Math 2250-004 Week 13 April 9-13
EP 7.6 - convolutions; 6.1-6.2 - eigenvalues, eigenvectors and diagonalizability; 7.1 - systems of differential equations.

$\underbrace{\text{6.1-6.2}}_{T,\,W}$ $\underbrace{\text{7.1}}_{F}$

Mon Apr 9
EP 7.6 Convolutions and Laplace transforms.
(& 10.5)

Announcements:

'til 10:46

Warm-up Exercise: Solve for $x(t)$

$$\begin{cases} x'' + x = 0 \\ x(0) = 0 \\ x'(0) = 1 \end{cases}$$

$f(t)$	$F(s)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$\sin kt$	$\frac{k}{s^2+k^2}$
1	$1/s$

Soln

$x(t) = \sin t$
(Also Chptr 5)

$\mathcal{L}$: $s^2X(s) - s\cdot 0 - 1 + X(s) = 0$

$X(s)(s^2+1) = 1$

$X(s) = \frac{1}{s^2+1}$

$x(t) = \sin t$

The convolution Laplace transform table entry says it's possible to find the inverse Laplace transform of a product of Laplace transforms. The answer is NOT the product of the inverse Laplace transforms, but a more complicated expression known as a "convolution". If you've had a multivariable Calculus class in which you studied interated integrals and changing the order of integration, you can verify that this table entry is true - I've included the proof as the last page of today's notes.

$f*g(t) := \int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	convolution integrals to invert Laplace transform products

<u>Overview:</u> This somewhat amazing table entry allows us to write down solution formulas for *any* constant coefficient nonhomogeneous differential equation, no matter how complicated the right hand side is. Let's focus our discussion on the sort of differential equation that arises in Math 2250, namely the second order case

$f(t)$	$F(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$c_1f_1+c_2f_2$	$c_1F_1+c_2F_2$

$$x''+a\,x'+b\,x=f(t)$$
$$x(0)=x_0$$
$$x'(0)=v_0\,.$$

Then in Laplace land, this equation is equivalent to

$$s^2X(s)-sx_0-v_0+a\left(s\,X(s)-x_0\right)+b\,X(s)=F(s)$$

$$\Rightarrow X(s)\left(s^2+a\,s+b\right)=F(s)+x_0\,s+v_0+a\,x_0.$$

$$\Rightarrow X(s)=F(s)\cdot\frac{1}{s^2+a\,s+b}+\frac{x_0\,s+v_0+a\,x_0}{s^2+a\,s+b}.$$

The inverse Laplace transform of the second fraction contains the initial value information, and its inverse Laplace transform will be a homogeneous solution for the differential equation, and will be zero if $x_0=v_0=0$. (Note that the Chapter 5 characteristic polynomial is exactly $p(r)=r^2+a\,r+b$, which coincides with the denominator $p(s)=s^2+a\,s+b$.)

The first fraction is a product of two Laplace transforms

$$F(s)\frac{1}{s^2+a\,s+b}=F(s)\,W(s)$$

for

$$W(s):=\frac{1}{s^2+a\,s+b}.$$

and so we can use the convolution table entry to write down an (integral) formula for the inverse Laplace transform. *No matter what* forcing function $f(t)$ appears on the right side of the differential equation, the corresponding solution (to the IVP with $x_0=v_0=0$) is always given by the integral

$$x(t)=f*w(t)=w*f(t)=\int_0^t w(\tau)f(t-\tau)\,d\tau,$$

where

$$\underline{w(t)}=\mathcal{L}^{-1}\{W(s)\}(t).$$

"weight fcn"

$$W(s) = \frac{1}{s^2 + as + b}$$

Let's look more closely at that solution formula: The solution to

$$x'' + a\,x' + b\,x = f(t)$$
$$x(0) = 0$$
$$x'(0) = 0$$

is given by

$$x(t) = f * w(t) = w * f(t) = \int_0^t w(\tau)\, f(t-\tau)\, d\tau = \int_0^t f(\tau)\, w(t-\tau)\, d\tau\,.$$

Exercise 1) The function $w(t) = \mathcal{L}^{-1}\{W(s)\}(t)$ is called the "weight function" for the differential equation. Verify that it is a solution to the homogeneous DE IVP

$\mathcal{L}$:

$$w'' + a\,w' + b\,w = 0$$
$$w(0) = 0$$
$$w'(0) = 1.$$

w''	$s^2 W(s) - s w(0) - w'(0)$
w'	$s W(s) - w(0)$

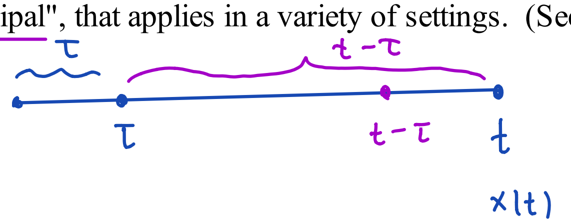
$$s^2 W(s) - 1 + a s W(s) + b W(s) = 0$$
$$W(s)\left(s^2 + as + b\right) = 1$$
$$W(s) = \frac{1}{s^2 + as + b} \quad \checkmark$$

Exercise 2 Interpret the convolution formula

$$x(t) = \int_0^t f(\tau)\, w(t-\tau)\, d\tau$$

$$\begin{cases} x'' + ax + b = f(t) \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

in terms of $x(t)$ being a result of the forces $f(\tau)$ for $0 \le \tau \le t$, and how responsive the system through the weight function, and the corresponding times $t \ge t - \tau \ge 0$. This is related to a general principle known as "Duhamal's Principal", that applies in a variety of settings. (See wikipedia.)



$x(t)$ depends on $f(\tau)$, $0 \le \tau \le t$

and on $w(t-\tau)$

$t \ge t - \tau \ge 0$

Exercise 3. Let's play the resonance game and practice convolution integrals, first with an old friend, but then with non-sinusoidal forcing functions. We'll stick with our earlier swing, but consider various forcing periodic functions $f(t)$.

$$\mathcal{L}: \quad \begin{aligned} x''(t) + x(t) &= f(t) \\ x(0) &= 0 \\ x'(0) &= 0 \end{aligned}$$

$x''(t)$	$s^2 X(s) - s\,x(0) - x'(0)$
$f * g(t) = \int_0^t f(\tau) g(t-\tau)\,d\tau$	$F(s)\,G(s)$

a) Find the weight function $w(t)$.
b) Write down the solution formula for $x(t)$ as a convolution integral.
c) Work out the special case of $X(s)$ when $f(t) = \cos(t)$, and verify that the convolution formula reproduces the answer we would've gotten from the table entry

$\frac{t}{2k}\sin(kt)$	$\frac{s}{(s^2+k^2)^2}$

$\int_0^t f(\tau) g(t-\tau)\,d\tau = \int_0^t g(\tau) f(t-\tau)\,d\tau$	$F(s)G(s)$

$$s^2 X(s) + X(s) = F(s)$$
$$X(s)(s^2+1) = F(s)$$
$$X(s) = F(s)\frac{1}{s^2+1} = F(s)W(s)$$

a) $w(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) = \sin t$

b) $x(t) = f * w(t) = \int_0^t f(\tau) w(t-\tau)\,d\tau$
$$= \int_0^t w(\tau) f(t-\tau)\,d\tau$$

check

$$\int_{\tau=0}^{t} f(\tau)\, g(t-\tau)\,d\tau \qquad \tilde{\tau} = t-\tau, \quad d\tilde{\tau} = -d\tau$$

$$= \int_{\tilde{\tau}=t}^{\tilde{\tau}=0} f(t-\tilde{\tau})\, g(\tilde{\tau})(-d\tilde{\tau})$$
$$= (-)(-)\int_0^t g(\tilde{\tau}) f(t-\tilde{\tau})\,d\tilde{\tau}$$

c). $\begin{cases} x'' + x = \cos t \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$

$\mathcal{L}$: $X(s)(s^2+1) = \frac{s}{s^2+1}$

$$X(s) = \frac{s}{(s^2+1)^2}$$

table $\boxed{x(t) = \frac{t}{2}\sin t}$

Convolution formula (b)

- function to integrate: sin(r)*cos(t-r)
- variable: r
- lower limit: 0
- upper limit: t

Definite integral: Step-by-step solution

$$\int_0^t \sin(r)\cos(t-r)\,dr = \frac{1}{2}t\sin(t)$$

Open code

- function to integrate: cos(r)*sin(t-r)
- variable: r
- lower limit: 0
- upper limit: t

Definite integral: Step-by-step solution

$$\int_0^t \cos(r)\sin(t-r)\,dr = \frac{1}{2}t\sin(t)$$

Open code

$x(t) = (\cos * \sin)(t)$
$$= \int_0^t (\cos\tau)\sin(t-\tau)\,d\tau$$

(in HW you compute some convolutions)

$$= \int_0^t (\cos\tau)\left[\cos t(-\sin\tau) + \sin t\cos\tau\right]d\tau \qquad \text{trig!}$$

$$= \cos t\int_0^t -\cos\tau\sin\tau\,d\tau + \sin t\int_0^t \underbrace{\cos^2\tau}_{\frac{1+\cos 2\tau}{2}}\,d\tau$$

$$= \cos t \left[-\frac{(\sin \tau)^2}{2} \right]_{\tau=0}^{t} + \sin t \left[\frac{\tau}{2} + \frac{\sin 2\tau}{4} \right]_0^t$$

$$= \cos t \cancel{\left[-\frac{\sin^2 t}{2} - 0 \right]} + \sin t \left[\frac{t}{2} \right] + \cancel{\frac{\sin 2t}{4} - 0} \Big]$$

$$= \frac{t}{2} \sin t$$

$$\frac{\sin t \sin 2t}{4} = \frac{(\sin t)\, 2 \sin t \cos t}{4} = \frac{1}{2} \sin^2 t \cos t$$

<u>Exercise 4</u> The solution $x(t)$ to

$$x'' + a\,x' + b\,x = f(t)$$
$$x(0) = 0$$
$$x'(0) = 0$$

is given by

$$x(t) = f * w(t) = w * f(t) = \int_0^t w(\tau) f(t-\tau)\, d\tau = \int_0^t f(\tau)\, w(t-\tau)\, d\tau .$$

We worked out that the solution to our DE IVP will be

$$x(t) = \int_0^t \sin(\tau) f(t-\tau)\, d\tau$$

$$\begin{cases} x'' + x = f \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

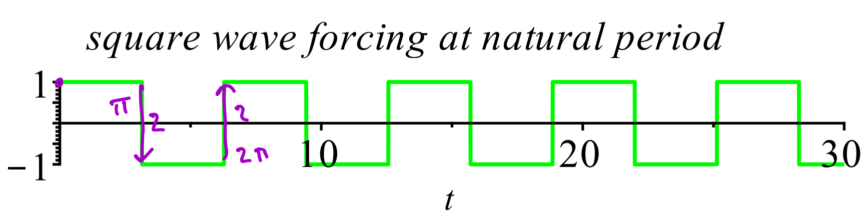
Since the unforced system has a natural angular frequency $\omega_0 = 1$, we expect resonance when the forcing function has the corresponding period of $T_0 = \frac{2\pi}{w_0} = 2\pi$. We will discover that there is the possibility for resonance if the period of f is a ***multiple*** of T_0. (Also, forcing at the natural period doesn't guarantee resonance...it depends what function you force with.)

<u>Example 1)</u> A square wave forcing function with amplitude 1 and period 2π. Let's talk about how we came up with the formula (which works until $t = 11\pi$).

```
> with(plots) :
> f1 := t -> -1 + 2·( Sum((-1)^n·Heaviside(t - n·Pi), n = 0..10) ) :
  plot1a := plot(f1(t), t = 0..30, color = green) :
  display(plot1a, title = `square wave forcing at natural period`);
```

$$x(t) = \int_0^t f(\tau)\, w(t-\tau)\, d\tau$$

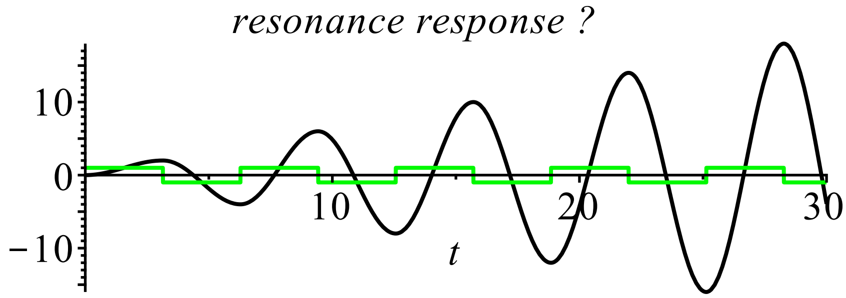
$$= -1 + 2u(0) - 2u(t-\pi) + 2u(t-2\pi) -$$



<u>1)</u> What's your vote? Is this square wave going to induce resonance, i.e. a response with linearly growing amplitude?

> $x1 := t \rightarrow \int_0^t \sin(\tau) \cdot f1(t - \tau)\, d\tau:$

$plot1b := plot(x1(t), t = 0\,..30, color = black):$
$display(\{plot1a, plot1b\}, title = \text{`resonance response ?`});$

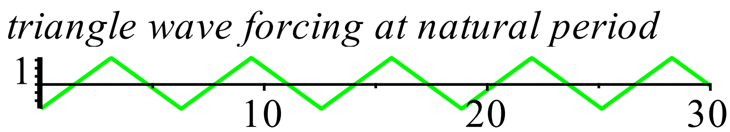


"$\pi/2$"

Example 2) A triangle wave forcing function, same period

> $f2 := t \rightarrow \int_0^t f1(s)\,\mathrm{d}s - 1.5$: *# this antiderivative of square wave should be triangle wave*

$plot2a := plot(f2(t), t = 0..30, color = green)$:
$display(plot2a, title = \text{`triangle wave forcing at natural period`})$;

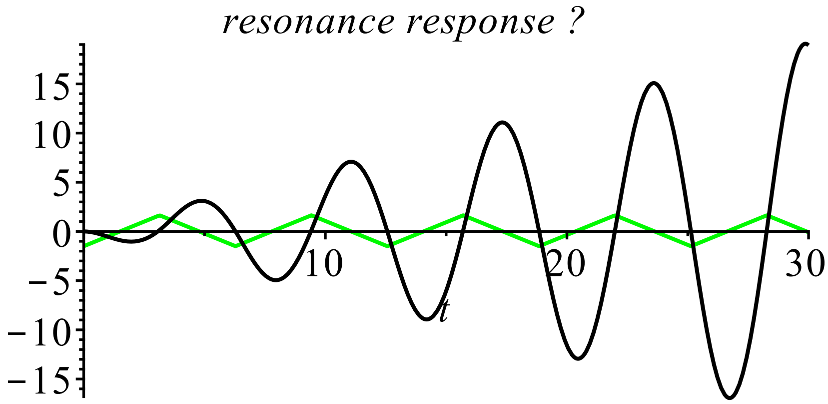


2) Resonance?

$$> x2 := t \rightarrow \int_0^t \sin(\tau) \cdot f2(t - \tau)\, \mathrm{d}\tau :$$

$plot2b := plot(x2(t), t = 0 ..30, color = black) :$

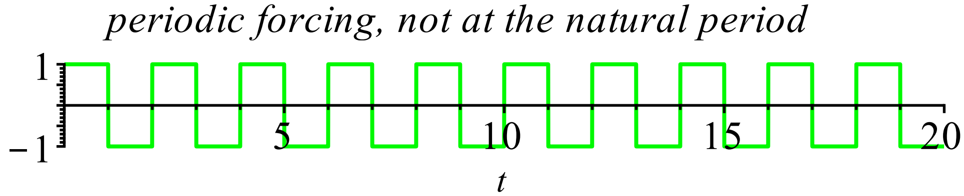
$display(\{plot2a, plot2b\}, title = \text{`resonance response ?`});$



Example 3) Forcing not at the natural period, e.g. with a square wave having period $T = 2$.

$$> f3 := t \to -1 + 2 \cdot \sum_{n=0}^{20} (-1)^n \cdot \text{Heaviside}(t - n) :$$

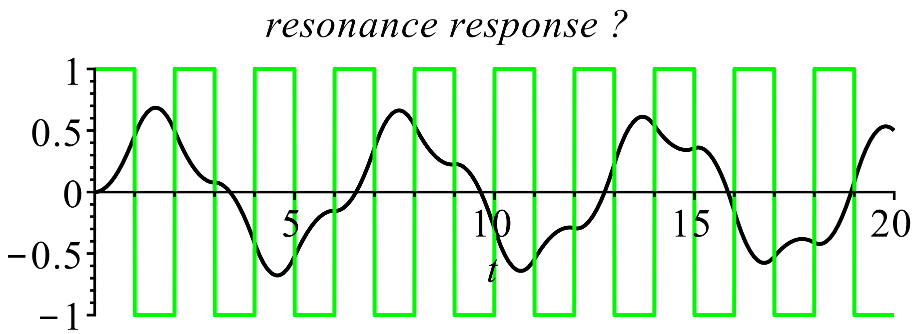
$plot3a := plot(f3(t), t = 0..20, color = green) :$
$display(plot3a, title = \text{`periodic forcing, not at the natural period`});$



3) Resonance?

> $x3 := t \rightarrow \int_0^t \sin(\tau) \cdot f3(t-\tau)\, \mathrm{d}\tau :$

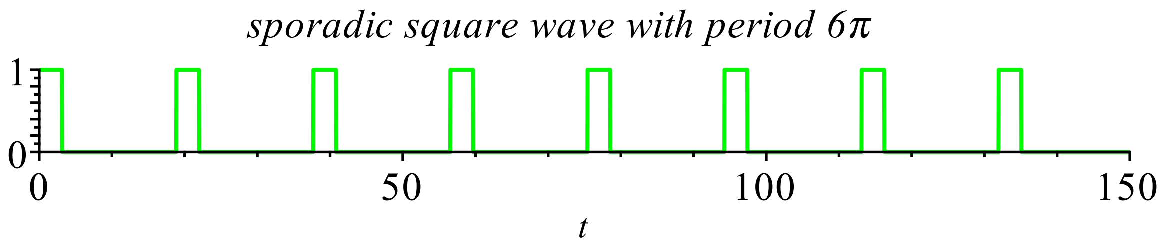
$plot3b := plot(x3(t), t = 0 ..20, color = black) :$
$display(\{plot3a, plot3b\}, title = `resonance\ response\ ?`);$



Example 4) Forcing not at the natural period, e.g. with a particular wave having period $T = 6\pi$.

> $f4 := t \rightarrow \sum_{n=0}^{10} \left(\text{Heaviside}(t - 6 \cdot n \cdot \pi) - \text{Heaviside}(t - (6 \cdot n + 1) \cdot \pi)\right) :$

$plot4a := plot(f4(t), t = 0..150, color = green) :$
$display(plot4a, title = sporadic\ square\ wave\ with\ period\ 6\pi);$

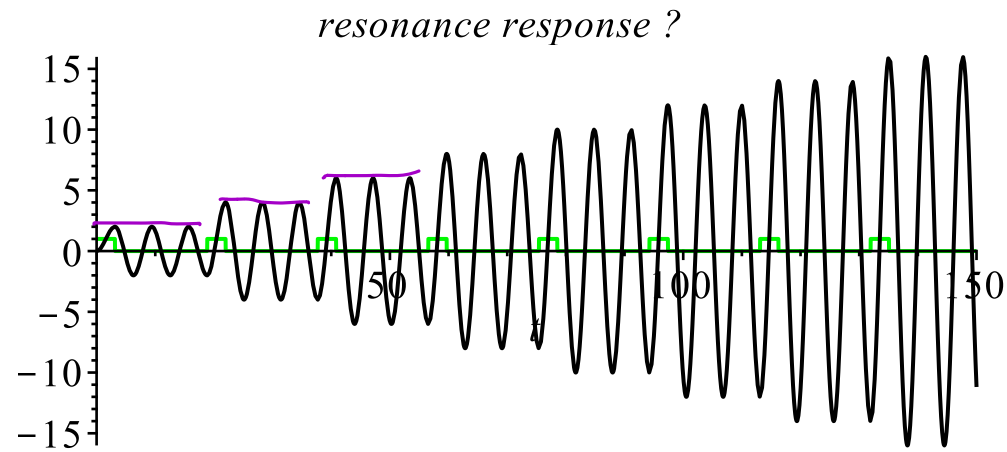


4) Resonance?

> $x4 := t \rightarrow \int_0^t \sin(\tau) \cdot f4(t-\tau)\, d\tau :$

$plot4b := plot(x4(t), t = 0..150, color = black) :$

$display(\{plot4a, plot4b\}, title = \text{`resonance response ?`});$



>

Hey, what happened???? How do we need to modify our thinking if we force a system with something which is not sinusoidal, in terms of worrying about resonance? In the case that this was modeling a swing (pendulum), how is it getting pushed?

Precise Answer: It turns out that any periodic function with period P is a (possibly infinite) superposition of a constant function with *cosine* and *sine* functions of periods $\left\{P, \frac{P}{2}, \frac{P}{3}, \frac{P}{4}, \ldots\right\}$. Equivalently, these functions in the superposition are

$\left\{1, \cos(\omega\, t), \sin(\omega\, t), \cos(2\,\omega\, t), \sin(2\,\omega\, t), \cos(3\,\omega\, t), \sin(3\,\omega\, t), \ldots\right\}$ with $\omega = \frac{2\cdot\pi}{P}$. This is the theory of Fourier series, which you will study in other courses, e.g. Math 3150, Partial Differential Equations. If the given periodic forcing function $f(t)$ has non-zero terms in this superposition for which $n\cdot\omega = \omega_0$ (the natural angular frequency) (equivalently $\frac{P}{n} = \frac{2\cdot\pi}{\omega_0} = T_0$), there will be resonance; otherwise, no resonance. We could already have understood some of this in Chapter 5, for example

Exercise 5) The natural period of the following DE is (still) $T_0 = 2\,\pi$. Notice that the period of the first forcing function below is $T = 6\,\pi$ and that the period of the second one is $T = T_0 = 2\,\pi$. Yet, it is the first DE whose solutions will exhibit resonance, not the second one. Explain, using Chapter 5 superposition ideas:

a)

$$x''(t) + x(t) = \cos(t) + \sin\left(\frac{t}{3}\right).$$

b)

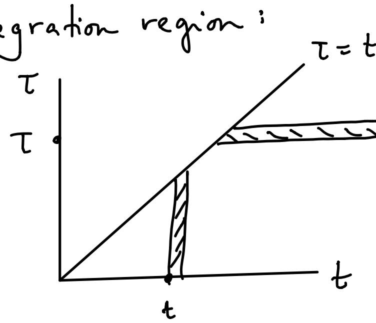
$$x''(t) + x(t) = \cos(2\,t) - 3\,\sin(3\,t).$$

proof of the convolution theorem:

$$\mathcal{L}\{f*g\}(s) = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)g(t-\tau)\,d\tau\right)dt$$

$$= \int_0^\infty\int_0^t e^{-st} f(\tau)g(t-\tau)\,d\tau\,dt \quad (1)$$

integration region:



$0 \le \tau < \infty$ (2)
$\tau \le t < \infty$

interchange limits:

$0 \le t < \infty$
$0 \le \tau \le t$
(1)

$$= \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)g(t-\tau)\,dt\,d\tau$$

$$= \int_0^\infty \int_\tau^\infty e^{-s\tau} f(\tau)\, e^{-s(t-\tau)} g(t-\tau)\,dt\,d\tau$$

pattern recognition

$$= \int_0^\infty e^{-s\tau} f(\tau)\left(\int_\tau^\infty e^{-s(t-\tau)} g(t-\tau)\,dt\right)d\tau$$

$\tilde{t} = t-\tau$
$d\tilde{t} = dt$

$$\left(\underbrace{\int_0^\infty e^{-s\tilde{t}} g(\tilde{t})\,d\tilde{t}}_{G(s)}\right)$$

$$= G(s)\int_0^\infty e^{-s\tau} f(\tau)\,d\tau$$

$$= G(s)F(s) \;!!$$

Tues Apr 10
6.1-6.2 Eigenvalues and eigenvectors for square matrices

T, W : matrix alg & vector spaces, F : begin systems of DE's, use T, W.

Announcements:
- pick up graded HW, quizzes, etc!
- quiz tomorrow "choice"
    - either Laplace transform : on-off, convolution, δ fcn.

    OR

    - evals - evects.

'til 10:46

Warm-up Exercise: Can you describe geometrically what the matrix transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ does:

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ x_2 \end{bmatrix} \quad ?$$

stretching in x_1-dir by factor of 3
leaving x_2-dir. alone ...

6.1-6.2 Eigenvalues and eigenvectors for square matrices.

The study of eigenvalues and eigenvectors is a return to matrix linear algebra, and the concepts we discuss will help us study linear systems of differential equations, in Chapter 7. Such systems of DE's arise naturally in the contexts of

- coupled input-output models, with several components.
- coupled mass-spring or RLC circuit loops, with several components.

To introduce the idea of eigenvalues and eigenvectors we'll first think geometrically.

Example Consider the matrix transformation T $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ with formula

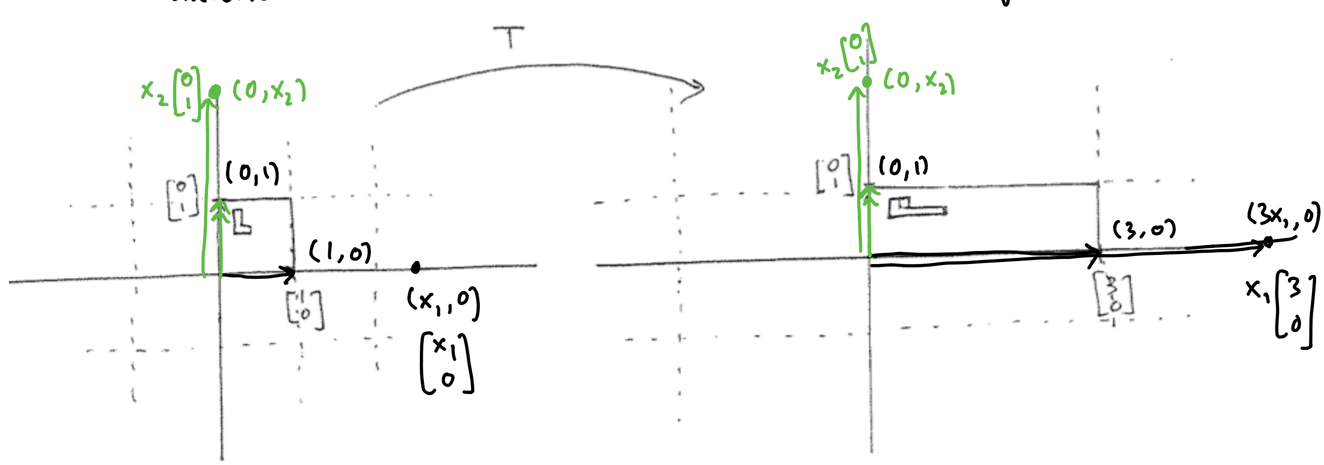
T for "transformation" but it's just a function.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ x_2 \end{bmatrix}$$

Notice that for the standard basis vectors $\underline{e}_1 = [1, 0]^T, \underline{e}_2 = [0, 1]^T$

$$T(\underline{e}_1) = 3\underline{e}_1$$
$$T(\underline{e}_2) = \underline{e}_2 \ .$$

In other words, looking at the equations above, T stretches by a factor of 3 in the $\underline{e}_1$ direction, and by a factor of 1 in the $\underline{e}_2$ direction, transforming a square grid in the domain into a parallel rectangular grid in the image:

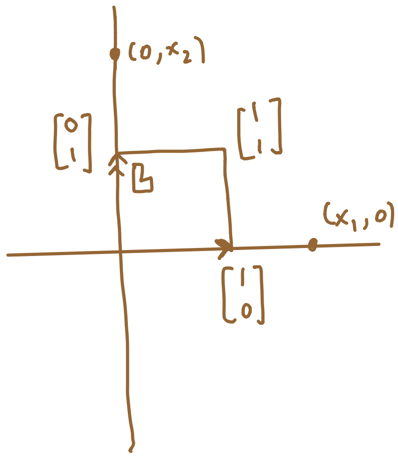


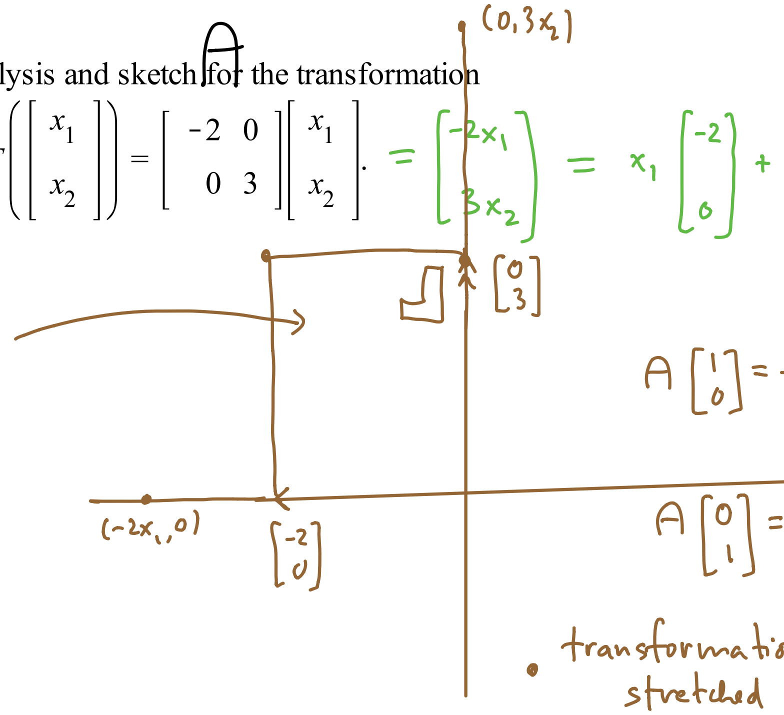
$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$A\begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = x_1\begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Exercise 1) Do a similar geometric analysis and sketch for the transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \overset{A}{\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. = \begin{bmatrix} -2x_1 \\ 3x_2 \end{bmatrix} = x_1 \begin{bmatrix} -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$





$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- transformation stretched by 2 horiz (& reflected across x_2-axis)
- stretched by 3 vertically

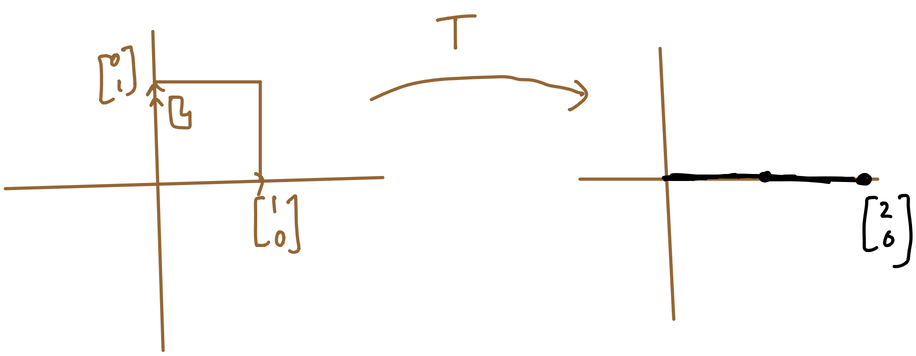
Exercise 2) And for the transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \overset{A}{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. = \begin{bmatrix} 2x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

stretched by 2 horiz. squashed to 0 vertically

$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



Definition: If $A_{n \times n}$ and if $A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$ for a scalar λ and a vector $\underline{\boldsymbol{v}} \neq \underline{\boldsymbol{0}}$ then $\underline{\boldsymbol{v}}$ is called an eigenvector of A, and λ is called the eigenvalue of $\underline{\boldsymbol{v}}$. (In some texts the words characteristic vector and characteristic value are used as synonyms for these words.)

$\tilde{e}_1, \tilde{e}_2$

- In the three examples above, the standard basis vectors (or multiples of them) were eigenvectors, and the corresponding eigenvalues were the diagonal matrix entries. A non-diagonal matrix may still have eigenvectors and eigenvalues, and this geometric information can still be important to find. But how do you find eigenvectors and eigenvalues for non-diagonal matrices? ...

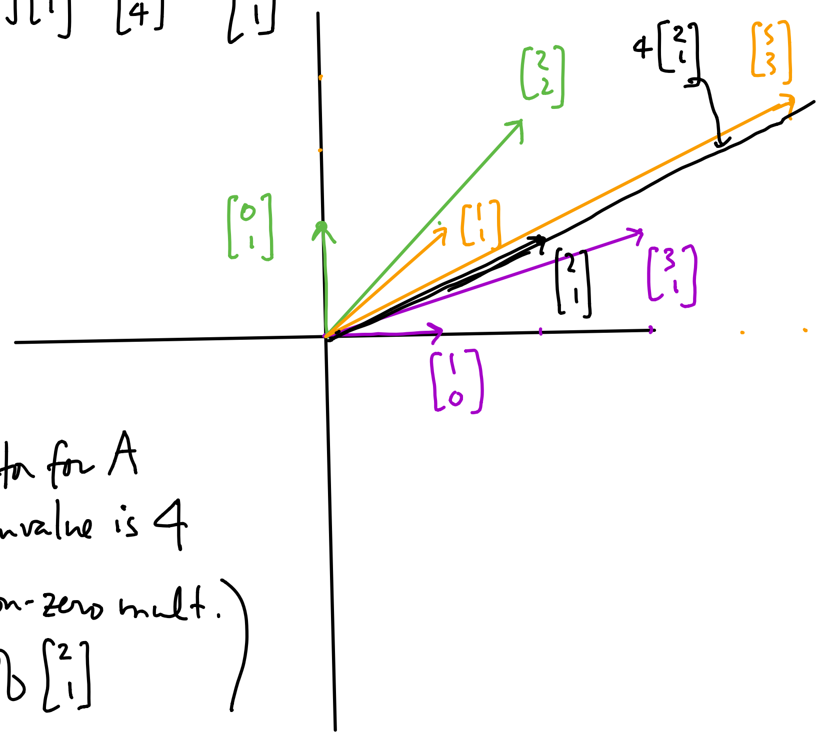
Exercise 3) Try to find eigenvectors and eigenvalues for the non-diagonal matrix, by just trying random input vectors $\underline{\boldsymbol{x}}$ and computing $A\,\underline{\boldsymbol{x}}$.

$$A\vec{v} = \lambda\vec{v}$$

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\vec{v}$	$A\vec{v}$	
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$	failed
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 5 \\ 3 \end{bmatrix}$	
$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 8 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	
$c\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$4\left(c\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$	



$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ eigenvector for A

eigenvalue is 4

(so is any non-zero mult. of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$)

How to find eigenvalues and eigenvectors (including eigenspaces) systematically:

If

$$A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$$
$$\Leftrightarrow A\,\underline{\boldsymbol{v}} - \boldsymbol{\lambda}\,\underline{\boldsymbol{v}} = \underline{\mathbf{0}}$$
$$\Leftrightarrow A\,\underline{\boldsymbol{v}} - \lambda\,I\,\underline{\boldsymbol{v}} = \underline{\mathbf{0}}$$

I = identity matrix

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ etc.

where I is the identity matrix.

$$\Leftrightarrow (A - \lambda\,I)\underline{\boldsymbol{v}} = \underline{\mathbf{0}}\,.$$

As we know, this last equation can have non-zero solutions $\underline{\boldsymbol{v}}$ if and only if the matrix $(A - \lambda\,I)$ is not invertible, i.e.

$$\Leftrightarrow det(A - \lambda\,I) = 0\,.$$

So, to find the eigenvalues and eigenvectors of matrix you can proceed as follows:

- Compute the polynomial in λ

1

$$p(\lambda) = det(A - \lambda\,I)\,.$$

$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix}$

If $A_{n \times n}$ then $p(\lambda)$ will be degree n. This polynomial is called the characteristic polynomial of the matrix A.

eigenvalues roots of $p(\lambda)$, i.e. solve $p(\lambda) = 0$

- λ_j can be an eigenvalue for some non-zero eigenvector $\underline{\boldsymbol{v}}$ if and only if it's a root of the characteristic polynomial, i.e. $p(\lambda_j) = 0$. For each such root, the homogeneous solution space of vectors $\underline{\boldsymbol{v}}$ solving

- $(A - \lambda_j\,I)\underline{\boldsymbol{v}} = \underline{\mathbf{0}}$

$(A\vec{v} = \lambda_j \vec{v})$

will be eigenvectors with eigenvalue λ_j. This subspace of eigenvectors will be at least one dimensional, since $(A - \lambda_j\,I)$ does not reduce to the identity and so the explicit homogeneous solutions will have free parameters. Find a basis of eigenvectors for this subspace. Follow this procedure for each eigenvalue, i.e. for each root of the characteristic polynomial.

Notation: The subspace of eigenvectors for eigenvalue λ_j is called the λ_j eigenspace, and denoted by $E_{\lambda = \lambda_j}$. (We include the zero vector in $E_{\lambda = \lambda_j}$.) The basis of eigenvectors is called an eigenbasis for $E_{\lambda = \lambda_j}$.

Exercise 4) a) Use the systematic algorithm to find the eigenvalues and eigenbases for the non-diagonal matrix of Exercise 3.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}.$$

b) Use your work to describe the geometry of the linear transformation in terms of directions that get stretched:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

① $|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 2$

$= (\lambda-3)(\lambda-2) - 2$

$= \lambda^2 - 5\lambda + 6 - 2$

$= \lambda^2 - 5\lambda + 4 = p(\lambda)$

eigenvalues are roots of p, i.e. $p(\lambda) = 0$

$= (\lambda - 4)(\lambda - 1) = 0$.

eigenvalues $\lambda = 4$
$\lambda = 1$.

$E_{\lambda=4} =$

$(A - 4I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$v_1 = 2t$
$v_2 = t$

$\vec{v} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$E_{\lambda=4} = span\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

shortcut:
$2\,col_1 + 1\cdot col_2 = \vec{0}$
so $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is soltn.

$E_{\lambda=1}$:

$(A - 1I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$\Downarrow$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$E_{\lambda=1} = span\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

$-1\cdot col_1 + 1\cdot col_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

long cut: backsolve
$v_1 = -t$
$v_2 = t$
$\vec{v} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

<u>Exercise 5)</u> Find the eigenvalues and eigenspace bases for

$$B := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}. \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(i) Find the characteristic polynomial and factor it to find the eigenvalues.
(ii) for each eigenvalue, find bases for the corresponding eigenspaces.
(iii) Can you describe the transformation $T(\underline{x}) = B\underline{x}$ geometrically using the eigenbases? Does $det(B)$ have anything to do with the geometry of this transformation?

① find eigenvalues. $|B - \lambda I|$ cubic roots

$$\begin{vmatrix} 4-\lambda & -2 & 1 \\ 2 & -\lambda & 1 \\ 2 & -2 & 3-\lambda \end{vmatrix} = 1\begin{vmatrix} 2 & -\lambda \\ 2 & -2 \end{vmatrix} - 1\begin{vmatrix} 4-\lambda & -2 \\ 2 & -2 \end{vmatrix} + (3-\lambda)\begin{vmatrix} 4-\lambda & -2 \\ 2 & -\lambda \end{vmatrix}$$

$$= 1(-4+2\lambda) - 1(\underbrace{-8+2\lambda+4}_{-4+2\lambda}) + (3-\lambda)\underbrace{[(4-\lambda)(-\lambda)+4]}_{[\lambda^2-4\lambda+4]}$$

$$= \cancel{2(\lambda-2)} - \underbrace{\cancel{2(\lambda-2)}}_{4-2\lambda}$$

got lucky

$$= (3-\lambda)(\lambda-2)^2, \quad \text{evals } \lambda = 3, \lambda = 2$$

$E_{\lambda=2}$

$(B-2I)\vec{v} = \vec{0}$

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ 2 & -2 & 1 & 0 \end{array}\right]$$

$\frac{R_1}{2} \to R_1$

$$\left[\begin{array}{ccc|c} 1 & -1 & .5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_1 = t_2 - .5t_3$
$v_2 = t_2$
$v_3 = t_3$

$$\vec{v} = t_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t_3\begin{bmatrix} -.5 \\ 0 \\ 1 \end{bmatrix}$$

$$E_{\lambda=2} = span\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}\right\}$$

$$\left(\begin{array}{l} 1\cdot\vec{col}_1 + 1\cdot\vec{col}_2 + 0\,\vec{col}_3 = \vec{0} \\ -1\cdot\vec{col}_1 \qquad\qquad + 2\,\vec{col}_3 = \vec{0} \end{array}\right)$$

$E_{\lambda=3}$:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & -3 & 1 & 0 \\ 2 & -2 & 0 & 0 \end{array}\right]$$

$-2R_1 + R_2 \to R_2$
$-2R_1 + R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right]$$

$-2R_2 + R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

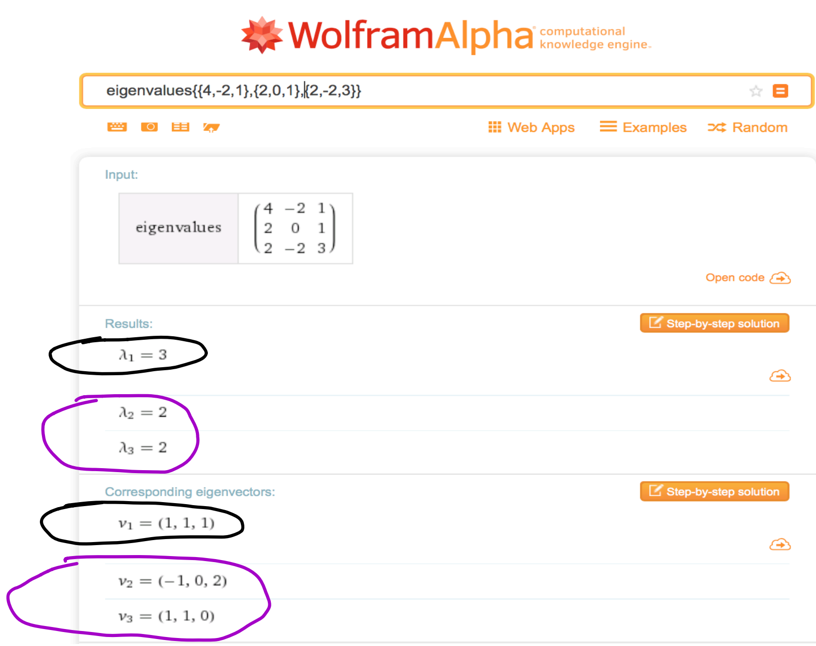
$2R_2 + R_1 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$E_{\lambda=3} = span\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$$

$$\left(\begin{array}{l} v_1 = t \\ v_2 = t \\ v_3 = t \end{array} \quad \vec{v} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

Your solution will be related to the output below:



In all of our examples so far, it turns out that by collecting bases from each eigenspace for the matrix $A_{n \times n}$, and putting them together, we get a basis for $\mathbb{R}^n$. This lets us understand the geometry of the transformation

$$T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$$

almost as well as if A is a diagonal matrix. This is actually something that does not always happen for a matrix A. When it does happen, we say that A is diagonalizable. We'll see tomorrow that not all matrices are diagonalizable.

Wed Apr 11
6.1-6.2 Eigenvalues and eigenvectors for square matrices; diagonalizability for matrices

Announcements:

- "Hand-in" folders are from wrong class, but hand in HW into them anyways. :)
- "choice" Quiz
- Tuesday notes expl.

'til 10:46

Warm-up Exercise: Find all eigenvalues, and a basis for each eigenspace, for the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$\lambda = 2, 3$

$E_{\lambda=2} = span\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$

$E_{\lambda=3} = span\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$

evect $\vec{v} \neq \vec{0}$:

$A\vec{v} = \lambda\vec{v}$

$A\vec{v} - \lambda\vec{v} = \vec{0}$

$(A - \lambda I)\vec{v} = \vec{0}$

① $\underbrace{|A - \lambda I|}_{p(\lambda)} = 0$

$p(\lambda) = 0$

find roots λ_j

② $E_{\lambda=\lambda_j}$ find basis

recipe

① $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)$

roots $\lambda = 2, 3$.

② $E_{\lambda=2}$

$(A - 2I)\vec{v} = \vec{0}$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$

backsolve

$v_1 = t$

$v_2 = 0$

$\vec{v} = t\begin{bmatrix}1\\0\end{bmatrix}$

or: $1 \cdot col_1 + 0\, col_2 = \vec{0}$

$\begin{bmatrix}1\\0\end{bmatrix}$ is evect

$E_{\lambda=3}$

$\left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$v_1 = 0$

$v_2 = t$

$\vec{v} = t\begin{bmatrix}0\\1\end{bmatrix}$

Recall from yesterday,

<u>Definition</u>: If $A_{n \times n}$ and if $A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$ for some scalar λ and vector $\underline{\boldsymbol{v}} \neq \underline{\mathbf{0}}$ then $\underline{\boldsymbol{v}}$ is called an <u>eigenvector</u> of $\underline{A}$, and λ is called the <u>eigenvalue</u> of $\underline{\boldsymbol{v}}$ (and an eigenvalue of A).

- For general matrices, the eigenvector equation $A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$ can be rewritten as

$$(A - \lambda I)\underline{\boldsymbol{v}} = \underline{\mathbf{0}} \ .$$

The only way such an equation can hold for $\underline{\boldsymbol{v}} \neq \underline{\mathbf{0}}$ is if the matrix $(A - \lambda I)$ does not reduce to the identity matrix. In other words - $det(A - \lambda I)$ must equal zero. Thus the only possible eigenvalues associated to a given matrix must be roots λ_j of the <u>characteristic polynomial</u>

$$p(\lambda) = det(A - \lambda I) \ .$$

- So, the first step in finding eigenvectors for A is actually to find the eigenvalues - by finding the characteristic polynomial and its roots λ_j .

- For each root λ_j the matrix $A - \lambda_j I$ will not reduce to the identity, and the solution space to

$$\left(A - \lambda_j I\right)\underline{\boldsymbol{v}} = \underline{\mathbf{0}}$$

will be at least one-dimensional, and have a basis of one or more eigenvectors. Find such a basis for this λ_j <u>eigenspace</u> $E_{\lambda = \lambda_j}$ by reducing the homogeneous matrix equation

$$\left(A - \lambda_j I\right)\underline{\boldsymbol{v}} = \underline{\mathbf{0}} \ ,$$

backsolving and extracting a basis. We can often "see" and eigenvector by realizing that homogeneous solutions to a matrix equation correspond to column dependencies.

- Finish any leftover exercises from Tuesday

Exercise 1) If your matrix A is diagonal, the general algorithm for finding eigenspace bases just reproduces the entries along the diagonal as eigenvalues, and the corresponding standard basis vectors as eigenspace bases. (Recall our diagonal matrix examples from yesterday, where the standard basis vectors were eigenvectors. This is typical for diagonal matrices.) Illustrate how this works for a 3×3 diagonal matrix, so that in the future you can just read of the eigendata if the matrix you're given is (already) diagonal:

$$A := \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}.$$

skip like warmup.

step 1) Find the roots of the characteristic polynomial $det(A - \lambda I)$.

step 2) Find the eigenspace bases, assuming the values of a_{11}, a_{22}, a_{33} are distinct (all different). What if $a_{11} = a_{22}$ but these values do not equal a_{33} ?

$$A\vec{e}_1 = a_{11}\vec{e}_1$$
$$A\vec{e}_2 = a_{22}\vec{e}_2$$
$$A\vec{e}_3 = a_{33}\vec{e}_3$$

$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ basis for $\mathbb{R}^3$

In most of our examples so far, it turns out that by collecting bases from each eigenspace for the matrix $A_{n \times n}$, and putting them together, we get a basis for $\mathbb{R}^n$. This lets us understand the geometry of the transformation

$$T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$$

almost as well as if A is a diagonal matrix, and so we call such matrices diagonalizable. Having such a basis of eigenvectors for a given matrix is also extremely useful for algebraic computations, and will give another reason for the word diagonalizable to describe such matrices.

Use the $\mathbb{R}^3$ basis made of out eigenvectors of the matrix B in Exercise 5 of Monday's notes, and put them into the columns of a matrix we will call P. We could order the eigenvectors however we want, but we'll put the $E_{\lambda=2}$ basis vectors in the first two columns, and the $E_{\lambda=3}$ basis vector in the third column. Here is the data from that problem:

$$B = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix} \qquad E_{\lambda=2} = span\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right\} \qquad E_{\lambda=3} = span\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

$$P := \begin{array}{c} \begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix} \end{array}.$$

Now do algebra (check these steps and discuss what's going on!)

$$\begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix} \left[\begin{array}{c|c|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} 0 & 2 & 3 \\ 2 & 0 & 3 \\ 4 & -4 & 3 \end{array}\right] \qquad \left[\begin{array}{c|c|c} 2\vec{v}_1 & 2\vec{v}_2 & 3\vec{v}_3 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{array}\right] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

In other words,

$$B\,P = P\,\mathrm{D}\,,$$

where D is the diagonal matrix of eigenvalues (for the corresponding columns of eigenvectors in P).
Equivalently (multiply on the right by P^{-1} or on the left by P^{-1}):

$$B = P\,\mathrm{D}\,P^{-1} \text{ and } P^{-1}BP = \mathrm{D}.$$

$$BP = PD$$
$$B = B\underbrace{PP^{-1}}_{I} = PDP^{-1}$$

$$BP = PD$$
$$P^{-1}BP = P^{-1}PD = D$$

Exercise 2) Use one of the the identities above to show how B^{100} can be computed with only two matrix multiplications!

$$D^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2^2 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 3^2 \end{bmatrix}$$

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} e & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & g \end{bmatrix} = \begin{bmatrix} ae & 0 & 0 \\ 0 & bf & 0 \\ 0 & 0 & cg \end{bmatrix}$$

$$D^{100} = \begin{bmatrix} 2^{100} & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 3^{100} \end{bmatrix}$$

$$B = PDP^{-1}$$

$$B^{100} = \underbrace{PDP^{-1}}\ \underbrace{PDP^{-1}}\ \underbrace{PDP^{-1}} \cdots \underbrace{PDP^{-1}}$$

$$I \qquad I \qquad I$$

$$= PD^{100}P^{-1}$$

Definition: Let $A_{n \times n}$. If there is an $\mathbb{R}^n$ (or $\mathbb{C}^n$) basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n$ consisting of eigenvectors of A, then A is called diagonalizable. This is precisely why:

Write $A\,\underline{\boldsymbol{v}}_j = \lambda_j\,\underline{\boldsymbol{v}}_j$ (some of these λ_j may be the same, as in the previous example). Let P be the matrix

$$P = \left[\underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2 | \cdots | \underline{\boldsymbol{v}}_n\right].$$

Then, using the various ways of understanding matrix multiplication, we see

$$A\,P = A\left[\underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2 | \cdots | \underline{\boldsymbol{v}}_n\right] = \left[\lambda_1\underline{\boldsymbol{v}}_1 | \lambda_2\underline{\boldsymbol{v}}_2 | \cdots | \lambda_n\underline{\boldsymbol{v}}_n\right]$$

$$= \left[\underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2 | \cdots | \underline{\boldsymbol{v}}_n\right] \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ : & : & \ldots & : \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

$$A\,P = P\,\mathrm{D}$$

$$A = P\,\mathrm{D}\,P^{-1}$$

$$P^{-1}A\,P = \mathrm{D}\,.$$

Unfortunately, not all matrices are diagonalizable:

Exercise 3) Show that

$$C := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

skip

is not diagonalizable.

$$|C - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)^2(3-\lambda)$$

$$E_{\lambda=2} = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

$$E_{\lambda=3} = \mathrm{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

<u>Facts about diagonalizability</u> (see text section 6.2 for complete discussion, with reasoning):

Let $A_{n \times n}$ have factored characteristic polynomial

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)^{k_1} (\lambda - \lambda_2)^{k_2} \ldots (\lambda - \lambda_m)^{k_m}$$

where like terms have been collected so that each λ_j is distinct (i.e different). Notice that

$$k_1 + k_2 + \ldots + k_m = n$$

because the degree of $p(\lambda)$ is n.

- Then $1 \le dim\left(E_{\lambda = \lambda_j}\right) \le k_j$. If $dim\left(E_{\lambda = \lambda_j}\right) < k_j$ then the λ_j eigenspace is called <u>defective.</u>
- The matrix A is diagonalizable if and only if each $dim\left(E_{\lambda = \lambda_j}\right) = k_j$. In this case, one obtains an $\mathbb{R}^n$

eigenbasis simply by combining bases for each eigenspace into one collection of n vectors. (Later on, the same definitions and reasoning will apply to complex eigenvalues and eigenvectors, and a basis of $\mathbb{C}^n$.)

- In the special case that A has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ each eigenspace is forced to be

1-dimensional since $k_1 + k_2 + \ldots + k_n = n$ so each $k_j = 1$. Thus A is automatically diagonalizable as a special case of the second bullet point.

<u>Exercise 4)</u> How do the examples from today and yesterday compare with the general facts about diagonalizability?

<u>a)</u> $A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$, $p(\lambda) = (\lambda - 4)(\lambda - 1)$; $E_{\lambda=4} = span\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$, $E_{\lambda=1} = span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$. diagonalizable

<u>b)</u> $B = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$, $p(\lambda) = -(\lambda - 2)^2(\lambda - 3)$, $E_{\lambda=2} = span\left\{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\}$, $E_{\lambda=3} = span\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$. diagonalizable

<u>c)</u> $C := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $p(\lambda) = -(\lambda - 2)^2(\lambda - 3)$, $E_{\lambda=2} = span\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$, $E_{\lambda=3} = span\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

<u>Not</u> diagonalizable

$E_{\lambda=2}$ is defective

because $p(\lambda)$ contains factor of $(\lambda-2)^2$

but $dim(E_{\lambda=2})$ only $=1$

Fri Apr 13

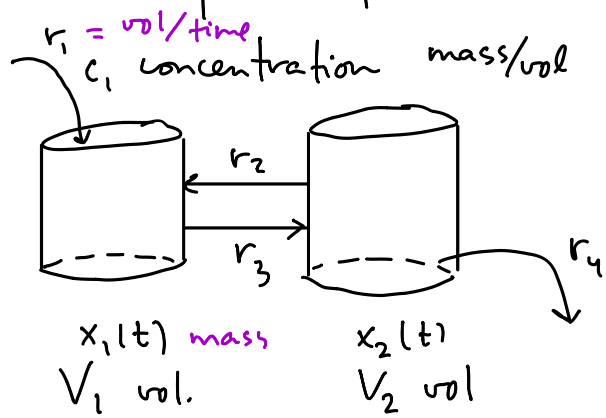
7.1 Systems of differential equations - to model multi-component systems via compartmental analysis:
http://en.wikipedia.org/wiki/Multi-compartment_model

Announcements:
- You'll be computing eigenvalues & eigenvectors for the rest of course
  purpose will be to solve systems of linear differential eqtns
- Today is sort of an overview of how that happens
  (Chapter 7, last chapter of course)

(warm-up for Lab #3)

'til 10:46

Warm-up Exercise: Let $x_1(t)$, $x_2(t)$ be solute amounts in this 2-component input-output model



$\frac{\text{vol}}{\text{time}}$

$V_1'(t) = 0$.

$V_1'(t) = r_1 + r_2 - r_3 = 0$

$V_2'(t) = 0$

$V_2'(t) = -r_2 - r_4 + r_3 = 0$

a) What condition on the rates r_1, r_2, r_3, r_4 (vol/time) keeps V_1, V_2 constant?

b) What are the DE's for $x_1(t)$ & $x_2(t)$?
(assume V_1, V_2 const)

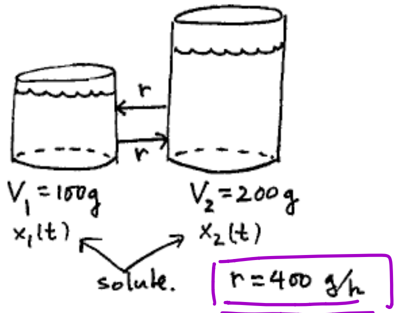
$$x_1'(t) = r_1 c_1 + r_2 \frac{x_2}{V_2} - r_3 \frac{x_1}{V_1}$$

$\frac{\text{mass}}{\text{time}}$ $\quad \frac{\cancel{\text{vol}}}{\text{time}} \frac{\text{mass}}{\cancel{\text{vol}}}$

↑ average concentration in tank (at time t)

$$x_2'(t) = r_3 \frac{x_1}{V_1} - (r_2 + r_4) \frac{x_2}{V_2}$$

Here's a relatively simple 2-tank problem to illustrate the ideas:



g = gallons
h = hours

Exercise 1) Find differential equations for solute amounts $x_1(t), x_2(t)$ above, using input-output modeling. Assume solute concentration is uniform in each tank. If $x_1(0) = b_1, x_2(0) = b_2$, write down the initial value problem that you expect would have a unique solution.

$$x_1'(t) = r \cdot \frac{x_2}{200} - r\frac{x_1}{100} = 400\frac{x_2}{200} - 400\frac{x_1}{100} = -4x_1 + 2x_2$$

$$x_2' = 400\frac{x_1}{100} - 400\frac{x_2}{200} = 4x_1 - 2x_2$$

IVP
$$\begin{cases} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \begin{bmatrix} x_1(0) \\ x_1'(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \end{cases}$$

answer (in matrix-vector form):

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Geometric interpretation of first order systems of differential equations.

The example on page 1 is a special case of the general initial value problem for a first order system of differential equations:

vector version of Chapter 1 existence-uniqueness

IVP $\left\{ \begin{array}{l} \underline{\boldsymbol{x}}'(t) = \underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t)) \\ \underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0 \end{array} \right.$

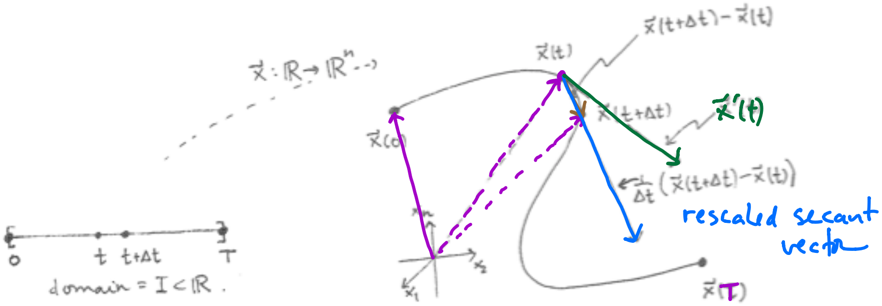
know "veloc." vect. in terms time & current location

know where we start

- We will see how any single differential equation (of any order), or any system of differential equations (of any order) is equivalent to a larger first order system of differential equations. And we will discuss how the natural initial value problems correspond.

Why we expect IVP's for first order systems of DE's to have unique solutions $\underline{\boldsymbol{x}}(t)$:

- From either a multivariable calculus course, or from physics, recall the geometric/physical interpretation of $\underline{\boldsymbol{x}}'(t)$ as the tangent/velocity vector to the parametric curve of points with position vector $\underline{\boldsymbol{x}}(t)$, as t varies. This picture should remind you of the discussion, but ask questions if this is new to you:



Analytically, the reason that the vector of derivatives $\underline{\boldsymbol{x}}'(t)$ computed component by component is actually a limit of scaled secant vectors (and therefore a tangent/velocity vector) is:

$$\underline{\boldsymbol{x}}'(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\begin{bmatrix} x_1(t+\Delta t) \\ x_2(t+\Delta t) \\ : \\ x_n(t+\Delta t) \end{bmatrix} - \begin{bmatrix} x_1(t) \\ x_2(t) \\ : \\ x_n(t) \end{bmatrix} \right)$$

$$= \lim_{\Delta t \to 0} \begin{bmatrix} \frac{1}{\Delta t}\left(x_1(t+\Delta t) - x_1(t)\right) \\ \frac{1}{\Delta t}\left(x_2(t+\Delta t) - x_2(t)\right) \\ : \\ \frac{1}{\Delta t}\left(x_n(t+\Delta t) - x_n(t)\right) \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ : \\ x_n'(t) \end{bmatrix},$$

provided each component function is differentiable. Therefore, the reason you expect a unique solution to the IVP for a first order system is that you know where you start ($\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$), and you know your "velocity" vector (depending on time and current location) $\Rightarrow$ you expect a unique solution! (Plus, you could use something like a vector version of Euler's method or the Runge-Kutta method to approximate it! You just convert the scalar quantities in the code into vector quantities. And this is what numerical solvers do.)

Exercise 2) Return to the page 1 tank example

$$\left.\begin{aligned} x_1'(t) &= -4\,x_1 + 2\,x_2 \\ x_2'(t) &= 4\,x_1 - 2\,x_2 \end{aligned}\right\}$$

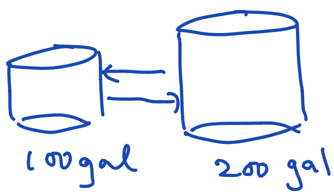
$x_1(0) = 9 \leftarrow$ 9 lbs in tank 1

$x_2(0) = 0 \leftarrow$ 0 lbs in tank 2.

2a) Interpret the parametric solution curve $[x_1(t), x_2(t)]^T$ to this IVP, as indicated in the pplane screen shot below. ("pplane" is the sister program to "dfield", that we were using in Chapters 1-2.) Notice how it follows the "velocity" vector field (which is time-independent in this example), and how the "particle motion" location $[x_1(t), x_2(t)]^T$ is actually the vector of solute amounts in each tank, at time *t*. If your system involved ten coupled tanks rather than two, then this "particle" is moving around in $\mathbb{R}^{10}$.
2b) What are the apparent limiting solute amounts in each tank?
2c) How could your smart-alec younger sibling have told you the answer to 2b without considering any differential equations or "velocity vector fields" at all?

$$\lim_{t\to\infty} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

2c)



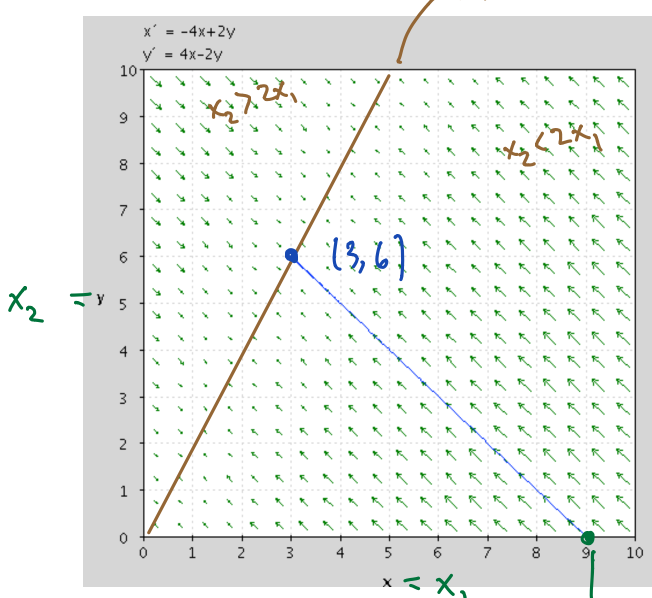
expect same concentrations in each tank as $t \to \infty$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

& expect total of 9 lbs & twice as much in 2nd tank because twice the vol.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = (\underbrace{2x_1 - x_2}_{0}) \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$x_2 = 2x_1$



$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

Chapter scalar eqtns

First order systems of differential equations of the form

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

are called linear homogeneous systems of DE's. (Think of rewriting the system as

$$\underline{\boldsymbol{x}}'(t) - A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$$

in analogy with how we wrote linear scalar differential equations.) Then the inhomogeneous system of first order DE's would be written as

$$\boxed{\underline{\boldsymbol{x}}'(t) - A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{f}}(t)}$$

or

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}} + \underline{\boldsymbol{f}}(t)$$

$x'(t) + p(t)x = 0$

$x'(t) + p(t)x(t) = q(t)$

Notice that the operator on vector-valued functions $\underline{\boldsymbol{x}}(t)$ defined by

$$L(\underline{\boldsymbol{x}}(t)) := \underline{\boldsymbol{x}}'(t) - A\,\underline{\boldsymbol{x}}(t)$$

is linear, i.e.

$$L(\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{y}}(t)) = L(\underline{\boldsymbol{x}}(t)) + L(\underline{\boldsymbol{y}}(t))$$
$$L(c\,\underline{\boldsymbol{x}}(t)) = c\,L(\underline{\boldsymbol{x}}(t)).$$

$$L(\vec{x}(t)+\vec{y}(t)) = (\vec{x}'+\vec{y}') - A(\vec{x}+\vec{y}) = (\vec{x}'-A\vec{x}) + (\vec{y}'-A\vec{y}) = L(\vec{x}) + L(\vec{y})$$

SO! The space of solutions to the homogeneous first order system of differential equations

$$\underline{\boldsymbol{x}}'(t) - A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$$

is a subspace. AND the general solution to the inhomogeneous system

$$\underline{\boldsymbol{x}}'(t) - A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{f}}(t)$$

will be of the form

$$\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}_P + \underline{\boldsymbol{x}}_H$$

where $\underline{\boldsymbol{x}}_P$ is any single particular solution and $\underline{\boldsymbol{x}}_H$ is the general homogeneous solution.

Exercise 3) In the case that A is a constant matrix (i.e. entries don't depend on t), consider the homogeneous problem

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}.$$

Look for solutions of the form

$$\boxed{\underline{\boldsymbol{x}}(t) = e^{\lambda t}\underline{\boldsymbol{v}},}$$

where $\underline{\boldsymbol{v}}$ is a constant vector. Show that $\underline{\boldsymbol{x}}(t) = e^{\lambda t}\underline{\boldsymbol{v}}$ solves the homogeneous DE system if and only if v is an eigenvector of A, with eigenvalue λ, i.e. $A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$.

Hint: In order for such an $\underline{\boldsymbol{x}}(t)$ to solve the DE it must be true that

$$\underline{\boldsymbol{x}}'(t) = \lambda\,e^{\lambda t}\underline{\boldsymbol{v}}$$

and

$$A\,\underline{\boldsymbol{x}}(t) = A\,e^{\lambda t}\underline{\boldsymbol{v}} = e^{\lambda t}A\,\underline{\boldsymbol{v}}$$

Set these two expressions equal.

$$\vec{x}'(t) = A\vec{x}$$

look for soltns of form $e^{\lambda t}\vec{v} = \vec{x}$ $\quad \vec{v}$ const. vector

$$\Rightarrow \quad \lambda e^{\lambda t}\vec{v} = \vec{x}'$$

$$A\vec{x} = A(e^{\lambda t}\vec{v}) = e^{\lambda t}(A\vec{v})$$

$$\vec{x}' = A\vec{x}$$

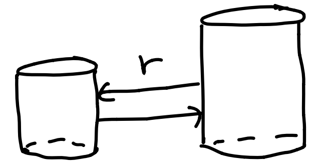
$$\lambda \cancel{e^{\lambda t}}\vec{v} = \cancel{e^{\lambda t}}A\vec{v}$$

$$\boxed{A\vec{v} = \lambda\vec{v}}$$

Exercise 4) Use the idea of Exercise 3 to solve the initial value problem of Exercise 2!! Compare your solution $\underline{\boldsymbol{x}}(t)$ to the parametric curve drawn by pplane, that we looked at a couple of pages back.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$



$x_1(t)$ $x_2(t)$

$V_1 = 100\text{ g}$ $V_2 = 200\text{ g}$

$r = 400\ \text{g/h}$

$$\begin{vmatrix} -4-\lambda & 2 \\ 4 & -2-\lambda \end{vmatrix} = (-4-\lambda)(-2-\lambda) - 8 = \cancel{8} + 6\lambda + \lambda^2 - \cancel{8} = \lambda(\lambda+6)$$

eigenvalues $\lambda = 0, -6$.

$E_{\lambda=0}$:

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \\ \hline 1 & -\tfrac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$E_{\lambda=0} = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$$

OR

$v_1 = \tfrac{1}{2}t$

$v_2 = t$

$\vec{v} = t \begin{bmatrix} \tfrac{1}{2} \\ 1 \end{bmatrix}$

$E_{\lambda=-6}$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$E_{\lambda=-6} = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \cancel{e^{0t}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

find c_1 & c_2

find $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

Exercise 5) Lessons learned from tank example: What condition on the matrix $A_{n \times n}$ will allow you to uniquely solve every initial value problem

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0 \in \mathbb{R}^n$$

using the method in Exercise 3-4 ? Hint: Chapter 6. (If that condition fails there are other ways to find the unique solutions.)