Wed March 7:
5.3 Solving constant coefficient homogeneous linear differential equations

Announcements:
- quiz today.
- Finish Tuesday's notes. Do today's!

'til 10:47

Warm-up Exercise: Find all solutions $y(x)$ to the homogeneous linear differential equation

like §5.1

$$L(y) = y'' + 5y' + 6y = 0$$

Hint: look for a basis of the 2-dimensional solution space made out of exponential functions, $y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}$.

try $y = e^{rx}$ find r's that work

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

$$L(y) = r^2 e^{rx} + 5(re^{rx}) + 6e^{rx} \overset{\text{want}}{=} 0$$

$$= e^{rx}\underbrace{[r^2 + 5r + 6]}_{\text{"characteristic polynomial" for the D.E.}} = 0$$

need $r^2 + 5r + 6 = 0$

$(r+3)(r+2) = 0$

$r = -2, -3$.

So e^{-2x}, e^{-3x} are solns

So since soln space is closed under + & scalar mult. (i.e. subspace),

$$y(x) = c_1 e^{-2x} + c_2 e^{-3x}$$

are solns.

Since soln space is 2-dim'l and since e^{-2x}, e^{-3x} are independent, they're a basis and these are all the solns!!

post pone until Wed.

helps with last part of lab problem

Theorem 4: All solutions to the <u>nonhomogeneous</u> second order linear DE

$$* \quad L(y) = y'' + p(x)y' + q(x)y = f(x)$$

are of the form $y = y_P + y_H$ where y_P is any single particular solution and y_H is some solution to the homogeneous DE. (y_H is called y_c, for complementary solution, in the text). Thus, if you can find a single particular solution to the nonhomogeneous DE, and all solutions to the homogeneous DE, you've actually found all solutions to the nonhomogeneous DE. (You had a homework problem related to this idea, 3.4.40, in homework 5, but in the context of matrix equations, a week or two ago. The same idea reappears in your current lab, in the last problem.)

proof: Make use of the fact that

$$L(y) := y'' + p(x)y' + q(x)y$$

is a linear operator. In other words, use the *linearity properties*

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(2) $L(c\,y) = c\,L(y),\ c \in \mathbb{R}$.

<u>proof</u>. Let $L(y_P) = f$

Let $L(y_H) = 0$

Then $L(y_P + y_H) = L(y_P) + L(y_H)$ (1)

$= f + 0$

$= f$

so $y_P + y_H$ solves $*$

Conversely, let

$L(y_Q) = f$

then $y_Q = y_P + \underbrace{(y_Q - y_P)}$

$L(y_Q - y_P) = L(y_Q) + L(-y_P)$ (1)

$= L(y_Q) - L(y_P)$ (2)

$= f - f$

$= 0$

so $y_Q - y_P$ <u>was</u> a homog. soln!

In Monday's notes we found that the general solution to the homogeneous differential equation

$$y'' - 2y' - 3y = 0$$

is

$$y_H = c_1 e^{-x} + c_2 e^{3x}.$$

Now consider the non-homogeneous differential equation

$$L(y) = y'' - 2y' - 3y = 6.$$

$y_p = A$
$y_p' = 0$
$y_p'' = 0$

want

$L(y_p) = 0 - 0 - 3A = 6$

$A = -2$

Notice that

$$y_P = -2$$

is one particular solution to the differential equation.

Exercise 1a) Solve the initial value problem

$$y'' - 2y' - 3y = 6.$$
$$y(0) = -1$$
$$y'(0) = -5$$

with a solution to the differential equation of the form

$$y = y_P + y_H = -2 + c_1 e^{-x} + c_2 e^{3x}.$$

by Thm 4!!

$$y' = 0 + -c_1 e^{-x} + 3c_2 e^{3x}$$

$$y(0) = -1 = -2 + c_1 + c_2$$
$$y'(0) = -5 = -c_1 + 3c_2$$

$c_1 + c_2 = 1$
$-c_1 + 3c_2 = -5$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 3 & -5 \end{array}\right]$$

$R_1 + R_2 \to R_2$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 4 & -4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \end{array}\right]$$

$-R_2 + R_1 \to R_1$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$$

$c_1 = 2$
$c_2 = -1$

$$\boxed{y(x) = -2 + 2e^{-x} - 1e^{3x}}$$

$y(0) = -2 + 2 - 1 = -1$ ✓
$y'(0) = 0 - 2 - 3 = -5$ ✓

1b) Notice that the same algebra shows you could solve every initial value problem

$$y'' - 2y' - 3y = 6.$$
$$y(0) = b_0$$
$$y'(0) = b_1$$

with a solution of the form

$$y = y_P + y_H = -2 + c_1 e^{-x} + c_2 e^{3x}$$

so by the uniqueness theorem for initial value problems, these ARE all the solutions to the differential equation even though we did not get them a direct method like we used for first order linear differential equations.

For the next two sections we focus homogneous linear differential equations with constant coefficients. Section 5.3 contains the algorithms we'll need and in section 5.4 we'll apply the general theory to the unforced mass-spring differential equation.

5.3: Algorithms for the basis and general (homogeneous) solution to

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

when the coefficients $\underline{a_{n-1}}, \underline{a_{n-2}}, \ldots \underline{a_1}, \underline{a_0}$ are all constant.

step 1) Try to find a basis for the $\underbrace{\text{solution space}}_{n\text{-dim'l}}$ made of exponential functions....try $y(x) = e^{rx}$. In this case

$y' = re^{rx}$
$y'' = r^2 e^{rx}$
$y''' = r^3 e^{rx}$

$$L(y) = e^{rx}\left(r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0\right) = e^{rx} p(r).$$

We call this polynomial $p(r)$ the characteristic polynomial for the differential equation, and can read off what it is directly from the expression for $L(y)$ if we want. For each root r_j of $p(r)$, we get a solution $e^{r_j x}$ to the homogeneous DE.

Case 1) If $p(r)$ has n distinct (i.e. different) real roots $r_1, r_2, \ldots, r_n$, then

$$e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}$$

is a basis for the solution space; i.e. the general solution is given by

$$y_H(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_n e^{r_n x}.$$

Example: The differential equation

$$y''' + 3y'' - y' - 3y = 0$$

has characteristic polynomial

$$p(r) = r^3 + 3r^2 - r - 3 = (r+3)\cdot(r^2 - 1) = (r+3)(r+1)\cdot(r-1)$$

so the general solution to

$$y''' + 3y'' - y' - 3y = 0$$

is

$r = -3, -1, 1$

solns e^x, e^{-x}, e^{-3x}

$$y_H(x) = c_1 e^x + c_2 e^{-x} + c_3 e^{-3x}.$$

up to now, you would've checked that $\{e^x, e^{-x}, e^{-3x}\}$ are a basis for soltn space, with Wronskian.

$y''' + 3y'' - y' - 3y = 0$
$y(0) = b_0$
$y'(0) = b_1$
$y''(0) = b_2$

$y(x) = c_1 e^x + c_2 e^{-x} + c_3 e^{-3x}$
$y'(x) = c_1 e^x + c_2(-e^{-x}) + c_3(-3e^{-3x})$
$y''(x) = c_1 e^x + c_2 e^{-x} + c_3(9e^{-3x})$

$$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = \underbrace{\begin{bmatrix} e^x & e^{-x} & e^{-3x} \\ e^x & -e^{-x} & -3e^{-3x} \\ e^x & e^{-x} & 9e^{-3x} \end{bmatrix}}_{W(e^x, e^{-x}, e^{-3x})} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

IVP

$$\begin{bmatrix} y(0) \\ y'(0) \\ y''(0) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -3 \\ 1 & 1 & 9 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}$$

det $\neq 0$!

Exercise 1) By construction, $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}$ all solve the differential equation. Show that they're linearly independent. This will be enough to verify that they're a basis for the solution space, since we know the solution space is n-dimensional. Hint: The easiest way to show this is to list your roots so that $r_1 < r_2 < \ldots < r_n$ and to use a limiting argument.

better way to show basis

- if n fcns are indep. in n-dim'l space, then they're a basis.

$$c_1 e^{x} + c_2 e^{-x} + c_3 e^{-3x} = 0 \quad x \in \mathbb{R}.$$

$$\div e^{x} \qquad c_1 + c_2 e^{-2x} + c_3 e^{-4x} = 0 \quad x \in \mathbb{R}$$

$$\lim_{x\to\infty}: \quad c_1 + 0 + 0 = 0 \qquad \text{so } c_1 = 0.$$

$$c_2 e^{-x} + c_3 e^{-3x} = 0$$

$$\div e^{-x} \ (\text{mult. by } e^{x}): \quad c_2 + c_3 e^{-2x} = 0$$

$$\lim_{x\to\infty} \quad c_2 + 0 = 0 \qquad \text{so } c_2 = 0$$

$$c_3 e^{-3x} = 0 \implies c_3 = 0$$

Case 2) Repeated real roots. In this case $p(r)$ has all real roots $r_1, r_2, \dots r_m$ $(m < n)$ with the r_j all different, but some of the factors $(r - r_j)$ in $p(r)$ appear with powers bigger than 1. In other words, $p(r)$ factors as

$$p(r) = (r - r_1)^{k_1}(r - r_2)^{k_2} \cdots (r - r_m)^{k_m}$$

with some of the $k_j > 1$, and $k_1 + k_2 + \dots + k_m = n$.

Start with a small example: The case of a second order DE for which the characteristic polynomial has a double root.

Exercise 2) Let r_1 be any real number. Consider the homogeneous DE

$$L(y) := y'' - 2\,r_1\,y' + r_1^2\,y = 0.$$

with $p(r) = r^2 - 2\,r_1\,r + r_1^2 = (r - r_1)^2$, i.e. r_1 is a double root for $p(r)$. Show that $e^{r_1 x}, x\,e^{r_1 x}$ are a basis for the solution space to $L(y) = 0$, so the general homogeneous solution is

$y_H(x) = c_1 e^{r_1 x} + c_2\,x\,e^{r_1 x}$. Start by checking that $x\,e^{r_1 x}$ actually (magically?) solves the DE.
(We may wish to study a special case $y'' + 6\,y' + 9\,y = 0$.)

$y'' + 6y' + 9y = 0$

you checked one like
this in your HW!

Here's the general algorithm: If

$$p(r) = \left(r - r_1\right)^{k_1}\left(r - r_2\right)^{k_2} \cdots \left(r - r_m\right)^{k_m}$$

then (as before) $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_m x}$ are independent solutions, but since $m < n$ there aren't enough of them to be a basis. Here's how you get the rest: For each $k_j > 1$, you actually get independent solutions

$$e^{r_j x}, x\, e^{r_j x}, x^2 e^{r_j x}, \ldots, x^{k_j - 1} e^{r_j x}.$$

This yields k_j solutions for each root r_j, so since $k_1 + k_2 + \ldots + k_m = n$ you get a total of n solutions to the differential equation. There's a good explanation in the text as to why these additional functions actually do solve the differential equation, see pages 316-318 and the discussion of "polynomial differential operators". I've also made a homework problem in which you can explore these ideas. Using the limiting method we discussed earlier, it's not too hard to show that all n of these solutions are indeed linearly independent, so they are in fact a basis for the solution space to $L(y) = 0$.

Exercise 3) Explicitly antidifferentiate to show that the solution space to the differential equation for $y(x)$

$$y^{(4)} - y^{(3)} = 0$$

agrees with what you would get using the repeated roots algorithm in Case 2 above. Hint: first find $v = y'''$, using $v' - v = 0$, then antidifferentiate three times to find y_H. When you compare to the repeated roots algorithm, note that it includes the possibility $r = 0$ and that $e^{0\,x} = 1$.

Case 3) Complex number roots - this will be our surprising and fun topic on Friday. Our analysis will explain exactly how and why trig functions and mixed exponential-trig-polynomial functions show up as solutions for some of the homogeneous DE's you worked with in your homework and lab for this past week. This analysis depends on Euler's formula, one of the most beautiful and useful formulas in mathematics:

$$e^{i\,\theta} = \cos(\theta) + i\sin(\theta)$$
$$\text{for } i^2 = -1.$$