

Math 2250-004

Week 10: March 12-16 5.4-5.6 applications of linear differential equations, and solution techniques for non-homogeneous problems.

Mon March 5:

5.3 - 5.4 Review of characteristic equation method for homogeneous DE solutions, and the unforced mass-spring-damper application.

Announcements:

Warm-up Exercise:

Review of characteristic equation algorithms... in section 5.4 our functions are usually $x(t)$, not $y(x)$ - don't let that trip you up.

Exercise 1) Use the characteristic polynomial and the complex roots algorithm, to find a basis for the solution space of functions $x(t)$ that solve

$$x''(t) + 6x'(t) + 13x(t) = 0.$$

Exercise 2) Suppose a 7^{th} order linear homogeneous DE has characteristic polynomial

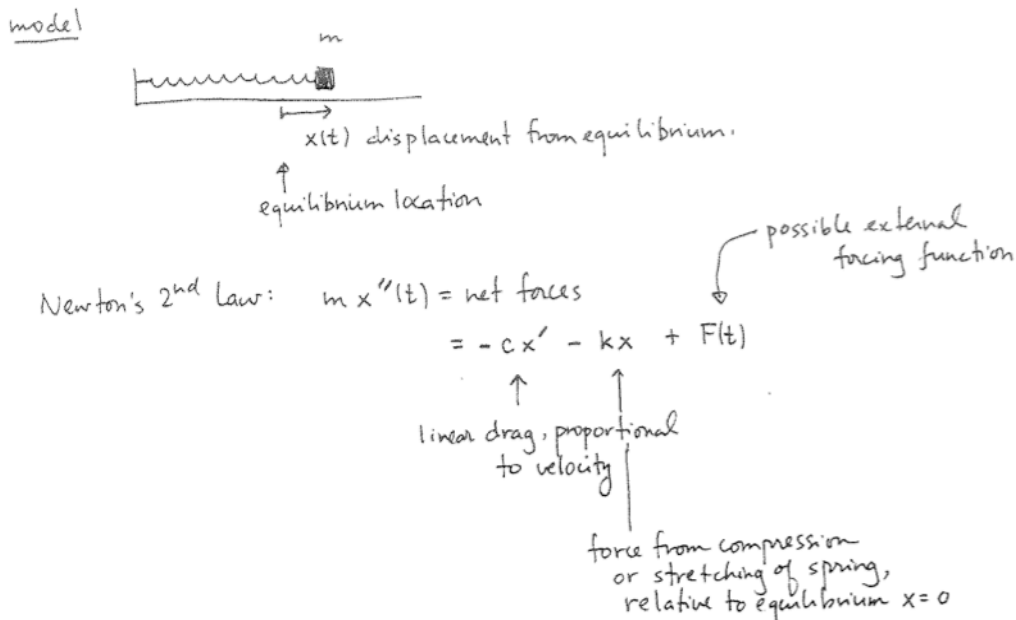
$$p(r) = (r^2 + 6r + 13)^2 (r - 2)^3.$$

What is the general solution $x(t)$ to the corresponding homogeneous DE for $x(t)$?

5.4: Applications of 2^{nd} order linear homogeneous DE's with constant coefficients, to unforced spring (and related) configurations.

In this section we study the differential equation below for functions $x(t)$:

$$m x'' + c x' + k x = 0.$$



In section 5.4 we assume the time dependent external forcing function $F(t) \equiv 0$. The expression for internal forces $-c x' - k x$ is a linearization model, about the constant solution $x = 0, x' = 0$, for which the net forces must be zero. Notice that $c \geq 0, k > 0$. The actual internal forces are probably not exactly linear, but this model is usually effective when $x(t), x'(t)$ are sufficiently small. k is called the Hooke's constant, and c is called the damping coefficient.

$$m x'' + c x' + k x = 0 .$$

This is a constant coefficient linear homogeneous DE, so we try $x(t) = e^{r t}$ and compute

$$L(x) := m x'' + c x' + k x = e^{r t} (m r^2 + c r + k) = e^{r t} p(r).$$

The different behaviors exhibited by solutions to this mass-spring configuration depend on what sorts of roots the characteristic polynomial $p(r)$ possesses...

Case 1) no damping ($c = 0$).

$$\begin{aligned} m x'' + k x &= 0 \\ x'' + \frac{k}{m} x &= 0 . \\ p(r) &= r^2 + \frac{k}{m} , \end{aligned}$$

has roots

$$r^2 = -\frac{k}{m} \quad \text{i.e.} \quad r = \pm i \sqrt{\frac{k}{m}} .$$

So the general solution is

$$x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}} t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}} t\right) .$$

We write $\sqrt{\frac{k}{m}} := \omega_0$ and call ω_0 the natural angular frequency . Notice that its units are radians per time. We also replace the linear combination coefficients c_1, c_2 by A, B . So, using the alternate letters, the general solution to

$$x'' + \omega_0^2 x = 0$$

is

$$x(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t) .$$

It's worth learning to recognize the undamped DE, and the trigonometric solutions, as it's easy to understand why they are solutions and you can then skip the characteristic polynomial step.

Exercise 1a Write down the general homogeneous solution $x(t)$ to the differential equation

$$x''(t) + 4 x(t) = 0.$$

1b) What is the general solution to $\theta(t)$ to

$$\theta''(t) + 10 \theta(t) = 0.$$

The motion exhibited by the solutions

$$x(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t)$$

to the undamped oscillator DE

$$x'' + \omega_0^2 x = 0$$

is called simple harmonic motion. The reason for this name is that $x(t)$ can be rewritten in "amplitude-phase form" as

$$x(t) = C \cos(\omega_0 t - \alpha) = C \cos(\omega_0(t - \delta))$$

in terms of an amplitude $C > 0$ and a phase angle α (or in terms of a time delay δ).

To see why this is so, equate the two forms and see how the coefficients A, B, C and phase angle α must be related:

$$x(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t) = C \cos(\omega_0 t - \alpha).$$

Exercise 2) Use the addition angle formula $\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$ to show that the two expressions above are equal provided

$$A = C \cos \alpha$$

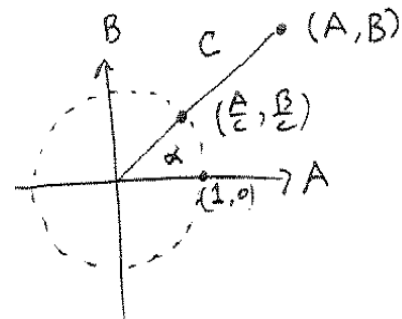
$$B = C \sin \alpha.$$

So if C, α are given, the formulas above determine A, B . Conversely, if A, B are given then

$$C = \sqrt{A^2 + B^2}$$

$$\frac{A}{C} = \cos(\alpha), \quad \frac{B}{C} = \sin(\alpha)$$

determine C, α . These correspondences are best remembered using a diagram in the $A - B$ plane:



It is important to understand the behavior of the functions

$$A \cos(\omega_0 t) + B \sin(\omega_0 t) = C \cos(\omega_0 t - \alpha) = C \cos(\omega_0(t - \delta))$$

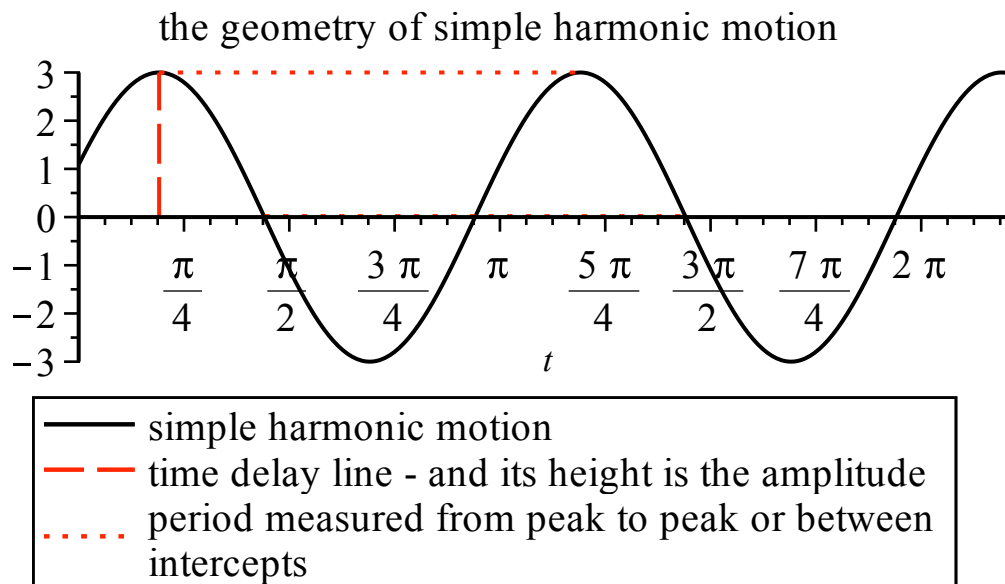
and the standard terminology:

The amplitude C is the maximum absolute value of $x(t)$. The *phase angle* α is the radians of $\omega_0 t$ on the unit circle, so that $\cos(\omega_0 t - \alpha)$ evaluates to 1. The time delay δ is how much the graph of $C \cos(\omega_0 t)$ is shifted to the right along the t -axis in order to obtain the graph of $x(t)$. Note that

$$\omega_0 = \text{angular velocity} \quad \text{units: radians/time}$$

$$f = \text{frequency} = \frac{\omega_0}{2\pi} \quad \text{units: cycles/time}$$

$$T = \text{period} = \frac{2\pi}{\omega_0} \quad \text{units: time/cycle.}$$



Exercise 3) A mass of 2 kg oscillates without damping on a spring with Hooke's constant $k = 18 \frac{N}{m}$. It is initially stretched 1 m from equilibrium, and released with a velocity of $\frac{3}{2} \frac{m}{s}$.

3a) Show that the mass' motion is described by $x(t)$ solving the initial value problem

$$x'' + 9x = 0$$

$$x(0) = 1$$

$$x'(0) = \frac{3}{2}.$$

3b) Solve the IVP in a, and convert $x(t)$ into amplitude-phase and amplitude-time delay form. Sketch the solution, indicating amplitude, period, and time delay. Check your work with the Wolfram alpha output on the next page

$x''(t) + 9x(t) = 0, x(0) = 1, x'(0) = \frac{3}{2}$

Web Apps
Examples
Random

Input:
$$\left\{ x''(t) + 9x(t) = 0, x(0) = 1, x'(0) = \frac{3}{2} \right\}$$
Open code

Autonomous equation:
$$x''(t) = -9x(t)$$
Autonomous equation »

ODE classification:

second-order linear ordinary differential equation

Alternate forms:
$$\left\{ x''(t) = -9x(t), x(0) = 1, x'(0) = \frac{3}{2} \right\}$$

$$\{ x''(t) + 9x(t) = 0, x(0) = 1, 2x'(0) = 3 \}$$

Differential equation solution:

☒ Step-by-step solution

$$x(t) = \frac{1}{2} \sin(3t) + \cos(3t)$$



plot cos(3*t)+.5*sin(3*t),t=0..Pi

Web Apps
Examples
Random

Input interpretation:

plot
cos(3 t) + 0.5 sin(3 t)
t = 0 to π

Open code

Plot:

Case 2: Unforced mass-spring system with damping: (This discussion is in Monday's notes but we'll likely need the start of Tuesday to finish it.)

- 3 possibilities that arise when the damping coefficient $c > 0$. There are three cases, depending on the roots of the characteristic polynomial:

$$m x'' + c x' + k x = 0$$

$$x'' + \frac{c}{m} x' + \frac{k}{m} x = 0$$

rewrite as

$$x'' + 2p x' + \omega_0^2 x = 0.$$

$(p = \frac{c}{2m}, \omega_0^2 = \frac{k}{m})$. The characteristic polynomial is

$$r^2 + 2p r + \omega_0^2 = 0$$

which has roots

$$r = -\frac{2p \pm \sqrt{4p^2 - 4\omega_0^2}}{2} = -p \pm \sqrt{p^2 - \omega_0^2}.$$

Case 2a) ($p^2 < \omega_0^2$, or $c^2 < 4mk$) underdamped. Complex roots

$$r = -p \pm \sqrt{p^2 - \omega_0^2} = -p \pm i \omega_1$$

with $\omega_1 = \sqrt{\omega_0^2 - p^2} < \omega_0$, the undamped angular frequency.

$$x(t) = e^{-p t} (A \cos(\omega_1 t) + B \sin(\omega_1 t)) = e^{-p t} C \cos(\omega_1 t - \alpha_1).$$

- solution decays exponentially to zero, but oscillates infinitely often, with exponentially decaying pseudo-amplitude $e^{-p t} C$ and pseudo-angular frequency ω_1 , and pseudo-phase angle α_1 .

$$r^2 + 2 p r + \omega_0^2 = 0$$

which has roots

$$r = -\frac{2 p \pm \sqrt{4 p^2 - 4 \omega_0^2}}{2} = -p \pm \sqrt{p^2 - \omega_0^2}.$$

Case 2b) ($p^2 > \omega_0^2$, or $c^2 > 4 m k$). overdamped. In this case we have two negative real roots

$$r_1 = -p - \sqrt{p^2 - \omega_0^2} < 0$$

$$r_1 < r_2 = -p + \sqrt{p^2 - \omega_0^2} < 0$$

and

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = e^{r_2 t} (c_1 e^{(r_1 - r_2) t} + c_2).$$

- solution converges to zero exponentially fast; solution passes through equilibrium location $x = 0$ at most once.

Case 2c) ($p^2 = \omega_0^2$, or $c^2 = 4 m k$) critically damped. Double real root $r_1 = r_2 = -p = -\frac{c}{2 m}$.

$$x(t) = e^{-p t} (c_1 + c_2 t).$$

- solution converges to zero exponentially fast, passing through $x = 0$ at most once, just like in the overdamped case. The critically damped case is the transition between overdamped and underdamped:

Exercise 4) Classify (underdamped, overdamped, critically damped), by finding the roots of the characteristic polynomial. Write down the general solution. Could you solve the IVP for $x(t)$?
4a)

$$x'' + 6x' + 9x = 0$$

$$x(0) = 1$$

$$x'(0) = \frac{3}{2} .$$

4b)

$$x'' + 10x' + 9x = 0$$

$$x(0) = 1$$

$$x'(0) = \frac{3}{2} .$$

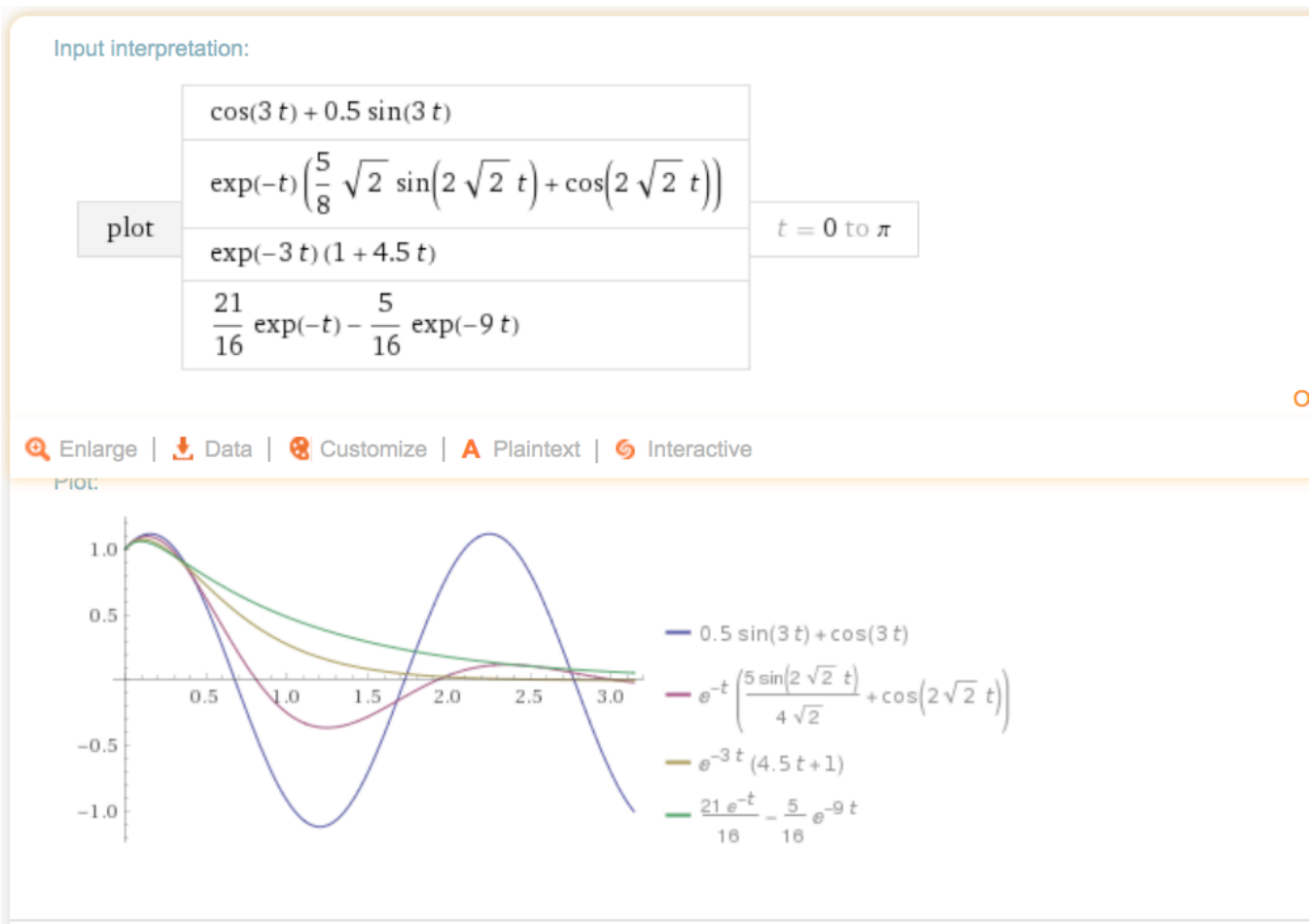
4c)

$$x'' + 2x' + 9x = 0$$

$$x(0) = 1$$

$$x'(0) = \frac{3}{2} .$$

The same initial displacement and velocity, and mass and spring constant - for an undamped, underdamped, overdamped, and critically damped mass-spring problem: (Courtesy Wolfram alpha).



Tues March 13:

5.4-5.5 Finish Monday's notes on 5.4, Then begin 5.5: Finding y_p for non-homogeneous linear differential equations

$$L(y) = f$$

(so that you can use the general solution $y = y_p + y_H$ to solve initial value problems). We're back to $y = y(x)$ in this section...

Announcements:

Warm-up Exercise:

Section 5.5: Finding y_p for non-homogeneous linear differential equations

$$L(y) = f$$

(so that you can use the general solution $y = y_p + y_H$ to solve initial value problems). In this section the text switches back to $y = y(x)$, from $x = x(t)$.

There are two methods we will use:

- The method of undetermined coefficients uses guessing algorithms, and works for constant coefficient linear differential equations with certain classes of functions $f(x)$ for the non-homogeneous term. The method seems magic, but actually relies on vector space theory. We've already seen simple examples of this, where we seemed to pick particular solutions out of the air. This method is the main focus of section 5.5.
- The method of variation of parameters is more general, and yields an integral formula for a particular solution y_p , assuming you are already in possession of a basis for the homogeneous solution space. This method has the advantage that it works for any linear differential equation and any (continuous) function f . It has the disadvantage that the formulas can get computationally messy especially for differential equations of order $n > 2$. We'll study the case $n = 2$ only.

The easiest way to explain the method of undetermined coefficients is with examples.

Roughly speaking, you make a "guess" with free parameters (undetermined coefficients) that "looks like" the right side. AND, you need to include all possible terms in your guess that could arise when you apply L to the terms you know you want to include.

We'll make this more precise later in the notes.

Exercise 1) Find a particular solution $y_p(x)$ for the differential equation

$$L(y) := y'' + 4y' - 5y = 10x + 3.$$

Hint: try $y_p(x) = d_1x + d_2$ because L transforms such functions into ones of the form $b_1x + b_2$. d_1, d_2 are your "undetermined coefficients", for the given right hand side coefficients $b_1 = 10, b_2 = 3$.

Exercise 2) Use your work in 1 and your expertise with homogeneous linear differential equations to find the general solution to

$$y'' + 4y' - 5y = 10x + 3$$

Exercise 3) Find a particular solution to

$$L(y) = y'' + 4y' - 5y = 14e^{2x}.$$

Hint: try $y_p = d e^{2x}$ because L transforms functions of that form into ones of the form $b e^{2x}$, i.e.

$L(d e^{2x}) = b e^{2x}$. " d " is your "undetermined coefficient" for $b = 14$.

Exercise 4a) Use superposition (linearity of the operator L) and your work from the previous exercises to find the general solution to

$$L(y) = y'' + 4y' - 5y = 14e^{2x} - 20x - 6.$$

4b) Solve (or at least set up the problem to solve) the initial value problem


$$y'' + 4y' - 5y = 14e^{2x} - 20x - 6$$

$$y(0) = 4$$

$$y'(0) = -4.$$

4c) Check your answer with technology.

$$y''(x) + 4y'(x) - 5y(x) = 14\exp(2x) - 20x - 6, y(0) = 4, y'(0) = -4$$

Web Apps Examples Random

Input:

$$\{y''(x) + 4y'(x) - 5y(x) = 14\exp(2x) - 20x - 6, y(0) = 4, y'(0) = -4\}$$
[Open code](#)

ODE classification:

second-order linear ordinary differential equation

Alternate forms:

$$\{y''(x) = -4y'(x) + 5y(x) - 20x + 14e^{2x} - 6, y(0) = 4, y'(0) = -4\}$$

$$\{5y(x) + 14e^{2x} = y''(x) + 4y'(x) + 20x + 6, y(0) = 4, y'(0) = -4\}$$

$$\{y''(x) + 4y'(x) - 5y(x) = 2(-10x + 7e^{2x} - 3), y(0) = 4, y'(0) = -4\}$$

[More](#)

Differential equation solution:

[Approximate form](#) [Step-by-step solution](#)

$$y(x) = 4x + \frac{8e^{-5x}}{5} - 4e^x + 2e^{2x} + \frac{22}{5}$$

Exercise 5) Find a particular solution to

$$L(y) := y'' + 4y' - 5y = 2 \cos(3x) .$$

Hint: To solve $L(y) = f$ we hope that f is in some finite dimensional subspace V that is preserved by L , i.e. $L : V \rightarrow V$.

- In Exercise 1 $V = \text{span}\{1, x\}$ and so we guessed $y_p = d_1 + d_2 x$.
- In Exercise 3 $V = \text{span}\{e^{2x}\}$ and so we guessed $y_p = d e^{2x}$.
- What's the smallest subspace V we can take in the current exercise? Can you see why $V = \text{span}\{\cos(3x)\}$ and a guess of $y_p = d \cos(3x)$ won't work?

$$y''(x)+4y'(x)-5y(x)=2\cos(3x)$$



Web Apps Examples Random

Input:

$$y''(x) + 4y'(x) - 5y(x) = 2\cos(3x)$$

Open code

ODE classification:

second-order linear ordinary differential equation

Alternate forms:

More

$$y''(x) = -4y'(x) + 5y(x) + 2\cos(3x)$$

$$5y(x) + 2\cos(3x) = y''(x) + 4y'(x)$$



$$y''(x) + 4y'(x) - 5y(x) = e^{-3ix} + e^{3ix}$$



Differential equation solution:

Approximate form

☒ Step-by-step solution

$$y(x) = c_1 e^{-5x} + c_2 e^x + \frac{6}{85} \sin(3x) - \frac{7}{85} \cos(3x)$$



All of the previous exercises rely on:

Method of undetermined coefficients (base case): If you wish to find a particular solution y_p , i.e.

$L(y_p) = f$ and if the non-homogeneous term f is in a finite dimensional subspace V with the properties that

- (i) $L : V \rightarrow V$, i.e. L transforms functions in V into functions which are also in V ; and
- (ii) The only function $g \in V$ for which $L(g) = 0$ is $g = 0$.

Then there is always a unique $y_p \in V$ with $L(y_p) = f$.

why: We already know this fact is true for matrix transformations $L(\underline{x}) = A_{n \times n} \underline{x}$ with $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (because if the only homogeneous solution is $\underline{x} = \underline{0}$ then A reduces to the identity, so also each matrix equation $A \underline{x} = \underline{b}$ has a unique solution \underline{x}). The theorem above is a generalization of that fact to general linear transformations. There is an "appendix" explaining this at the end of today's notes, for students who'd like to understand the details.

Exercise 6) Use the method of undetermined coefficients to guess the form for a particular solution $y_p(x)$ for a constant coefficient differential equation

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = f$$

(assuming the only such solution in your specified subspace that would solve the homogeneous DE is the zero solution):

6a) $L(y) = x^3 + 6x - 5$

6b) $L(y) = 4e^{2x} \sin(3x)$

6c) $L(y) = x \cos(2x)$

$$y''(x) + 4 \cdot y'(x) - 5 \cdot y(x) = x \cdot \cos(2 \cdot x)$$

Differential equation solution:

Approximate form

☒ Step-by-step solution

$$y(x) = c_1 e^{-5x} + c_2 e^x + \frac{1}{13} x \sin(x) + \frac{29 \sin(x)}{338} - \frac{3}{26} x \cos(x) + \frac{\cos(x)}{169}$$



Wed March 14:

5.5: Finding y_p for non-homogeneous linear differential equations, continued. Introduction of 5.6: forced oscillation problems, to get ready for lab.

$$L(y) = f$$

(so that you can use the general solution $y = y_p + y_H$ to solve initial value problems).

Announcements:

Warm-up Exercise:

On Tuesday we discussed the base case of undetermined coefficients:

Method of undetermined coefficients (base case): If you wish to find a particular solution y_p , i.e.

$L(y_p) = f$ and if the non-homogeneous term f is in a finite dimensional subspace V with the properties that

- (i) $L : V \rightarrow V$, i.e. L transforms functions in V into functions which are also in V ; and
- (ii) The only function $g \in V$ for which $L(g) = 0$ is $g = 0$.

Then there is always a unique $y_p \in V$ with $L(y_p) = f$.

why: We already know this fact is true for matrix transformations $L(\mathbf{x}) = A_{n \times n} \mathbf{x}$ with $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$

(because if the only homogeneous solution is $\mathbf{x} = \mathbf{0}$ then A reduces to the identity, so also each matrix equation $A \mathbf{x} = \mathbf{b}$ has a unique solution \mathbf{x} . The theorem above is a generalization of that fact to general linear transformations. There is an "appendix" explaining this at the end of today's notes, for students who'd like to understand the details.

Exercise 1) Use the method of undetermined coefficients to guess the form for a particular solution $y_p(x)$ for a constant coefficient differential equation

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = f$$

(assuming the only such solution in your specified subspace that would solve the homogeneous DE is the zero solution):

1a) $L(y) = x^3 + 6x - 5$

1b) $L(y) = 4e^{2x} \sin(3x)$

1c) $L(y) = x \cos(2x)$

BUT LOOK OUT

Exercise 2a) Find a particular solution to

$$y'' + 4y' - 5y = 4e^x.$$

Hint: since $y_H = c_1 e^x + c_2 e^{-5x}$, a guess of $y_P = a e^x$ will not work (and $\text{span}\{e^x\}$ does not satisfy the "base case" conditions for undetermined coefficients). Instead try

$$y_P = d x e^x$$

and factor $L = D^2 + 4D - 5 = [D + 5] \circ [D - 1]$.

2b) check work with technology

The screenshot shows a web application for solving differential equations. At the top, the equation $y''(x) + 4y'(x) - 5y(x) = 4e^x$ is entered into a search bar. Below the search bar, there are navigation links: "Web Apps", "Examples", and "Random". The main content area displays the input equation $y''(x) + 4y'(x) - 5y(x) = 4e^x$ and an "Open code" button. Below this, the "ODE classification" is shown as "second-order linear ordinary differential equation". The "Alternate forms" section shows two equivalent equations: $y''(x) = -4y'(x) + 5y(x) + 4e^x$ and $5y(x) + 4e^x = y''(x) + 4y'(x)$. The "Differential equation solution" section shows the solution $y(x) = c_1 e^{-5x} + c_2 e^x + \frac{2}{3} e^x x$. There are buttons for "Approximate form" and "Step-by-step solution".

A vector space theorem like the one for the base case, except for $L : V \rightarrow W$, combined with our understanding of how to factor constant coefficient differential operators (as in lab you're working on this week for homogeneous DE's) leads to an extension of the method of undetermined coefficients, for right hand sides which can be written as sums of functions having the indicated forms below. See the discussion in pages 341-346 of the text, and the table on page 346, reproduced here.

Method of undetermined coefficients (extended case): Finding y_p for non-homogeneous linear differential equations

$$L(y) = f$$

If L has a factor $(D - r)^s$ and e^{rx} is also associated with (a portion of) the right hand side $f(x)$ then the corresponding guesses you would have made in the "base case" need to be multiplied by x^s , as in Exercise 2. (This is like your current lab problem, and if you understand that, you have an inkling of why this recipe works. On the other hand, if you don't understand that problem, there's another one this week so you get a second chance. :-) You may also need to use superposition, as in our earlier exercises, if different portions of $f(x)$ are associated with different exponential functions.

Extended case of undetermined coefficients

| $f(x)$ | y_p | $s > 0$ when $p(r)$ has these roots: |
|---|---|--------------------------------------|
| $P_m(x) = b_0 + b_1x + \dots + b_mx^m$ | $x^s(c_0 + c_1x + c_2x^2 + \dots + c_mx^m)$ | $r = 0$ |
| $b_1 \cos(\omega x) + b_2 \sin(\omega x)$ | $x^s(c_1 \cos(\omega x) + c_2 \sin(\omega x))$ | $r = \pm i\omega$ |
| $e^{ax}(b_1 \cos(\omega x) + b_2 \sin(\omega x))$ | $x^s e^{ax}(c_1 \cos(\omega x) + c_2 \sin(\omega x))$ | $r = a \pm i\omega$ |
| $b_0 e^{ax}$ | $x^s c_0 e^{ax}$ | $r = a$ |
| $(b_0 + b_1x + \dots + b_mx^m)e^{ax}$ | $x^s(c_0 + c_1x + c_2x^2 + \dots + c_mx^m)e^{ax}$ | $r = a$ |

Exercise 3) Set up the undetermined coefficients particular solutions for the examples below. When necessary use the extended case to modify the undetermined coefficients form for y_p . Use technology to check if your "guess" form was right.

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = f$$

3a) $y'''' + 2y'' = x^2 + 6x$

(So the characteristic polynomial for $L(y) = 0$ is $r^3 + 2r^2 = r^2(r+2) = (r-0)^2(r+2)$.)

Differential equation solution:

$$y(x) = c_3 x^2 + c_2 x + c_1 + \frac{x^5}{180} + \frac{x^4}{12}$$

3b) $y'' - 4y' + 13y = 4e^{2x}\sin(3x)$

(So the characteristic polynomial for $L(y) = 0$ is

$$r^2 - 4r + 13 = (r-2)^2 + 9 = (r-2+3i)(r-2-3i).$$

Differential equation solution:

$$y(x) = c_1 e^{2x} \sin(3x) + c_2 e^{2x} \cos(3x) + \frac{4}{37} e^x \sin(3x) + \frac{24}{37} e^x \cos(3x)$$

3c) $y'' + 5y' + 4y = 5 \cos(2x) + 4e^x + 5e^{-x}$.

(So the characteristic polynomial for $L(y) = 0$ is $p(r) = r^2 + 5r + 4 = (r + 4)(r + 1)$.)

Differential equation solution:

$$y(x) = c_1 e^{-4x} + c_2 e^{-x} + \frac{5e^{-x}x}{3} + \frac{2e^x}{5} + \sin(x) \cos(x)$$

Variation of Parameters: This is an alternate method for finding particular solutions. Its advantage is that it always provides a particular solution, even for non-homogeneous problems in which the right-hand side doesn't fit into a nice finite dimensional subspace preserved by L , and even if the linear operator L is not constant-coefficient. The formula for the particular solutions can be somewhat messy to work with, however, once you start computing.

Here's the formula: Let $y_1(x), y_2(x), \dots, y_n(x)$ be a basis of solutions to the homogeneous DE

$$L(y) := y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_1(x)y' + p_0(x)y = 0.$$

Then $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + \dots + u_n(x)y_n(x)$ is a particular solution to

$$L(y) = f$$

provided the coefficient functions (aka "varying parameters") $u_1(x), u_2(x), \dots, u_n(x)$ have derivatives satisfying the matrix equation

$$\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = [W(y_1, y_2, \dots, y_n)]^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f \end{bmatrix}$$

where $[W(y_1, y_2, \dots, y_n)]$ is the Wronskian matrix.

Here's how to check this fact when $n = 2$: Write

$$y_p = y = u_1 y_1 + u_2 y_2.$$

Thus

$$y' = u_1 y_1' + u_2 y_2' + (u_1' y_1 + u_2' y_2).$$

Set

$$(u_1' y_1 + u_2' y_2) = 0.$$

Then

$$y'' = u_1 y_1'' + u_2 y_2'' + (u_1' y_1' + u_2' y_2').$$

Set

$$(u_1' y_1' + u_2' y_2') = f.$$

Notice that the two (...) equations are equivalent to the matrix equation

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

which is equivalent to the $n = 2$ version of the claimed condition for y_p . Under these conditions we compute

$$\begin{aligned} & p_0 [y = u_1 y_1 + u_2 y_2] \\ & + p_1 [y' = u_1 y_1' + u_2 y_2'] \\ & + 1 [y'' = u_1 y_1'' + u_2 y_2'' + f] \\ & L(y) = u_1 L(y_1) + u_2 L(y_2) + f \\ & L(y) = 0 + 0 + f = f \end{aligned}$$

Appendix: The following two theorems justify the method of undetermined coefficients, in both the "base case" and the "extended case." We will not discuss these in class, but I'll be happy to chat about the arguments with anyone who's interested, outside of class. They only use ideas we've talked about already, although they are abstract.

Theorem 0:

- Let V and W be vector spaces. Let V have dimension $n < \infty$ and let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis for V .
- Let $L : V \rightarrow W$ be a linear transformation, i.e. $L(\mathbf{u} + \mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$ and $L(c\mathbf{u}) = cL(\mathbf{u})$ holds $\forall \mathbf{u}, \mathbf{v} \in V, c \in \mathbb{R}$.) Consider the range of L , i.e.

$$\begin{aligned} \text{Range}(L) &:= \{L(d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \dots + d_n\mathbf{v}_n) \in W, \text{ such that each } d_j \in \mathbb{R}\} \\ &= \{d_1L(\mathbf{v}_1) + d_2L(\mathbf{v}_2) + \dots + d_nL(\mathbf{v}_n) \in W, \text{ such that each } d_j \in \mathbb{R}\} \\ &= \text{span}\{L(\mathbf{v}_1), L(\mathbf{v}_2), \dots, L(\mathbf{v}_n)\}. \end{aligned}$$

Then $\text{Range}(L)$ is $n - \text{dimensional}$ if and only if the only solution to $L(\mathbf{v}) = \mathbf{0}$ is $\mathbf{v} = \mathbf{0}$.

proof:

(i) \Leftarrow : The only solution to $L(\mathbf{v}) = \mathbf{0}$ is $\mathbf{v} = \mathbf{0}$ implies $\text{Range}(L)$ is $n - \text{dimensional}$:

If we can show $L(\mathbf{v}_1), L(\mathbf{v}_2), \dots, L(\mathbf{v}_n)$ are linearly independent, then they will be a basis for $\text{Range}(L)$ and this subspace will have dimension n . So, consider the dependency equation:

$$d_1L(\mathbf{v}_1) + d_2L(\mathbf{v}_2) + \dots + d_nL(\mathbf{v}_n) = \mathbf{0}.$$

Because L is a linear transformation, we can rewrite this equation as

$$L(d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \dots + d_n\mathbf{v}_n) = \mathbf{0}.$$

Under our assumption that the only homogeneous solution is the zero vector, we deduce

$$d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \dots + d_n\mathbf{v}_n = \mathbf{0}.$$

Since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are a basis they are linearly independent, so $d_1 = d_2 = \dots = d_n = 0$. □

(ii) \Rightarrow : $\text{Range}(L)$ is $n - \text{dimensional}$ implies the only solution to $L(\mathbf{v}) = \mathbf{0}$ is $\mathbf{v} = \mathbf{0}$: Since the range of L is $n - \text{dimensional}$, $L(\mathbf{v}_1), L(\mathbf{v}_2), \dots, L(\mathbf{v}_n)$ must be linearly independent. Now, let $\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \dots + d_n\mathbf{v}_n$ be a homogeneous solution, $L(\mathbf{v}) = \mathbf{0}$. In other words,

$$\begin{aligned} L(d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \dots + d_n\mathbf{v}_n) &= \mathbf{0} \\ \Rightarrow d_1L(\mathbf{v}_1) + d_2L(\mathbf{v}_2) + \dots + d_nL(\mathbf{v}_n) &= \mathbf{0} \\ \Rightarrow d_1 = d_2 = \dots = d_n = 0 &\Rightarrow \mathbf{v} = \mathbf{0}. \end{aligned}$$

□

Theorem 1 Let V and W be vector spaces, both with the same dimension $n < \infty$. Let $L : V \rightarrow W$ be a linear transformation. Let the only solution to $L(\mathbf{v}) = \mathbf{0}$ be $\mathbf{v} = \mathbf{0}$. Then for each $\mathbf{w} \in W$ there is a unique $\mathbf{v} \in V$ with $L(\mathbf{v}) = \mathbf{w}$.

proof: By Theorem 0, the dimension of $\text{Range}(L)$ is $n - \text{dimensional}$. Therefore it must be all of W . So for each $\mathbf{w} \in W$ there is at least one $\mathbf{v}_p \in V$ with $L(\mathbf{v}_p) = \mathbf{w}$. But the general solution to $L(\mathbf{v}) = \mathbf{w}$ is $\mathbf{v} = \mathbf{v}_p + \mathbf{v}_H$ where \mathbf{v}_H is the general solution to the homogeneous equation. By assumption, $\mathbf{v}_H = \mathbf{0}$, so the particular solution is unique. □

Remark: In the base case of undetermined coefficients, $W = V$. In the extended case, W is the space in which f lies, and $V = x^s W$, i.e. the space of all functions which are obtained from ones in W by multiplying them by x^s . This is because if L factors as

$$L = (D - r_1 I)^{k_1} \circ (D - r_2 I)^{k_2} \circ \dots \circ (D - r_m I)^{k_m}$$

and if f is in a subspace W associated with the characteristic polynomial root r_m , then for $s = k_m$ the factor

$(D - r_m I)^{k_m}$ of L will transform the space $V = x^s W$ back into W , and not transform any non-zero function in V into the zero function. And the other factors of L will then preserve W , also without transforming any non-zero elements to the zero function.

Math 2250-004

Friday Mar 16

- Section 5.6: Forced mechanical vibrations.

Announcements:

Warm-up Exercise:

Section 5.6: forced oscillations in mechanical (and electrical) systems. We will continue to discuss section 5.6 on the Monday after spring break. Today we'll discuss what happens when there is no damping - $c = 0$. We'll deal with the damped case after spring break.

But here is an Overview for all cases:

$$m x'' + c x' + k x = F_0 \cos(\omega t)$$

using section 5.5 undetermined coefficients algorithms.

- undamped ($c = 0$) :

In this case the complementary homogeneous differential equation for $x(t)$ is

$$m x'' + k x = 0$$

$$x'' + \frac{k}{m} x = 0$$

$$x'' + \omega_0^2 x = 0$$

which has simple harmonic motion solutions $x_H(t) = C \cos(\omega_0 t - \alpha)$. So for the non-homogeneous DE the method of undetermined coefficients implies we can find particular and general solutions as follows:

- $\omega \neq \omega_0 := \sqrt{\frac{k}{m}} \Rightarrow x_P = A \cos(\omega t)$ because only even derivatives, we don't need $\sin(\omega t)$ terms !!

$$\Rightarrow x = x_P + x_H = A \cos(\omega t) + C_0 \cos(\omega_0 t - \alpha_0).$$

- $\omega \neq \omega_0$ but $\omega \approx \omega_0$, $C \approx C_0$ Beating!

- $\omega = \omega_0 \Rightarrow x_P = t(A \cos(\omega_0 t) + B \sin(\omega_0 t))$ (Case II of undetermined coefficients.)

$$\Rightarrow x = x_P + x_H = C t \cos(\omega_0 t - \alpha) + C_0 \cos(\omega_0 t - \alpha_0).$$

("pure" resonance!)

.....

$$m x'' + c x' + k x = F_0 \cos(\omega t)$$

- damped ($c > 0$): in all cases $x_P = A \cos(\omega t) + B \sin(\omega t) = C \cos(\omega t - \alpha)$ (because the roots of the characteristic polynomial are never $\pm i \omega$ when $c > 0$).

- underdamped: $x = x_P + x_H = C \cos(\omega t - \alpha) + e^{-p t} C_1 \cos(\omega_1 t - \alpha_1).$

- critically-damped: $x = x_P + x_H = C \cos(\omega t - \alpha) + e^{-p t} (c_1 t + c_2).$

- over-damped: $x = x_P + x_H = C \cos(\omega t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}.$

- in all three damped cases on the previous page, $x_H(t) \rightarrow 0$ exponentially and is called the transient solution $x_{tr}(t)$ (because it disappears as $t \rightarrow \infty$). And in these damped cases $x_p(t)$ as above is called the steady periodic solution $x_{sp}(t)$ (because it is what persists as $t \rightarrow \infty$, and because it's periodic).

- if c is small enough and $\omega \approx \omega_0$ then the amplitude C of $x_{sp}(t)$ can be large relative to $\frac{F_0}{m}$, and the system can exhibit practical resonance. This can be an important phenomenon in electrical circuits, where amplifying signals is important.

forced undamped oscillations:

Exercise 1a) Solve the initial value problem for $x(t)$:

$$x'' + 9x = 80 \cos(5t)$$

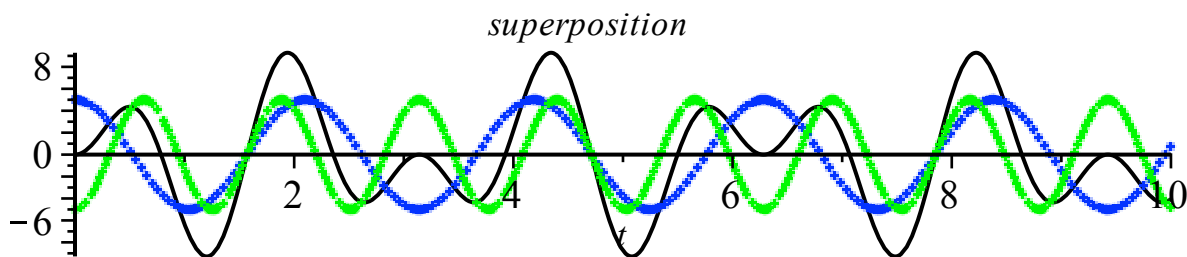
$$x(0) = 0$$

$$x'(0) = 0.$$

1b) This superposition of two sinusoidal functions is periodic because there is a common multiple of their (shortest) periods. What is this (common) period?

1c) Compare your solution and reasoning with the display at the bottom of this page.

```
> with(plots):
> plot1 := plot(-5*cos(5*t), t=0..10, color=green, style=point):
  plot2 := plot(5*cos(3*t), t=0..10, color=blue, style=point):
  plot3 := plot(-5*cos(5*t) + 5*cos(3*t), t=0..10, color=black):
  display({plot1, plot2, plot3}, title='superposition');
```



In general: Use the method of undetermined coefficients to solve the initial value problem for $x(t)$, in the case $\omega \neq \omega_0 = \sqrt{\frac{k}{m}}$:

$$\begin{aligned}x''(t) + \frac{k}{m}x(t) &= \frac{F_0}{m} \cos(\omega t) \\x(0) &= x_0 \\x'(0) &= v_0\end{aligned}$$

Solution:

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)} (\cos(\omega_0 t) - \cos(\omega t)) + x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t)$$

There is an interesting beating phenomenon for $\omega \approx \omega_0$ (but still with $\omega \neq \omega_0$). This is explained analytically via trig identities, and is familiar to musicians in the context of superposed sound waves (which satisfy the homogeneous linear "wave equation" partial differential equation):

$$\begin{aligned}\cos(\alpha - \beta) - \cos(\alpha + \beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \\&\quad - (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) \\&= 2 \sin(\alpha)\sin(\beta) .\end{aligned}$$

Set $\alpha = \frac{1}{2}(\omega + \omega_0)t$, $\beta = \frac{1}{2}(\omega - \omega_0)t$ in the identity above, to rewrite the first term in $x(t)$ as a product rather than a difference:

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)} 2 \sin\left(\frac{1}{2}(\omega + \omega_0)t\right) \sin\left(\frac{1}{2}(\omega - \omega_0)t\right) + x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t) .$$

In this product of sinusoidal functions, the first one has angular frequency and period close to the original angular frequencies and periods of the original sum. But the second sinusoidal function has small angular frequency and long period, given by

$$\text{angular frequency: } \frac{1}{2}(\omega - \omega_0), \quad \text{period: } \frac{4\pi}{|\omega - \omega_0|} .$$

We will call half that period the beating period, as explained by the next exercise:

$$\text{beating period: } \frac{2\pi}{|\omega - \omega_0|}, \quad \text{beating amplitude: } \frac{2F_0}{m|\omega^2 - \omega_0^2|}.$$

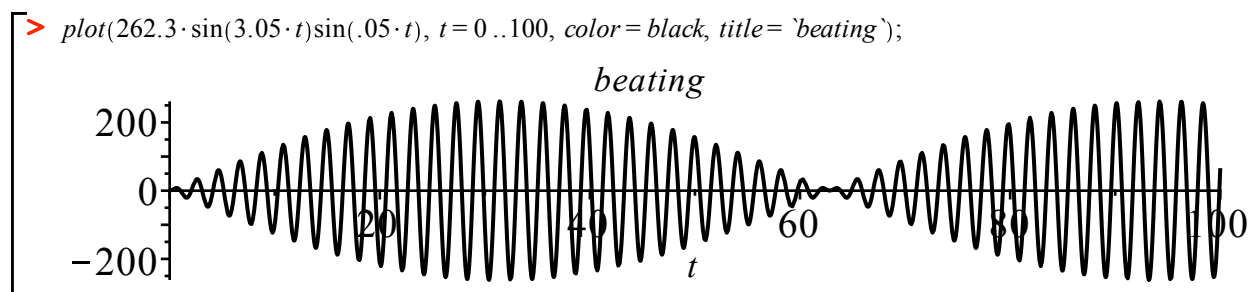
Exercise 2a) Use one of the formulas on the previous page to write down the IVP solution $x(t)$ to

$$x'' + 9x = 80 \cos(3.1t)$$

$$x(0) = 0$$

$$x'(0) = 0.$$

2b) Compute the beating period and amplitude. Compare to the graph shown below.



Resonance:

Resonance! $\omega = \omega_0$ (and the limit as $\omega \rightarrow \omega_0$)

$$\begin{cases} x'' + \omega_0^2 x = \frac{F_0}{m} \cos \omega_0 t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

using 5.5, guess

$$\begin{aligned} + \omega_0^2 (& x_p = t (A \cos \omega_0 t + B \sin \omega_0 t)) \\ 0 (& x_p' = t (-A \omega_0 \sin \omega_0 t + B \omega_0 \cos \omega_0 t) + A \cos \omega_0 t + B \sin \omega_0 t) \\ + 1 (& x_p'' = t (-A \omega_0^2 \cos \omega_0 t - B \omega_0^2 \sin \omega_0 t) + [-A \omega_0 \sin \omega_0 t + B \omega_0 \cos \omega_0 t] 2) \end{aligned}$$

$$L(x_p) = t(0) + 2 [-A \omega_0 \sin \omega_0 t + B \omega_0 \cos \omega_0 t] \stackrel{\text{want}}{=} \frac{F_0}{m} \cos \omega_0 t$$

$$\text{Deduce } \begin{aligned} A &= 0 \\ B &= \frac{F_0}{2m\omega_0} \end{aligned}$$

$$x_p(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

sats $x(0)=0$, $x'(0)=0$, so IVP soln is

$$x(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t + x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t$$

You can also get this solution by letting $\omega \rightarrow \omega_0$ in the beating formula. We will probably do it that way in class, on the next page.

in the case $\omega \neq \omega_0 = \sqrt{\frac{k}{m}}$ we copy the IVP solution in both forms, from previous page

$$\begin{aligned}x''(t) + \frac{k}{m}x(t) &= \frac{F_0}{m}\cos(\omega t) \\ x(0) &= x_0 \\ x'(0) &= v_0\end{aligned}$$

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)} (\cos(\omega_0 t) - \cos(\omega t)) + x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t)$$

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)} 2 \sin\left(\frac{1}{2}(\omega + \omega_0)t\right) \sin\left(\frac{1}{2}(\omega - \omega_0)t\right) + x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t) .$$

If we let $\omega \rightarrow \omega_0$ this solution will converge to the resonance IVP solution on the previous page....

Exercise 3a) Solve the IVP

$$x'' + 9x = 80 \cos(3t)$$

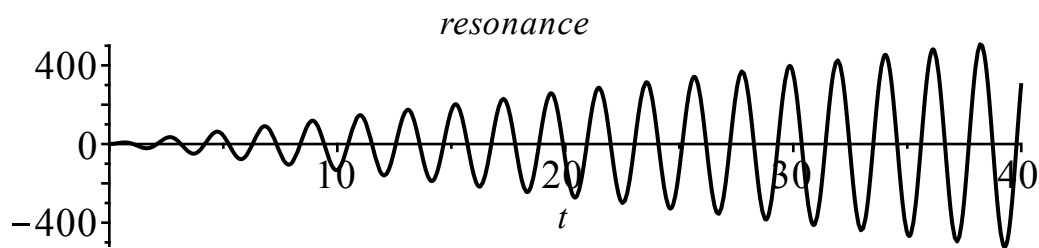
$$x(0) = 0$$

$$x'(0) = 0.$$

First just use the general solution formula above this exercise and substitute in the appropriate values for the various terms. Then, if time, use variation of parameters (see the last pages of today's notes), to check a particular solution and to illustrate this alternate method for finding particular solutions.

3b) Compare the solution graph below with the beating graph in exercise 2.

```
> plot( (40/3) * t * sin(3 * t), t = 0..40, color = black, title = `resonance` );
```



```
>
```

- After finishing the discussion of undamped forced oscillations, we will discuss the physics and mathematics of damped forced oscillations

$$m x'' + c x' + k x = F_0 \cos(\omega t) .$$

Here are some links which address how these phenomena arise, also in more complicated real-world applications in which the dynamical systems are more complex and have more components. Our baseline cases are the starting points for understanding these more complicated systems. We'll also address some of these more complicated applications when we move on to systems of differential equations, in a few weeks.

http://en.wikipedia.org/wiki/Mechanical_resonance (wikipedia page with links)

http://www.nset.org.np/nset/php/pubaware_shaketable.php (shake tables for earthquake modeling)

http://www.youtube.com/watch?v=M_x2jOKAhZM (an engineering class demo shake table)

<http://www.youtube.com/watch?v=j-zczJXSxw> (Tacoma narrows bridge)

http://en.wikipedia.org/wiki/Electrical_resonance (wikipedia page with links)

http://en.wikipedia.org/wiki/Crystal_oscillator (crystal oscillators)