basis for the solution space to his homogeneous system of differential equations from the eigenvector information above:

2d) Solve the initial value problem.

Here are some pictures to help understand what the model is predicting ... you could also construct these graphs using pplane.

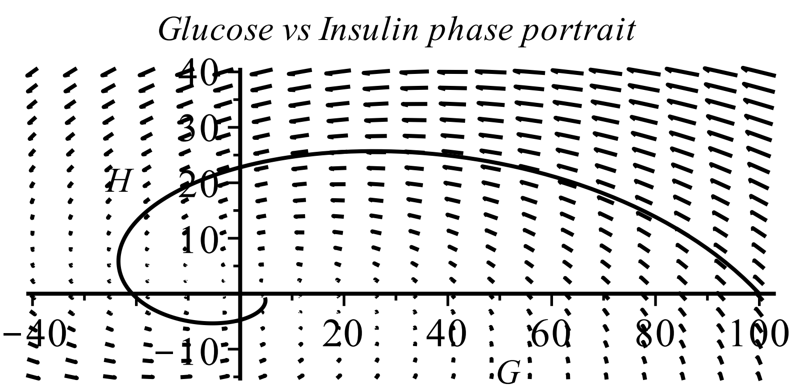
(1) Plots of glucose vs. insulin, at time t hours later:

```
> with(plots) :
> G := t→100·exp(-.1·t)·cos(.2·t) :
  H := t→50·exp(-.1·t)·sin(.2·t) :
  plot1 := plot(G(t), t = 0 ..30, color = green) :
  plot2 := plot(H(t), t = 0 ..30, color = brown) :
  display({plot1, plot2}, title = `underdamped glucose-insulin interactions`);
```



2) A phase portrait of the glucose-insulin system:

```
> pict1 := fieldplot([-.1·G - .4·H, .1·G - .1·H], G =-40 ..100, H =-15 ..40) :
  soltn := plot([G(t), H(t), t = 0 ..30], color = black) :
  display({pict1, soltn}, title = `Glucose vs Insulin phase portrait`);
```



• The example we just worked is linear, and is vastly simplified. But mathematicians, doctors, bioengineers, pharmacists, are very interested in (especially more realistic) problems like these.

Solutions to homogeneous linear systems of DE's when matrix has complex eigenvalues:

$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}}$$

Let A be a real number matrix. Let

$$\lambda = a + b\,i \in \mathbb{C}$$
$$\underline{\mathbf{v}} = \underline{\boldsymbol{\alpha}} + i\,\underline{\boldsymbol{\beta}} \in \mathbb{C}^n$$

satisfy $A\,\underline{\mathbf{v}} = \lambda\,\underline{\mathbf{v}}$, with $a, b \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}^n$.

- Then $\underline{\mathbf{z}}(t) = e^{\lambda t}\underline{\mathbf{v}}$ is a complex solution to

$$\underline{\mathbf{x}}'(t) = A\,\mathbf{x}$$

because $\underline{\mathbf{z}}'(t) = \lambda e^{\lambda t}\underline{\mathbf{v}}$ and this is equal to $A\,\underline{\mathbf{z}} = \boldsymbol{A}\,e^{\lambda t}\underline{\mathbf{v}} = e^{\lambda t}A\,\underline{\mathbf{v}} = e^{\lambda t}\lambda\,\underline{\mathbf{v}}$.

- But if we write $\underline{\mathbf{z}}(t)$ in terms of its real and imaginary parts,

$$\underline{\mathbf{z}}(t) = \underline{\mathbf{x}}(t) + i\,\underline{\mathbf{y}}(t)$$

then the equality

$$\underline{\mathbf{z}}'(t) = A\,\underline{\mathbf{z}}$$
$$\Rightarrow \underline{\mathbf{x}}'(t) + i\,\underline{\mathbf{y}}'(t) = A(\underline{\mathbf{x}}(t) + i\,\underline{\mathbf{y}}(t)) = A\,\underline{\mathbf{x}}(t) + i\,A\,\underline{\mathbf{y}}(t)\ .$$

Equating the real and imaginary parts on each side yields

$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}}(t)$$
$$\underline{\mathbf{y}}'(t) = A\,\underline{\mathbf{y}}(t)$$

i.e. the real and imaginary parts of the complex solution are each real solutions.

- If $A\,(\underline{\boldsymbol{\alpha}} + i\,\underline{\boldsymbol{\beta}}) = (a + b\,i)(\underline{\boldsymbol{\alpha}} + i\,\underline{\boldsymbol{\beta}})$ then it is straightforward to check that $A\,(\underline{\boldsymbol{\alpha}} - i\,\underline{\boldsymbol{\beta}}) = (a - b\,i)(\underline{\boldsymbol{\alpha}} - i\,\underline{\boldsymbol{\beta}})$. Thus the complex conjugate eigenvalue yields the complex conjugate eigenvector. The corresponding complex solution to the system of DEs

$$e^{(a\,-\,i\,b)t}(\underline{\boldsymbol{\alpha}} - i\,\underline{\boldsymbol{\beta}}) = \underline{\mathbf{x}}(t) - i\,\underline{\mathbf{y}}(t)$$

so yields the same two real solutions (except with a sign change on the second one). Another way to understand how we get the two real solutions is to take the two complex solutions

$$\underline{\mathbf{z}}(t) = \underline{\mathbf{x}}(t) + i\,\underline{\mathbf{y}}(t)$$
$$\underline{\mathbf{w}}(t) = \underline{\mathbf{x}}(t) - i\,\underline{\mathbf{y}}(t)$$

and recover $\underline{\mathbf{x}}(t), \underline{\mathbf{y}}(t)$ as linear combinations of these homogeneous solutions:

$$\underline{\mathbf{x}}(t) = \frac{1}{2}\left(\underline{\mathbf{z}}(t) + \underline{\mathbf{w}}(t)\right)$$
$$\underline{\mathbf{y}}(t) = \frac{1}{2\,i}\left(\underline{\mathbf{z}}(t) - \underline{\mathbf{w}}(t)\right).$$

- The Glucose-insulin example is linearized, and is vastly simplified. But mathematicians, doctors, bioengineers, pharmacists, are very interested in (especially more realistic) problems like these. Prof. Fred Adler and a recent graduate student Chris Remien in the Math Department, and collaborating with the University Hospital recently modeled liver poisoning by acetominophen (brand name Tyleonol), by studying a non-linear system of 8 first order differential equations. They came up with a state of the art and very useful diagnostic test:

http://unews.utah.edu/news_releases/math-can-save-tylenol-overdose-patients-2/

Here's a link to their published paper. For fun, I copied and pasted the non-linear system of first order differential equations from a preprint of their paper, below:

http://onlinelibrary.wiley.com/doi/10.1002/hep.25656/full

http://www.math.utah.edu/~korevaar/2250spring12/adler-remien-preprint.pdf

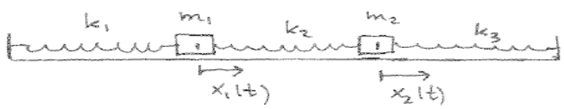
APAP	$\frac{dA}{dt} = -\frac{\alpha}{H_{max}} AH - \delta_a A$
NAPQI	$\frac{dN}{dt} = \frac{qp\alpha}{H_{max}} A - \gamma NG$
GSH	$\frac{dG}{dt} = \kappa - \gamma NG - \delta_g G$
Functional Hepatocytes	$\frac{dH}{dt} = rH\left(1 - \frac{H+Z}{H_{max}}\right) - \eta NH$
Damaged Hepatocytes	$\frac{dZ}{dt} = \eta NH - \delta_z Z$
AST	$\frac{dS}{dt} = \frac{d_z \beta_s}{\theta H_{max}} Z - \delta_s (S - S_{min})$
ALT	$\frac{dL}{dt} = \frac{d_z \beta_l}{\theta H_{max}} Z - \delta_l (L - L_{min})$
Clotting Factor	$\frac{dF}{dt} = \beta_f \left(\frac{H}{H_{max}} - F\right)$

Fri Apr 20
7.4 Mass-spring systems and untethered mass-spring trains.

Announcements:

Warm-up Exercise:

7.4 Mass-spring systems and untethered mass-spring trains.
In your homework and lab for this week you study special cases of the spring systems below, with no damping. Although we draw the pictures horizontally, they would also hold in vertical configuration if we measure displacements from equilibrium in the underlying gravitational field.

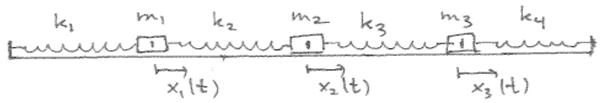


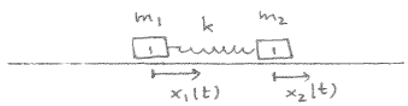
Let's make sure we understand why the natural system of DEs and IVP for this system is

$$m_1 x_1''(t) = -k_1 x_1 + k_2(x_2 - x_1)$$
$$m_2 x_2''(t) = -k_2(x_2 - x_1) - k_3 x_2$$
$$x_1(0) = a_1, \quad x_1'(0) = a_2$$
$$x_2(0) = b_1, \quad x_2'(0) = b_2$$

<u>Exercise 1a)</u> What is the dimension of the solution space to this homogeneous linear system of differential equations? Why?

1b) What if one had a configuration of n masses in series, rather than just 2 masses? What would the dimension of the homogeneous solution space be in this case? Why? Examples:





We can write the system of DEs for the system at the top of page 1 in matrix-vector form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 - k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We denote the diagonal matrix on the left as the "mass matrix" M, and the matrix on the right as the spring constant matrix K (although to be completely in sync with Chapter 5 it would be better to call the spring matrix $-K$). All of these configurations of masses in series with springs can be written as

$$M\underline{\mathbf{x}}''(t) = K\underline{\mathbf{x}}.$$

If we divide each equation by the reciprocal of the corresponding mass, we can solve for the vector of accelerations:

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{k_1 + k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2 + k_3}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

which we write as

$$\underline{\mathbf{x}}''(t) = A\underline{\mathbf{x}}.$$

(You can think of A as the "acceleration" matrix.)
Notice that the simplification above is mathematically identical to the algebraic operation of multiplying the first matrix equation by the (diagonal) inverse of the diagonal mass matrix M. In all cases:

$$M\underline{\mathbf{x}}''(t) = K\underline{\mathbf{x}} \quad \Rightarrow \underline{\mathbf{x}}''(t) = A\underline{\mathbf{x}}\ , \text{with } A = M^{-1}K.$$

How to find a basis for the solution space to conserved-energy mass-spring systems of DEs

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}} \;. \qquad (*)$$

Based on our previous experiences, the natural thing for this homogeneous system of linear differential equations is to try and find a basis of solutions of the form

$$\underline{\boldsymbol{x}}(t) = f(t)\underline{\boldsymbol{v}} \qquad (**)$$

You might guess that $f(t) = e^{\lambda t}$ but that turns out to not be the best way to go. Let's see what $f(t)$ should equal by substituting in our guess! (We would maybe also think about first converting the second order system to an equivalent first order system of twice as many DE's, one for for each position function and one for each velocity function, and then the exponential guess would work, but they'd end up being complex exponentials.) Substituting (**) into (*) yields

$$f''(t)\,\underline{\boldsymbol{v}} = A\,(f(t)\underline{v}) = f(t)\,A\,\underline{\boldsymbol{v}}\,.$$

Since for each *t*, the left side is a scalar multiple of the constant vector $\underline{\boldsymbol{v}}$, so must be the right side. So $\underline{\boldsymbol{v}}$ must be an eigenvector of *A*,

$$A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}\,,$$

and if $f(t)$ is a real function and if $\underline{\boldsymbol{v}}$ is a real (as opposed to complex) vector, then λ is also real. Then

$$f''(t)\,\underline{\boldsymbol{v}} = A\,(f(t)\underline{v}) = f(t)\,\lambda\,\underline{\boldsymbol{v}}\,.$$

So we must have

$$f''(t) - \lambda f(t) = 0.$$

So possible $f(t)$'s are

Case 1)

$$f(t) = c_1 + c_2 t \qquad \text{if } \lambda = 0$$

Case 2)

$$f(t) = c_1\cos(\omega\,t) + c_2\,\sin(\omega\,t) \qquad \text{if } \lambda < 0, \lambda = -\omega^2 \quad \omega = \sqrt{-\lambda}$$

Case 3)

$$f(t) = c_1 e^{\sqrt{\lambda}\,t} + c_2\,e^{-\sqrt{\lambda}\,t} \qquad \text{if } \lambda > 0\,.$$

Case 3 will never happen for our mass-spring configurations, because of conservation of energy!

This leads to the

Solution space algorithm: Consider a very special case of a homogeneous system of linear differential equations,

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}} \ .$$

If $A_{n \times n}$ is a diagonalizable matrix and if all of its eigenvalues are non-positive then for each eigenpair $\left(\lambda_j, \underline{\boldsymbol{v}}_j\right)$ with $\lambda_j < 0$ there are two linearly independent sinusoidal solutions to $\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$ given by

$$\boldsymbol{x}_j(t) = \cos\left(\omega_j\, t\right)\underline{\boldsymbol{v}}_j \qquad \underline{\boldsymbol{y}}_j(t) = \sin\left(\omega_j\, t\right)\underline{\boldsymbol{v}}_j$$

with

$$\omega_j = \sqrt{-\lambda_j} \ .$$

And for an eigenpair $\left(\lambda_j, \underline{\boldsymbol{v}}_j\right)$ with $\lambda_j = 0$ there are two indpendent solutions given by constant and linear functions

$$\boldsymbol{x}_j(t) = \underline{\boldsymbol{v}}_j \qquad \underline{\boldsymbol{y}}_j(t) = t\,\underline{\boldsymbol{v}}_j$$

This procedure constructs $2\,n$ independent solutions to the system $\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$, i.e. a basis for the solution space.

Remark: What's amazing is that the fact that if the system is conservative, the acceleration matrix will always be diagonalizable, and all of its eigenvalues will be non-positive. In fact, if the system is tethered to at least one wall (as in the first two diagrams on page 1), all of the eigenvalues will be strictly negative, and the algorithm above will always yield a basis for the solution space. (If the system is not tethered and is free to move as a train, like the third diagram on page 1, then $\lambda = 0$ will be one of the eigenvalues, and will yield the constant velocity and displacement contribution to the solution space, $\left(c_1 + c_2\, t\right)\underline{\boldsymbol{v}}$, where $\underline{\boldsymbol{v}}$ is the corresponding eigenvector. Together with the solutions from strictly negative eigenvalues this will still lead to the general homogeneous solution.)

Exercise 2) Consider the special case of the configuration on page one for which $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$. In this case, the equation for the vector of the two mass accelerations reduces to

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{k}{m} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

a) Find the eigendata for the matrix

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

b) Deduce the eigendata for the acceleration matrix A which is $\frac{k}{m}$ times this matrix.

c) Find the $4-$ dimensional solution space to this two-mass, three-spring system.

solution The general solution is a superposition of two "fundamental modes". In the slower mode both masses oscillate "in phase", with equal amplitudes, and with angular frequency $\omega_1 = \sqrt{\frac{k}{m}}$. In the faster mode, both masses oscillate "out of phase" with equal amplitudes, and with angular frequency $\omega_2 = \sqrt{\frac{3\,k}{m}}$. The general solution can be written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1 \cos\left(\omega_1 t - \alpha_1\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos\left(\omega_2 t - \alpha_2\right) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \left(c_1 \cos\left(\omega_1 t\right) + c_2 \sin\left(\omega_1 t\right)\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(c_3 \cos\left(\omega_2 t\right) + c_4 \sin\left(\omega_2 t\right)\right) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Exercise 3) Show that the general solution above lets you uniquely solve each IVP uniquely. This should reinforce the idea that the solution space to these two second order linear homgeneous DE's is four dimensional.

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{2\,k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2\,k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

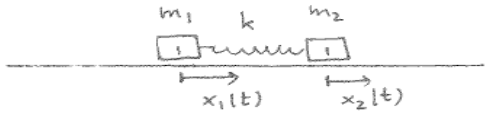
$$x_1(0) = a_1, \quad x_1'(0) = a_2$$
$$x_2(0) = b_1, \quad x_2'(0) = b_2$$

Experiment: Although we probably won't have time in class to measure the spring constants, I've measured them earlier. We can predict the numerical values for the two fundamental modes of the vertical mass-spring configuration corresponding to Exercise 2, and then check our predictions like we did for the single mass-spring configuration, I have brought along a demonstration so that we can see these two vibrations.

> $Digits := 5 :$
$$k := \frac{.05 \cdot 9.806}{.153};$$
$$\omega 1 := \sqrt{\frac{k}{.05}}; \; T1 := evalf\left(\frac{2 \cdot \pi}{\omega 1}\right);$$
$$\omega 2 := \sqrt{3.0} \cdot \omega 1; \; T2 := evalf\left(\frac{2 \cdot \pi}{\omega 2}\right);$$

$$k := 3.2046$$
$$\omega 1 := 8.0057$$
$$T1 := 0.78483$$
$$\omega 2 := 13.867$$
$$T2 := 0.45311 \qquad \textbf{(1)}$$

Exercise 4) Consider a train with two cars connected by a spring:



4a) Verify that the linear system of DEs that governs the dynamics of this configuration (it's actually a special case of what we did before, with two of the spring constants equal to zero) is

$$x_1'' = \frac{k}{m_1}(x_2 - x_1)$$
$$x_2'' = -\frac{k}{m_2}(x_2 - x_1)$$

4b). Use the eigenvalues and eigenvectors computed below to find the general solution. For $\lambda = 0$ and its corresponding eigenvector $\underline{\boldsymbol{v}}$ remember that you get two solutions

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{v}} \text{ and } \underline{\boldsymbol{x}}(t) = t\,\underline{\boldsymbol{v}},$$

rather than the expected $\cos(\omega t)\underline{\boldsymbol{v}}$, $\sin(\omega t)\underline{\boldsymbol{v}}$. Interpret these solutions in terms of train motions. You will use these ideas in some of your homework problems and in your lab exercise about molecular vibrations.

> *Eigenvectors* $\left(\begin{bmatrix} -\frac{k}{m_1} & \frac{k}{m_1} \\ \frac{k}{m_2} & -\frac{k}{m_2} \end{bmatrix}\right)$;

$$\begin{bmatrix} 0 \\ -\frac{k(m_1 + m_2)}{m_2 m_1} \end{bmatrix}, \begin{bmatrix} 1 & -\frac{m_2}{m_1} \\ 1 & 1 \end{bmatrix} \qquad (2)$$