Math 2250-004   Week 12 April 2-6
continue 10.1-10.3; also cover parts of 10.4-10.5, EP 7.6

Mon Apr 2:
10.1-10.3 Laplace transform and initial value problems like we studied in Chapter 5

Announcements:

Warm-up Exercise:

Recall,

- The Laplace transform is a linear transformation "$\mathcal{L}$" that converts piecewise continuous functions $f(t)$, defined for $t \geq 0$ and with at most exponential growth ($|f(t)| \leq Ce^{Mt}$ for some values of C and M), into functions $F(s)$ defined by the transformation formula

$$F(s) = \mathcal{L}\{f(t)\}(s) := \int_0^\infty f(t)e^{-st}\, dt .$$

- Notice that the integral formula for $F(s)$ is only defined for sufficiently large s, and certainly for $s > M$, because as soon as $s > M$ the integrand is decaying exponentially, so the improper integral from $t = 0$ to ∞ converges.

- The convention is to use lower case letters for the input functions and (the same) capital letters for their Laplace transforms, as we did for $f(t)$ and $F(s)$ above. Thus if we called the input function $x(t)$ then we would denote the Laplace transform by $X(s)$.

Exercise 1) (to review) Use the table entries we computed last Wednesday, to compute

1a) $\mathcal{L}\{4-5\cos(3t)+2e^{-4t}\sin(12t)\}(s)$

1b) $\mathcal{L}^{-1}\left\{\frac{2}{s-2}+\frac{1}{s^2+2s+5}\right\}(t)$.

$f(t)$ $\lvert f(t)\rvert \le Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st}\,dt$ for $s > M$
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$
1	$\frac{1}{s}$ $(s > 0)$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$ $(s > \Re(a))$
$\cos(kt)$	$\frac{s}{s^2+k^2}$ $(s > 0)$
$\sin(kt)$	$\frac{k}{s^2+k^2}$ $(s > 0)$
$e^{at}\cos(kt)$	$\frac{(s-a)}{(s-a)^2+k^2}$ $(s > a)$
$e^{at}\sin(kt)$	$\frac{k}{(s-a)^2+k^2}$ $(s > a)$
$f'(t)$	$s\,F(s) - f(0)$
$f''(t)$	$s^2F(s) - s\,f(0) - f'(0)$

Laplace transform table

Exercise 2) (to review) Use Laplace transforms to solve the IVP for an underdamped, unforced oscillator DE. Compare to Chapter 5 method.

$$x''(t) + 6\,x'(t) + 34\,x(t) = 0$$
$$x(0) = 3$$
$$x'(0) = 1$$

We'll fill in more table entries today. (Compare to front cover of your text, which contains this information but maybe more compactly.)

$f(t)$, with $\lvert f(t)\rvert \le C e^{M t}$	$F(s) := \int_0^\infty f(t) e^{-s t}\, dt$ for $s > M$	↓ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	☐
1 t t^2 $t^n,\ n \in \mathbb{N}$	$\frac{1}{s}$ $(s > 0)$ $\frac{1}{s^2}$ $\frac{2}{s^3}$ $\frac{n!}{s^{n+1}}$	☐
$e^{\alpha t}$	$\frac{1}{s - \alpha}$ $(s > \Re(\alpha))$	☐
$\cos(k t)$ $\sin(k t)$ $\cosh(k t)$ $\sinh(k t)$ $e^{a t}\cos(k t)$ $e^{a t}\sin(k t)$	$\frac{s}{s^2 + k^2}$ $(s > 0)$ $\frac{k}{s^2 + k^2}$ $(s > 0)$ $\frac{s}{s^2 - k^2}$ $(s > k)$ $\frac{k}{s^2 - k^2}$ $(s > k)$ $\frac{(s - a)}{(s - a)^2 + k^2}$ $(s > a)$ $\frac{k}{(s - a)^2 + k^2}$ $(s > a)$	☐ ☐ ☐ ☐
$f'(t)$ $f''(t)$ $f^{(n)}(t),\ n \in \mathbb{N}$ $\int_0^t f(\tau)\, d\tau$	$s\, F(s) - f(0)$ $s^2 F(s) - s\, f(0) - f'(0)$ $s^n\, F(s) - s^{n-1} f(0) - \ldots - f^{(n-1)}(0)$ $\frac{F(s)}{s}$	☐ ☐
$t\, f(t)$ $t^2\, f(t)$ $t^n\, f(t),\ n \in \mathbb{Z}$ $\frac{f(t)}{t}$	$-F'(s)$ $F''(s)$ $(-1)^n\, F^{(n)}(s)$ $\int_s^\infty F(\sigma)\, d\sigma$	
$t\, \cos(k t)$ $\frac{1}{2 k}\, t\, \sin(k t)$	$\frac{s^2 - k^2}{(s^2 + k^2)^2}$ $\frac{s}{(s^2 + k^2)^2}$	

$\frac{1}{2k^3}(\sin(kt) - kt\cos(kt))$	$\frac{1}{(s^2+k^2)^2}$	
$e^{at}f(t)$ $t\,e^{at}$ $t^n e^{at},\ n \in \mathbb{Z}$	$F(s-a)$ $\frac{1}{(s-a)^2}$ $\frac{n!}{(s-a)^{n+1}}$	

Laplace transform table

work down the table ...

3a) $\mathcal{L}\{cosh(kt)\}(s) = \frac{s}{s^2 - k^2}$

3b) $\mathcal{L}\{\sinh(kt)\}(s) = \frac{k}{s^2 - k^2}$.

Exercise 4) Recall we used integration by parts on Wednesday to derive

$$\mathcal{L}\{g'(t)\}(s) = s\mathcal{L}\{g(t)\}(s) - g(0).$$

Use that identity to show

a) $\mathcal{L}\{f''(t)\}(s) = s^2F(s) - s\,f(0) - f'(0)$,

b) $\mathcal{L}\{f'''(t)\}(s) = s^3F(s) - s^2f(0) - s\,f'(0) - f''(0)$,

c) $\mathcal{L}\{f^{(n)}(t)\}(s) = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0),\ n \in \mathbb{N}$.

d) $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}(s) = \frac{F(s)}{s}$.

These are the identities that make Laplace transform work so well for initial value problems such as we studied in Chapter 5.... with Laplace transforms the "free parameters" when you write down the solution $x = x_P + x_H$ are exactly the initial values for the differential equation, rather than the linear combinations coefficients in the general homogeneous solution, so you definitely save a step there, in solving IVPs.

Exercise 5) Find $\mathcal{L}^{-1}\left\{\frac{1}{s\left(s^2+4\right)}\right\}(t)$

a) using the result of 4d.

b) using partial fractions.

Exercise 6) Show $\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}$, $n \in \mathbb{N}$, using the results of 4, namely

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0),\ n \in \mathbb{N}.$$

Math 2250-004
Tues Apr 3

10.2-10.3 Laplace transform, and application to DE IVPs, including Chapter 5.
Today we'll continue to fill in the Laplace transform table, and to use the table entries to solve linear differential equations. One focus today will be to review partial fractions, since the table entries are set up precisely to show the inverse Laplace transforms of the components of partial fraction decompositions.

Announcements:

Warm-up Exercise:

$$F(s) = \mathcal{L}\{f(t)\}(s) := \int_0^\infty f(t)e^{-st}\, dt\,.$$

Exercise 1) Check why this table entry is true - notice that it generalizes how the Laplace transforms of $\cos(k\,t)$, $\sin(k\,t)$ are related to those of $e^{a\,t}\cos(k\,t)$, $e^{a\,t}\sin(k\,t)$:

$e^{a\,t} f(t)$	$F(s-a)$

Exercise 2) Use the table entry

$t^n,\ n \in \mathbb{Z}$	$\dfrac{n!}{s^{n+1}}$

and Exercise 1 to get the table entry

$t^n e^{a\,t}$	$\dfrac{n!}{(s-a)^{n+1}}$

A harder table entry to understand is the following one - go through this computation and see why it seems reasonable, even though there's one step that we don't completely justify. The table entry is

$tf(t)$	$-F'(s)$

We recognize that it will be helpful for application problems where resonance occurs.

Here's how we get it:

$$F(s) = \mathcal{L}\{f(t)\}(s) := \int_0^\infty f(t)e^{-s\,t}\,dt$$

$$\Rightarrow \frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty f(t)e^{-s\,t}\,dt = \int_0^\infty \frac{d}{ds}f(t)e^{-s\,t}\,dt\,.$$

It's this last step which is true, but needs more justification. We know that the derivative of a sum is the sum of the derivatives, and the integral is a limit of Riemann sums, so this step does at least seem reasonable.

The rest is straightforward:

$$\int_0^\infty \frac{d}{ds}f(t)e^{-s\,t}\,dt = \int_0^\infty f(t)(-t)e^{-s\,t}\,dt = -\mathcal{L}\{tf(t)\}(s) \qquad \square.$$

For resonance and other applications ...

<u>Exercise 4)</u> Use $\mathcal{L}\{t f(t)\}(s) = -F'(s)$

<u>a)</u> $\mathcal{L}\{t\cos(k t)\}(s) = \dfrac{s^2 - k^2}{\left(s^2 + k^2\right)^2}$

<u>b)</u> $\mathcal{L}\left\{\dfrac{1}{2k}\, t\sin(k t)\right\}(s) = \dfrac{s}{\left(s^2 + k^2\right)^2}$

<u>c)</u> Then use <u>a</u> and the identity

$$\frac{1}{\left(s^2+k^2\right)^2} = \frac{1}{2k^2}\left(\frac{s^2+k^2}{\left(s^2+k^2\right)^2} - \frac{s^2-k^2}{\left(s^2+k^2\right)^2}\right)$$

to verify the table entry

$$\mathcal{L}^{-1}\left\{\frac{1}{\left(s^2+k^2\right)^2}\right\}(t) = \frac{1}{2k^2}\left(\frac{1}{k}\sin(k t) - t\cos(k t)\right).$$

Notice how the Laplace transform table is set up to use partial fraction decompositions. And be amazed at how it lets you quickly deduce the solutions to important DE IVPs, like this resonance problem:

Exercise 5a) Use Laplace transforms to write down the solution to

$$x''(t) + \omega_0^2\, x(t) = F_0 \sin(\omega_0\, t)$$
$$x(0) = x_0$$
$$x'(0) = v_0\,.$$

what phenomenon do the solutions to this IVP exhibit? (Compare, in your homework you will re-solve the IVP when the forcing is $F_0 \cos(\omega_0\, t)$. We worked pretty hard in Chapter 5, to derive this general solution formula.)

5b) Use Laplace transforms to solve the general undamped forced oscillation problem, when $\omega \neq \omega_0$:

$$x''(t) + \omega_0^2\, x(t) = F_0 \sin(\omega\, t)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

what phenomenon to the solutions to this IVP exhibit when $\omega \approx \omega_0$ (but $\omega \neq \omega_0$) ? (In your homework, you'll do this with forcing $F_0 \sin(\omega\, t)$. This will mirror work we did in Chapter 5.)

$f(t)$, with $\lvert f(t)\rvert \le Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st}\,dt$ for $s > M$	↓ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	☐
1 t t^2 $t^n,\ n \in \mathbb{N}$	$\frac{1}{s}$ $(s > 0)$ $\frac{1}{s^2}$ $\frac{2}{s^3}$ $\frac{n!}{s^{n+1}}$	☐
$e^{\alpha t}$	$\frac{1}{s-\alpha}$ $(s > \Re(a))$	☐
$\cos(kt)$ $\sin(kt)$ $\cosh(kt)$ $\sinh(kt)$ $e^{at}\cos(kt)$ $e^{at}\sin(kt)$ $e^{at}f(t)$	$\frac{s}{s^2+k^2}$ $(s > 0)$ $\frac{k}{s^2+k^2}$ $(s > 0)$ $\frac{s}{s^2-k^2}$ $(s > k)$ $\frac{k}{s^2-k^2}$ $(s > k)$ $\frac{(s-a)}{(s-a)^2+k^2}$ $(s > a)$ $\frac{k}{(s-a)^2+k^2}$ $(s > a)$ $F(s-a)$	☐ ☐ ☐ ☐ ☐ ☐
$f'(t)$ $f''(t)$ $f^{(n)}(t),\ n \in \mathbb{N}$ $\int_0^t f(\tau)\,d\tau$	$s\,F(s) - f(0)$ $s^2F(s) - s\,f(0) - f'(0)$ $s^n\,F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$ $\frac{F(s)}{s}$	☐ ☐ ☐ ☐
$t\,f(t)$ $t^2\,f(t)$ $t^n\,f(t),\ n \in \mathbb{Z}$ $\frac{f(t)}{t}$	$-F'(s)$ $F''(s)$ $(-1)^n\,F^{(n)}(s)$ $\int_s^\infty F(\sigma)\,d\sigma$	
$t\cos(kt)$	$\frac{s^2-k^2}{(s^2+k^2)^2}$	

$\frac{1}{2k} t \sin(k t)$	$\frac{s}{(s^2 + k^2)^2}$	
$\frac{1}{2k^3} (\sin(k t) - k t \cos(k t))$	$\frac{1}{(s^2 + k^2)^2}$	
$t e^{a t}$	$\frac{1}{(s - a)^2}$	
$t^n e^{a t}, n \in \mathbb{Z}$	$\frac{n!}{(s - a)^{n+1}}$	

Laplace transform table

The pendulum application that we didn't cover carefully in Chapter 5....we'll use this for a sequence of examples using Laplace transform over the next several lectures.

1) pendulum

$\theta = \theta(t)$

conservative system $KE + PE = \text{const.}$

$$\frac{1}{2}mv^2 + mgh = \text{const}$$

$$s = L\theta$$

$$v = \frac{ds}{dt} = L\theta'(t) \qquad h = L - L\cos\theta = L(1-\cos\theta)$$

So, $\frac{1}{2}mL^2(\theta'(t))^2 + mgL(1-\cos(\theta(t))) \equiv \text{const}$

D_t: $mL^2\theta'\theta'' + mgL(\sin\theta)\theta' \equiv 0$

$mL\theta'(L\theta'' + g\sin\theta) \equiv 0$

$mL\theta' \neq 0$ except at isolated times

$\rightsquigarrow$ deduce eqtn of motion is

$$\boxed{\theta'' + \frac{g}{L}\sin\theta = 0}$$

linearize:

$$\boxed{\theta'' + \frac{g}{L}\theta = 0}$$

$$\omega_0 = \sqrt{\frac{g}{L}}$$

$$\theta(t) = C\cos(\omega_0 t - \alpha)$$

non-linear DE

but $\sin\theta = \theta - \frac{\theta^3}{3!} + \dots$

$\sin\theta \approx \theta$ θ small is excellent approx (alternating series test)

Math 2250-4
Wed Apr 4
10.3 partial fractions; 10.5 piecewise forcing.

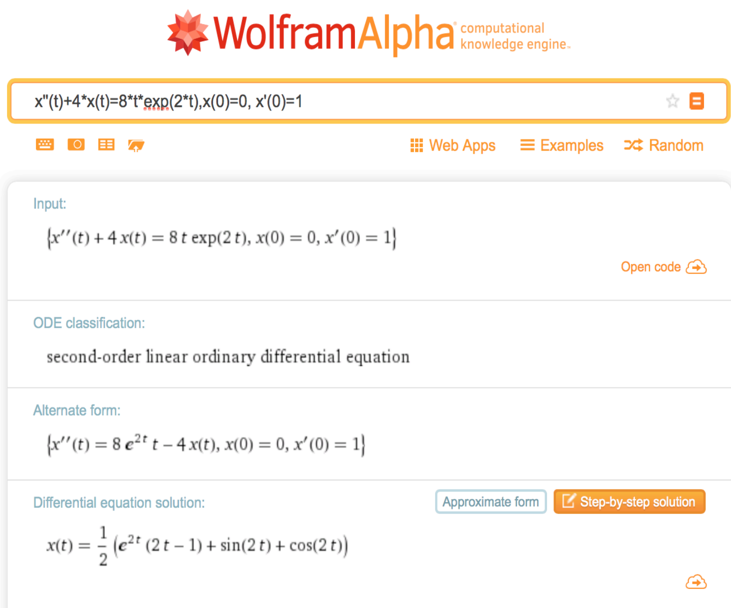
Announcements:

Warm-up Exercise:

partial fractions occur naturally when solving initial value problems with Laplace transforms, as we've already seen. Here's a moderately-involved example:

<u>Exercise 1)</u> Solve the following IVP. Use this example to recall the general partial fractions algorithm.

$$x''(t) + 4\,x(t) = 8\,t\,e^{2\,t}$$
$$x(0) = 0$$
$$x'(0) = 1$$



Wolfram alpha can check most of your steps, once you've set up the problem. Or, if it's a ridiculous problem don't try to even work it by hand:

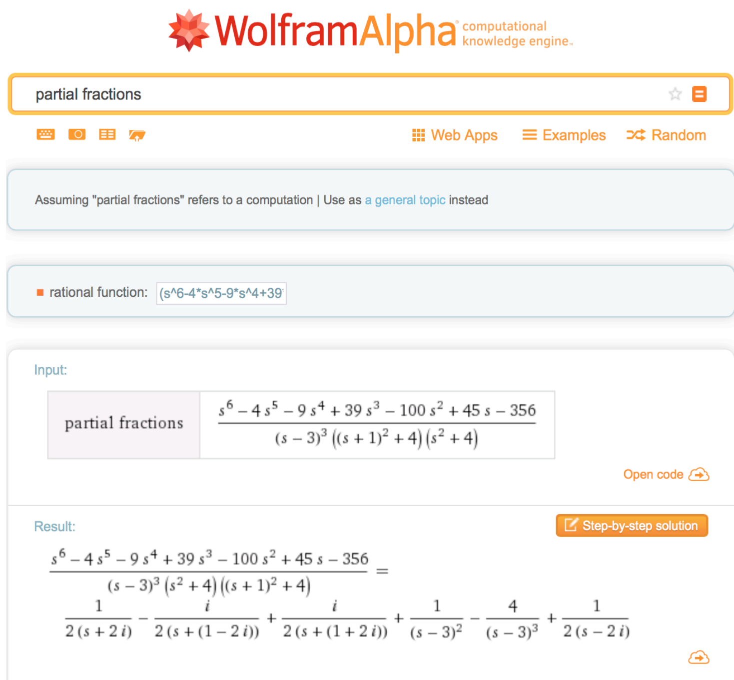
<u>Exercise 2a)</u> What is the *form* of the partial fractions decomposition for

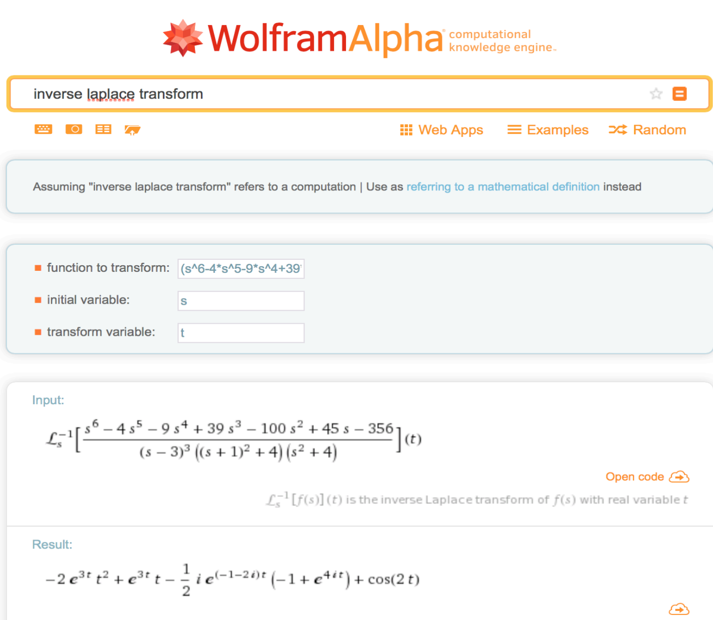
$$X(s) = \frac{-356 + 45\,s - 100\,s^2 - 4\,s^5 - 9\,s^4 + 39\,s^3 + s^6}{(s-3)^3\left((s+1)^2+4\right)\left(s^2+4\right)}\ .$$

<u>2b)</u> Check exact numbers with Wolfram alpha

<u>2c)</u> What is $x(t) = \mathcal{L}^{-1}\{X(s)\}(t)$?

<u>2d)</u> Have Wolfram alpha compute the inverse Laplace transform directly. Notice that being fluid with Euler's formula is useful.





10.5 Piecewise continuous forcing functions.....e.g. turning the forcing on and off.

• The following Laplace transform material is useful in systems where we turn forcing functions on and off, and when we have right hand side "forcing functions" that are more complicated than what undetermined coefficients can handle. We will continue this discussion on Friday, with a few more table entries including "the delta (impulse) function".

$f(t)$ with $\|f(t)\| \le C e^{M t}$	$F(s) := \int_0^{\infty} f(t) e^{-s t}\, dt$ for $s > M$	comments
$u(t-a)$ unit step function	$\frac{e^{-a s}}{s}$	for turning components on and off at $t = a$.
$f(t-a)\, u(t-a)$	$e^{-a s} F(s)$	more complicated on/off
$\int_0^{t} f(t-\tau) f(\tau)\, d\tau$	$F(s) G(s)$	"convolution" for inverting products of Laplace transforms

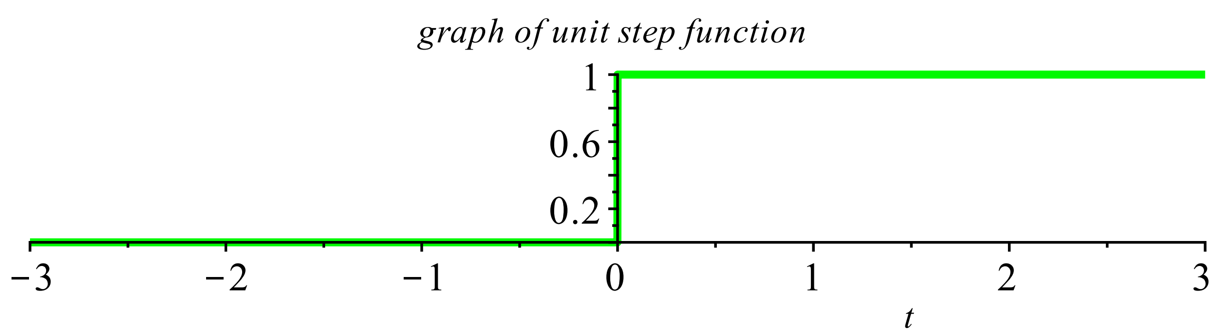
The unit step function with jump at $t = 0$ is defined to be

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \ge 0 \end{cases} .$$

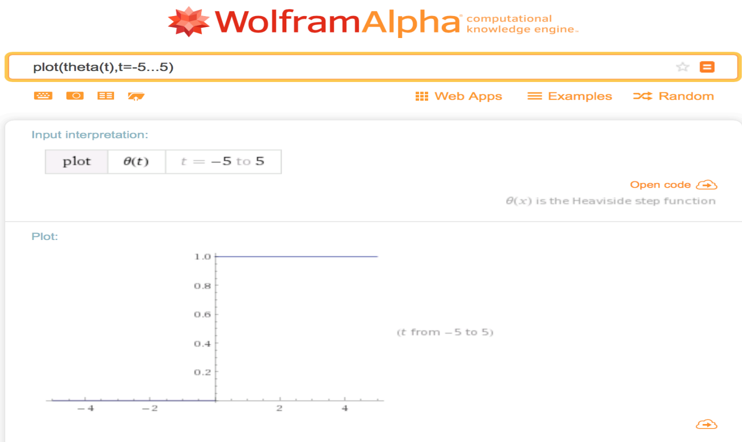
IThis function is also called the "Heaviside" function, e.g. in Maple and Wolfram alpha. In Wolfram alpha it's also called the "theta" function. Oliver Heaviside was a an accomplished physicist in the 1800's. The name is not because the graph is heavy on one side. :-)

http://en.wikipedia.org/wiki/Oliver_Heaviside

```
> with(plots) :
  plot(Heaviside(t), t=-3..3, color = green, title = `graph of unit step function`);
```



Notice that technically the vertical line should not be there - a more precise picture would have a solid point at $(0, 1)$ and a hollow circle at $(0, 0)$, for the graph of $u(t)$. In terms of Laplace transform integral definition it doesn't actually matter what we define $u(0)$ to be.



Then

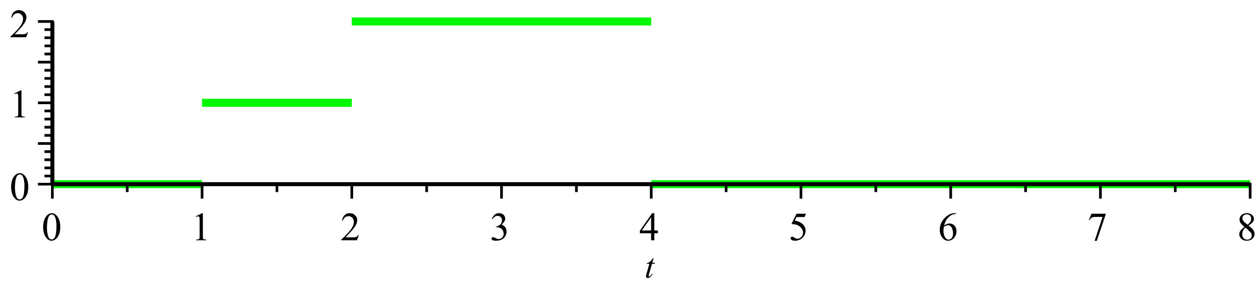
$$u(t-a)=\begin{cases}0, & t-a<0;\ i.e.\ t<a\\ 1, & t-a\geq 0;\ i.e.\ t\geq a\end{cases}$$

and has graph that is a horizontal translation by a to the right, of the original graph, e.g. for $a=2$:

<u>Exercise 3)</u> Verify the table entries

$u(t-a)$ unit step function	$\frac{e^{-a\,s}}{s}$	for turning components on and off at $t=a$.
$f(t-a)\ u(t-a)$	$e^{-a\,s}F(s)$	more complicated on/off

Exercise 4) Consider the function $f(t)$ which is zero for $t > 4$ and with the following graph. Use linearity and the unit step function entry to compute the Laplace transform $F(s)$. This should remind you of a homework problem from the assignment due tomorrow - although you're asked to find the Laplace transform of that step function directly from the definition. In your next week's homework assignment you will re-do that problem using unit step functions. (Of course, you could also check your answer in this week's homework with this method.)



Setup: an under-employed mathematician/engineer/scientist (your choice)

likes to take his/her child to the swings...

recall pendulum (linearized) eqtn, without forcing, for $\theta = \theta(t)$

$$L\theta'' + g\theta = 0$$

$$\dashrightarrow x'' + g\frac{x}{L} = 0$$

$$\dashrightarrow m x'' + \frac{mg}{L} x = F_0 \cos \omega t \quad \leftarrow \text{parent forcing (?!)}$$

$x(t) = L \sin\theta(t)$
$\doteq L\theta$ for small θ
so $x'' \doteq L\theta''$

$$\dashrightarrow x'' + \frac{g}{L} x = \frac{F_0}{m} \cos \omega t$$

$$x'' + \omega_0^2 x = \frac{F_0}{m} \cos \omega t$$

parent pushes sinusoidally for exactly 5 cycles, and with $\frac{F_0}{m} = .2$ and then releases:

for resonance $\omega = \omega_0$
construct swing with $L = g \approx 9.8$ m.
so $\omega_0^2 = 1$, $T_0 = 2\pi \approx 6.2$ seconds

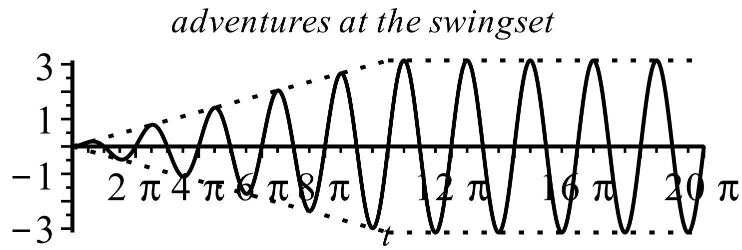
Exercise 5a) Explain why the description above leads to the differential equation initial value problem for $x(t)$

$$x''(t) + x(t) = .2 \cos(t)\left(1 - u(t - 10\pi)\right)$$
$$x(0) = 0$$
$$x'(0) = 0$$

5b) Find $x(t)$. Show that after the parent stops pushing, the child is oscillating with an amplitude of exactly π meters (in our linearized model).

Pictures for the swing:

```
> plot1 := plot(.1·t·sin(t), t = 0..10·Pi, color = black) :
  plot2 := plot(Pi·sin(t), t = 10·Pi..20·Pi, color = black) :
  plot3 := plot(Pi, t = 10·Pi..20·Pi, color = black, linestyle = 2) :
  plot4 := plot(-Pi, t = 10·Pi..20·Pi, color = black, linestyle = 2) :
  plot5 := plot(.1·t, t = 0..10·Pi, color = black, linestyle = 2) :
  plot6 := plot(-.1·t, t = 0..10·Pi, color = black, linestyle = 2) :
  display({plot1, plot2, plot3, plot4, plot5, plot6}, title = `adventures at the swingset`);
```



Alternate approach via Chapter 5:

<u>step 1)</u> solve

$$x''(t) + x(t) = .2\cos(t)$$
$$x(0) = 0$$
$$x'(0) = 0$$

for $0 \le t \le 10\pi$.

<u>step 2)</u> Then solve

$$y''(t) + y(t) = 0$$
$$y(0) = x(10\pi)$$
$$y'(0) = x'(10\pi)$$

and set $x(t) = y(t - 10)$ for $t > 10$.

$f(t)$, with $\lvert f(t)\rvert \le Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st}\,dt$ for $s > M$	↓ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	☐
1 t t^2 $t^n,\ n \in \mathbb{N}$	$\frac{1}{s}$ $(s > 0)$ $\frac{1}{s^2}$ $\frac{2}{s^3}$ $\frac{n!}{s^{n+1}}$	☐ ☐ ☐ ☐
$e^{\alpha t}$	$\frac{1}{s-\alpha}$ $(s > \Re(a))$	☐
$\cos(k t)$ $\sin(k t)$ $\cosh(k t)$ $\sinh(k t)$ $e^{a t}\cos(k t)$ $e^{a t}\sin(k t)$ $e^{a t} f(t)$	$\frac{s}{s^2+k^2}$ $(s > 0)$ $\frac{k}{s^2+k^2}$ $(s > 0)$ $\frac{s}{s^2-k^2}$ $(s > k)$ $\frac{k}{s^2-k^2}$ $(s > k)$ $\frac{(s-a)}{(s-a)^2+k^2}$ $(s > a)$ $\frac{k}{(s-a)^2+k^2}$ $(s > a)$ $F(s-a)$	☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐
$u(t-a)$ $f(t-a)\,u(t-a)$ $\delta(t-a)$	$\frac{e^{-a s}}{s}$ $e^{-a s}F(s)$ $e^{-a s}$	
$f'(t)$ $f''(t)$ $f^{(n)}(t),\ n \in \mathbb{N}$	$s\,F(s) - f(0)$ $s^2F(s) - s\,f(0) - f'(0)$ $s^n\,F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$	☐ ☐ ☐

$\int_0^t f(\tau)\, d\tau$	$\frac{F(s)}{s}$	☐
$t\, f(t)$ $t^2\, f(t)$ $t^n\, f(t),\ n \in \mathbb{Z}$ $\frac{f(t)}{t}$	$-F'(s)$ $F''(s)$ $(-1)^n\, F^{(n)}(s)$ $\int_s^{\infty} F(\sigma)\, d\sigma$	☐ ☐ ☐
$t\cos(k\,t)$ $\frac{1}{2\,k}\, t\sin(k\,t)$ $\frac{1}{2\,k^3}\,(\sin(k\,t) - k\,t\cos(k\,t))$ $t\, e^{a\,t}$ $t^n\, e^{a\,t},\ n \in \mathbb{Z}$	$\frac{s^2 - k^2}{(s^2 + k^2)^2}$ $\frac{s}{(s^2 + k^2)^2}$ $\frac{1}{(s^2 + k^2)^2}$ $\frac{1}{(s-a)^2}$ $\frac{n!}{(s-a)^{n+1}}$	☐ ☐ ☐ ☐ ☐
$\int_0^t f(\tau) g(t-\tau)\, d\tau$	$F(s)G(s)$	
$f(t)$ with period p	$\frac{1}{1 - e^{-ps}} \int_0^p f(t) e^{-s\,t}\, dt$	

Laplace transform table

Math 2250-4
Fri Apr 6
10.5, EP7.6 piecewise and implulse forcing.

Announcements:

Warm-up Exercise:

Laplace table entries for today.

$f(t)$ with $\lvert f(t)\rvert \le Ce^{Mt}$	$F(s) := \int_0^{\infty} f(t)e^{-st}\,dt$ for $s > M$	comments
$u(t-a)$ unit step function	$\frac{e^{-as}}{s}$	for turning components on and off at $t = a$.
$f(t-a)\,u(t-a)$	$e^{-as}F(s)$	more complicated on/off
$\delta(t-a)$	e^{-as}	unit impulse/delta "function"

EP 7.6 impulse functions and the δ operator.

Consider a force $f(t)$ acting on an object for only on a very short time interval $a \le t \le a+\varepsilon$, for example as when a bat hits a ball. This impulse p of the force is defined to be the integral

$$p := \int_a^{a+\varepsilon} f(t)\,dt$$

and it measures the net change in momentum of the object since by Newton's second law

$$m\,v'(t) = f(t)$$

$$\Rightarrow \int_a^{a+\varepsilon} m\,v'(t)\,dt = \int_a^{a+\varepsilon} f(t)\,dt = p$$

$$\Rightarrow m\,v(t)\Big]_{t=a}^{a+\varepsilon} = p\,.$$

Since the impulse p only depends on the integral of $f(t)$, and since the exact form of f is unlikely to be known in any case, the easiest model is to replace f with a constant force having the same total impulse, i.e. to set

$$f = p\,d_{a,\varepsilon}(t)$$

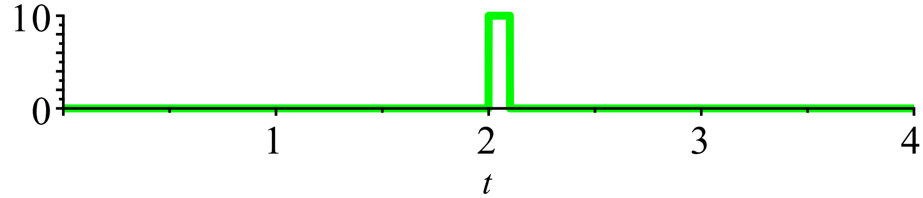
where $d_{a,\varepsilon}(t)$ is the unit impulse function given by

$$d_{a,\varepsilon}(t) = \begin{cases} 0, & t < a \\ \frac{1}{\varepsilon}, & a \le t < a+\varepsilon \\ 0, & t \ge a+\varepsilon \end{cases}.$$

Notice that

$$\int_a^{a+\varepsilon} d_{a,\varepsilon}(t)\,dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\,dt = 1\,.$$

Here's a graph of $d_{2,.1}(t)$, for example:


10
0
1
2
3
4
t

Since the unit impulse function is a linear combination of unit step functions, we could solve differential equations with impulse functions so-constructed. As far as Laplace transform goes, it's even easier to take the limit as $\varepsilon \to 0$ for the Laplace transforms $\mathcal{L}\{d_{a,\varepsilon}(t)\}(s)$, and this effectively models impulses on very short time scales.

$$d_{a,\varepsilon}(t) = \frac{1}{\varepsilon}\left[u(t-a) - u(t-(a+\varepsilon))\right]$$

$$\Rightarrow \mathcal{L}\{d_{a,\varepsilon}(t)\}(s) = \frac{1}{\varepsilon}\left(\frac{e^{-a s}}{s} - \frac{e^{-(a+\varepsilon)s}}{s}\right)$$

$$= e^{-a s}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right).$$

In Laplace land we can use L'Hopital's rule (in the variable ε) to take the limit as $\varepsilon \to 0$:

$$\lim_{\varepsilon \to 0} e^{-a s}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right) = e^{-a s}\lim_{\varepsilon \to 0}\left(\frac{s\, e^{-\varepsilon s}}{s}\right) = e^{-a s}.$$

The result in time t space is not really a function but we call it the "delta function" $\delta(t-a)$ anyways, and visualize it as a function that is zero everywhere except at $t = a$, and that it is infinite at $t = a$ in such a way that its integral over any open interval containing a equals one. As explained in EP7.6, the delta "function" can be thought of in a rigorous way as a linear transformation, not as a function. It can also be thought of as the derivative of the unit step function $u(t-a)$, and this is consistent with the Laplace table entries for derivatives of functions. In any case, this leads to the very useful Laplace transform table entry

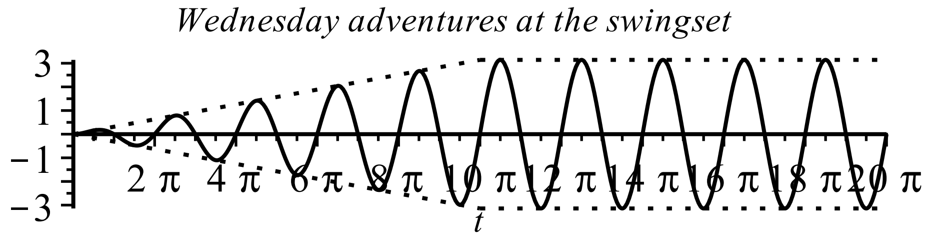
$\delta(t-a)$ unit impulse function	$e^{-a s}$	for impulse forcing

Exercise 1) Revisit the swing from Wednesday's notes and solve the IVP below for $x(t)$. In this case the parent is providing an impulse each time the child passes through equilibrium position after completing a cycle.

$$x''(t) + x(t) = .2\,\pi\left[\delta(t) + \delta(t-2\pi) + \delta(t-4\pi) + \delta(t-6\pi) + \delta(t-8\pi)\right]$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

> *with(plots)* :

> *plot1* ≔ *plot*(.1 · *t* · sin(*t*), *t* = 0 ..10 · Pi, *color* = *black*) :
plot2 ≔ *plot*(Pi · sin(*t*), *t* = 10 · Pi ..20 · Pi, *color* = *black*) :
plot3 ≔ *plot*(Pi, *t* = 10 · Pi ..20 · Pi, *color* = *black*, *linestyle* = 2) :
plot4 ≔ *plot*(−Pi, *t* = 10 · Pi ..20 · Pi, *color* = *black*, *linestyle* = 2) :
plot5 ≔ *plot*(.1 · *t*, *t* = 0 ..10 · Pi, *color* = *black*, *linestyle* = 2) :
plot6 ≔ *plot*(−.1 · *t*, *t* = 0 ..10 · Pi, *color* = *black*, *linestyle* = 2) :
display({*plot1*, *plot2*, *plot3*, *plot4*, *plot5*, *plot6*}, *title* = `Wednesday adventures at the swingset`);



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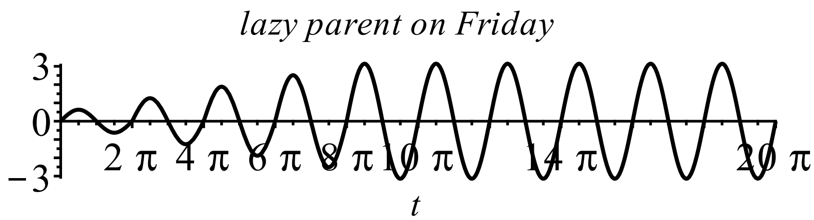
impulse solution: five equal impulses to get same final amplitude of π meters - Exercise 1:

> *f* ≔ *t* → .2 · Pi · *sum*(Heaviside(*t* − *k* · 2 · Pi) · sin(*t* − *k* · 2 · Pi), *k* = 0 ..4) :

> *plot*(*f*(*t*), *t* = 0 ..20 · Pi, *color* = *black*, *title* = `lazy parent on Friday`);

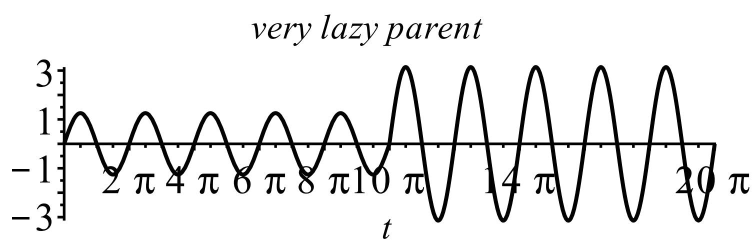


>

Or, an impulse at $t = 0$ and another one at $t = 10\,\pi$.

> *g* ≔ *t* → .2 · Pi · (2 · sin(*t*) + 3 · Heaviside(*t* − 10 · Pi) · sin(*t* − 10 · Pi)) :

> *plot*(*g*(*t*), *t* = 0 ..20 · Pi, *color* = *black*, *title* = `very lazy parent`);



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