Tues March 6:

5.1-5.2 Second order and n^{th} order linear differential equations, and vector space theory connections.

<u>Announcements:</u>

- office hours 4:30–6:00 JWB 240 (Chapter 4 is hard).
- finish Monday's & Tuesday's notes

'til 10:47

<u>Warm-up Exercise:</u> a) Solve the IVP

a lot like §5.1 HW

$$\begin{cases} y'' - 2y' - 3y = 0 \\ y(0) = 1 \\ y'(0) = -5 \end{cases}$$

using the D.E. solutions we verified on Monday,

$$y(x) = c_1 e^{3x} + c_2 e^{-x}$$

warmup

$$\boxed{= -e^{3x} + 2e^{-x}}$$

(find c_1, c_2)

b) Would you have been able to solve with arbitrary initial values

$$y(0) = b_1 \quad ? \quad \text{yes.}$$
$$y'(0) = b_2$$

$$y(x) = c_1 e^{3x} + c_2 e^{-x}$$
$$y'(x) = c_1(3e^{3x}) + c_2(-e^{-x})$$

(reducing augmented matrix faster)

this would've been any $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$\begin{bmatrix} y(x) \\ y'(x) \end{bmatrix} = \begin{bmatrix} e^{3x} & e^{-x} \\ 3e^{3x} & -e^{-x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Wronskian Matrix, $W(e^{3x}, e^{-x})$

$$W(y_1, y_2) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

For IVP $x = 0$

@ $x = 0$

$$\begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

inverse matrix

$$\frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} = -\frac{1}{4} \begin{bmatrix} 4 \\ -8 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

so $\boxed{y(x) = -e^{3x} + 2e^{-x}}$

Since each IVP can be solved this way, using $c_1 e^{3x} + c_2 e^{-x} = y$, the existence-uniquess thm for IVP's tells us these are <u>all</u> to solutns to DE!!

Exercise 3a) Check the linearity properties (1),(2) for the general second order differential operator L and general functions $y_1(x), y_2(x)$. Compare to matrix multiplication properties. In other words show that the operator L defined by

$$\boxed{L(y) := y'' + p(x)y' + q(x)y}$$

satisfies the so-called *linearity properties*

Chptr 3-4 analog.

$A(\vec{x}+\vec{z}) = A\vec{x} + A\vec{z}$

$A(c\vec{x}) = cA\vec{x}$

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(2) $L(cy) = cL(y), c \in \mathbb{R}$.

(1) $L(y_1+y_2) = (y_1+y_2)'' + p(x)(y_1+y_2)' + q(x)(y_1+y_2)$

$= (y_1'' + y_2'') + p(x)(y_1' + y_2') + q(x)(y_1+y_2)$

$= y_1'' + p(x)y_1' + q(x)y_1 + y_2'' + p(x)y_2' + q(x)y_2$

$= L(y_1) + L(y_2)$

Tuesday:

(2) $L(cy) = (cy)'' + p(x)(cy)' + q(x)(cy)$

$= cy'' + p(x)cy' + q(x)cy = c[y'' + p(x)y' + q(x)y]$

$= cL(y)$

3b) Use these properties to show that what happened in Exercise 2b was no accident:

Theorem 2: the solution space to the *homogeneous* second order linear DE

$$L(y) = y'' + p(x)y' + q(x)y = 0$$

is a subspace. Notice that this is the analogous proof we used earlier to show that the solution space to a homogeneous matrix equation is a subspace.

α) closed under +

β) closed under scalar mult

proof α) let $L(y_1) = 0$

$L(y_2) = 0$

so $L(y_1+y_2) = L(y_1) + L(y_2) = 0 + 0 = 0$!

so y_1+y_2 solves the homog. DE too

β) Let $L(y) = 0$

$L(cy) = cL(y) = c \cdot 0 = 0$ so cy solves the DE too.

Chptr 4

Soln space to $A\vec{x} = \vec{0}$

Unlike in the previous example, and unlike what was true for the first order linear differential equation

$$y' + p(x)y = q(x)$$

there is <u>not</u> a clever integrating factor formula that will always work to find the general solution of the second order linear differential equation

$$y'' + p(x)y' + q(x)y = f(x).$$

Rather, we will usually resort to vector space theory and algorithms based on clever guessing, as in the following example:

<u>Exercise 4)</u> Consider the homogeneous linear DE for $y(x)$

$$y'' - 2y' - 3y = 0$$

<u>4a)</u> Find two exponential functions $y_1(x) = e^{rx}$, $y_2(x) = e^{\rho x}$ that solve this DE.

$y' + ay = 0$

$y' = -ay$

$y = Ce^{-ax}$

exponentials worked for 1st order, const coeff homog. linear DE's, so why not try them for higher order.

try e^{rx}

$y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

$y'' - 2y' - 3y = r^2 e^{rx} - 2(re^{rx}) - 3e^{rx}$

$= e^{rx}\left[r^2 - 2r - 3\right] \overset{\text{want}}{=} 0$ ← zero fun

$= e^{rx}\left[(r-3)(r+1)\right]$

$r = 3, -1$ will work.

So e^{3x}, e^{-x} are soltns.

Since soltn space is sub vector space.

$c_1 e^{3x}$, $c_2 e^{-x}$, $c_1 e^{3x} + c_2 e^{-x}$ are soltns!

$y(x) = c_1 e^{3x} + c_2 e^{-x}$

Monday warmup explained.

4b) Show that every IVP

Tuesday's warm-up

$$y'' - 2y' - 3y = 0$$
$$y(0) = b_0$$
$$y'(0) = b_1$$

can be solved with a unique linear combination $y(x) = c_1 y_1(x) + c_2 y_2(x)$, (where c_1, c_2 depend on b_0, b_1).

$= c_1 e^{3x} + c_2 e^{-x}$

Then use the uniqueness theorem to deduce that y_1, y_2 span the solution space to this homogeneous differential equation. Since these two functions are not constant multiples of each other, they are linearly independent and a basis for the *2-dimensional* solution space!

Some important facts about spanning sets, independence, bases and dimension follow from one key fact, and then logic. We will want to use these facts going forward, in our study of differential equations.

key fact: If n vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ span a vector space W then any collection $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_N$ of vectors in W with $N > n$ will always be linearly dependent. (This is explained on pages 254-255 of the text, and has to to do with matrix facts that we already know.) Notice too that this fact fits our intuition based on what we know in the special cases that we've studied, in particular $W = \mathbb{R}^n$.)

Thus:
1) If a finite collection of vectors in W is linearly independent, then no collection with fewer vectors can span all of W. (This is because if the smaller collection did span, the larger collection wouldn't have been linearly independent after all, by the key fact.)

2) Every basis of W has the same number of vectors, so the concept of dimension is well-defined and doesn't depend on choice of basis. (This is because if $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are a basis for W then every larger collection of vectors is dependent by the key fact and every smaller collection fails to span by (1), so only collections with exactly n vectors have a chance to be bases.)

3) Let the dimension of W be the number n, i.e. there is some basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ for W. Then if vectors $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$ span W then they're automatically linearly independent and thus a basis. (If they were dependent we could delete one of the $\underline{\boldsymbol{w}}_j$ that was a linear combination of the others and still have a spanning set. This would violate (1) since $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly independent.)

4) Let the dimension of W be the number n, i.e. there is some basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ for W. Then if $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$ are in W and are linearly independent, they automatically span W and thus are a basis. (If they didn't span W we could augment with a vector $\boldsymbol{w}_{n+1}$ not in their span and have a collection of $n+1$ still independent* vectors in W, violating the key fact.

* Check: If $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$ are linearly independent, and $\underline{\boldsymbol{w}}_{n+1} \notin span\{\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n\}$, then $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n, \underline{\boldsymbol{w}}_{n+1}$ are also linearly independent. This fact generalizes the ideas we used when we figured out all possible subspaces of $\mathbb{R}^3$. Here's how it goes:

To show the larger collection is still linearly independent study the equation

$$c_1\underline{\boldsymbol{w}}_1 + c_2\underline{\boldsymbol{w}}_2 + \ldots + c_n\underline{\boldsymbol{w}}_n + d\,\underline{\boldsymbol{w}}_{n+1} = \underline{\mathbf{0}}.$$

Since $\underline{\boldsymbol{w}} \notin span\{\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n\}$ it must be that $d = 0$ (since otherwise we could solve for $\underline{\boldsymbol{w}}_{n+1}$ as a linear combination of $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$). But once $d = 0$, we have

$$c_1\underline{\boldsymbol{w}}_1 + c_2\underline{\boldsymbol{w}}_2 + \ldots + c_n\underline{\boldsymbol{w}}_n = \underline{\mathbf{0}}$$

which implies $c_1 = c_2 = \ldots = c_n = 0$ by the independence of $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$.

Theorem 3: The solution space to the second order homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = 0$$

is 2-dimensional on any interval I for which the hypotheses of the existence-uniqueness theorem hold.

proof:

Pick any $x_0 \in I$. Find solutions $y_1(x), y_2(x)$ to initial value problems at x_0 so that the so-called Wronskian matrix for y_1, y_2 at x_0

$$W(y_1, y_2)(x_0) = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}$$

is invertible (i.e. $\begin{bmatrix} y_1(x_0) \\ y_1'(x_0) \end{bmatrix}, \begin{bmatrix} y_2(x_0) \\ y_2'(x_0) \end{bmatrix}$ are a basis for $\mathbb{R}^2$, or equivalently so that the determinant of the Wronskian matrix (called just the Wronskian) is non-zero at x_0).

- You may be able to find suitable y_1, y_2 by a method like we used in the previous example, but the existence-uniqueness theorem guarantees they exist even if you don't know how to find formulas for them.

Under these conditions, the solutions y_1, y_2 are actually a basis for the solution space! Here's why:

• span: the condition that the Wronskian matrix is invertible at x_0 means we can solve each IVP there with a linear combination $y = c_1 y_1 + c_2 y_2$: In that case, to solve the IVP

$$\begin{aligned} y'' + p(x)y' + q(x)y &= 0 \\ y(x_0) &= b_0 \\ y'(x_0) &= b_1 \end{aligned}$$

we set

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

$y'(x) = c_1 y_1' + c_2 y_2'$

At x_0 we wish to find c_1, c_2 so that

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= b_0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= b_1. \end{aligned}$$

This system is equivalent to the the matrix equation

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}.$$

When the Wronskian matrix at x_0 has an inverse, the unique solution $[c_1, c_2]^T$ is given by

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}^{-1} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}.$$

Since the uniqueness theorem says each IVP has a unique solution, we've found it!

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

- <u>Span</u>: Since each solution $y(x)$ to the differential equation solves *some* initial value problem at x_0, this gives all solutions, as we let $[b_0, b_1]^T$ vary freely in $\mathbb{R}^2$. So each solution $y(x)$ *is* a linear combination of y_1, y_2. Thus $\{y_1, y_2\}$ <u>spans</u> the solution space.

- <u>Linear independence:</u> If we have the identity

$$c_1 y_1(x) + c_2 y_2(x) = 0$$

then by differentiating each side with respect to x we also have

$$c_1 y_1'(x) + c_2 y_2'(x) = 0.$$

Evaluating at $x = x_0$ this is the system

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= 0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= 0 \end{aligned}$$

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The theory for n^{th} order linear differential equations is conceptually the same as for second order...

<u>Definition:</u> An nth order linear differential equation for a function $y(x)$ is a differential equation that can be written in the form

$$A_n(x)y^{(n)} + A_{n-1}(x)y^{(n-1)} + \ldots + A_1(x)y' + A_0(x)y = F(x) .$$

We search for solution functions $y(x)$ defined on some specified interval I of the form $a < x < b$, or (a, ∞), $(-\infty, a)$ or (usually) the entire real line $(-\infty, \infty)$. In this chapter we assume the function $A_n(x) \neq 0$ on I, and divide by it in order to rewrite the differential equation in the standard form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f .$$

($a_{n-1}, \ldots a_1, a_0, f$ are all functions of x, and the DE above means that equality holds for all value of x in the interval I.)

<u>Theorem 1</u> (Existence-Uniqueness Theorem): Let $a_{n-1}(x), a_{n-2}(x), \ldots a_1(x), a_0(x), f(x)$ be specified continuous functions on the interval I, and let $x_0 \in I$. Then there is a unique solution $y(x)$ to the <u>initial value problem</u>

$$\begin{gathered} y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f \\ y(x_0) = b_0 \\ y'(x_0) = b_1 \\ y''(x_0) = b_2 \\ \vdots \\ y^{(n-1)}(x_0) = b_{n-1} \end{gathered}$$

and $y(x)$ exists and is n times continuously differentiable on the entire interval I.

The differential equation

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$

is called <u>linear</u> because the operator L defined by

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y$$

satisfies the so-called <u>linearity properties</u>

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(2) $L(c\,y) = c\,L(y),\ c \in \mathbb{R}$.

• *The proof that L satisfies the linearity proporties is just the same as it was for the case when* $n = 2$, which we checked.

The following two theorems only use the linearity properties of the operator L. I've kept the same numbering we used for the case $n = 2$.

<u>Theorem 2:</u> The solution space to the homogeneous linear DE

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

is a subspace.

<u>Theorem 4:</u> The general solution to the <u>nonhomogeneous</u> n^{th} order linear DE

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$

is $y = y_P + y_H$ where y_P is any single particular solution and y_H is the general solution to the homogeneous DE. (y_H is called y_c, for complementary solution, in the text).

<u>Theorem 3:</u> The solution space to the nth order homogeneous linear differential equation

$$y^{(n)} + a_{n-1}\, y^{(n-1)} + \ldots + a_1\, y' + a_0\, y \equiv 0$$

is n-dimensional. Thus, any n independent solutions $y_1, y_2, \ldots y_n$ will be a basis, and all homogeneous solutions will be uniquely expressible as linear combinations

$$y_H = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$

<u>proof:</u> By the existence half of <u>Theorem 1</u>, we know that there are solutions for each possible initial value problem for this (homogenenous case) of the IVP for n^{th} order linear DEs. So, pick solutions $y_1(x), y_2(x), \ldots. y_n(x)$ so that their vectors of initial values (which we'll call initial value vectors)

$$\begin{bmatrix} y_1(x_0) \\ y_1'(x_0) \\ y_1''(x_0) \\ \vdots \\ y_1^{(n-1)}(x_0) \end{bmatrix}, \begin{bmatrix} y_2(x_0) \\ y_2'(x_0) \\ y_2''(x_0) \\ \vdots \\ y_2^{(n-1)}(x_0) \end{bmatrix}, \ldots, \begin{bmatrix} y_n(x_0) \\ y_n'(x_0) \\ y_n''(x_0) \\ \vdots \\ y_n^{(n-1)}(x_0) \end{bmatrix}$$

are a basis for $\mathbb{R}^n$ (i.e. these n vectors are linearly independent and span $\mathbb{R}^n$. (Well, you may not know how to "pick" such solutions, but you know they exist because of the existence theorem.)

<u>Claim:</u> In this case, the solutions $y_1, y_2, \ldots y_n$ are a basis for the solution space. In particular, every solution to the homogeneous DE is a unique linear combination of these n functions and the dimension of the solution space is n .... discussion on next page.

- Check that $y_1, y_2, \ldots y_n$ **span** the solution space: Consider any solution $y(x)$ to the DE. We can compute its vector of initial values

$$\begin{bmatrix} y(x_0) \\ y'(x_0) \\ y''(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix} := \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}.$$

Now consider a linear combination $z = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$. Compute its initial value vector, and notice that you can write it as the product of the Wronskian matrix at x_0 times the vector of linear combination coefficients:

$$\begin{bmatrix} z(x_0) \\ z'(x_0) \\ \vdots \\ z^{(n-1)}(x_0) \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y'_n(x_0) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

We've chosen the $y_1, y_2, \ldots y_n$ so that the Wronskian matrix at x_0 has an inverse, so the matrix equation

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y'_n(x_0) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}$$

has a unique solution $\underline{\mathbf{c}}$. For this choice of linear combination coefficients, the solution $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ has the same initial value vector at x_0 as the solution $y(x)$. By the uniqueness half of the existence-uniqueness theorem, we conclude that

$$y(x) = z(x) = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n .$$

Thus $y_1, y_2, \ldots y_n$ span the solution space.

- **linear independence**: If a linear combination $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \equiv 0$, then differentiate this identity $n - 1$ times, and then substitute $x = x_0$ into the resulting n equations. This yields the Wronskian matrix equation above, with $[b_0, b_1, \ldots b_{n-1}]^T = [0, 0, \ldots, 0]^T$. So the matrix equation above implies that $[c_1, c_2, \ldots c_n]^T = \mathbf{0}$. So $y_1, y_2, \ldots y_n$ are also linearly independent.

- Thus $y_1, y_2, \ldots y_n$ are a basis for the solution space and the general solution to the homogeneous DE can be written as

$$y_H = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n .$$

Let's do some new exercises that tie these ideas together. (We may do these exercises while or before we wade through the general discussions on the previous pages!)

<u>Exercise 2)</u> Consider the 3rd order linear homogeneous DE for $y(x)$:

$$L(y) = y''' + 3y'' - y' - 3y = 0.$$

Find a basis for the 3-dimensional solution space, and the general solution. Use the Wronskian matrix (or determinant) to verify you have a basis. Hint: try exponential functions.

<u>try</u>

$y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

$y''' = r^3 e^{rx}$

$$L(y) = r^3 e^{rx} + 3r^2 e^{rx} - re^{rx} - 3e^{rx} \equiv 0$$

$$= e^{rx}\left[r^3 + 3r^2 - r - 3\right]$$

$$= e^{rx}\left[r^2(r+3) - (r+3)\right]$$

"cheat"

$$= \cancel{e^{rx}}\left[(r+3)(r^2-1)\right] \equiv 0$$

$$(r+3)(r-1)(r+1) = 0$$

$$r = 1, -1, -3$$

e^{x}, e^{-x}, e^{-3x} are solns

So $y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{-3x}$ are solns

these are all!