Tues March 6:
5.1-5.2 Second order and $n^{th}$ order linear differential equations, and vector space theory connections.

Announcements:

- Office hours 4:30-6:00 JW3 240 (Chapter 4 is hand).
- Finish Monday's & Tuesday's notes.

Warm-up Exercise:

a) Solve the IVP

\[
\begin{cases}
  y'' - 2y' - 3y = 0 \\
  y(0) = 1 \\
  y'(0) = -5
\end{cases}
\]

using the D.E. solutions we verified on Monday:

\[
y(x) = c_1 e^{3x} + c_2 e^{-x}
\]

(finding $c_1, c_2$): 

\[
y(x) = -e^{3x} + 2e^{-x}
\]

b) Would you have been able to solve with arbitrary initial values?

\[
y(0) = b_1 \\
y'(0) = b_2
\]

\[
y(x) = c_1 e^{3x} + c_2 e^{-x}
\]

\[
y'(x) = c_1 (3e^{3x}) + c_2 (-e^{-x})
\]

For IVP:

\[
\begin{bmatrix}
  y(x) \\
  y'(x)
\end{bmatrix} =
\begin{bmatrix}
  e^{3x} & e^{-x} \\
  3e^{3x} & -e^{-x}
\end{bmatrix}
\begin{bmatrix}
  c_1 \\
  c_2
\end{bmatrix}
\]

Wronskian Matrix:

\[
W(e^{3x}, e^{-x}) =
\begin{bmatrix}
  y_1 & y_2 \\
  y_1' & y_2'
\end{bmatrix}
\]

Since each IVP can be solved this way, using $c_1 e^{3x} + c_2 e^{-x} = y$,
the existence-uniqueness theorem for IVP's tells us these are all solutions to DE!!
Exercise 3a) Check the linearity properties (1),(2) for the general second order differential operator $L$ and general functions $y_1(x), y_2(x)$. Compare to matrix multiplication properties. In other words show that the operator $L$ defined by

$$L(y) := y'' + p(x)y' + q(x)y$$

satisfies the so-called linearity properties

1. $L(y_1 + y_2) = L(y_1) + L(y_2)$

2. $L(c y) = c L(y), c \in \mathbb{R}$.

$$L(y_1 + y_2) = (y_1 + y_2)'' + p(x)(y_1 + y_2)' + q(x)(y_1 + y_2)$$

$$= (y_1'' + y_2'') + p(x)(y_1' + y_2') + q(x)(y_1 + y_2)$$

$$= y_1'' + p(x)y_1' + q(x)y_1 + y_2'' + p(x)y_2' + q(x)y_2 = L(y_1) + L(y_2)$$

Tuesday:

$$L(c y) = (c y)' = c y' + p(x)(c y)' + q(x)(c y)$$

$$= c y'' + p(x)c y' + q(x)c y = c [y'' + p(x)y' + q(x)y] = c L(y)$$

3b) Use these properties to show that what happened in Exercise 2b was no accident:

**Theorem 2:** the solution space to the homogeneous second order linear DE

$$L(y) = y'' + p(x)y' + q(x)y = 0$$

is a subspace. Notice that this is the analogous proof we used earlier to show that the solution space to a homogeneous matrix equation is a subspace.

**Proof**

a) Let $L(y_1) = 0$, $L(y_2) = 0$.

So $L(y_1 + y_2) = L(y_1) + L(y_2) = 0 + 0 = 0$.

So $y_1 + y_2$ satisfies the homog. DE too.

b) Let $L(y) = 0$.

$L(c y) = c L(y) = c \cdot 0 = 0$ so $c y$ solves the DE too.
Unlike in the previous example, and unlike what was true for the first order linear differential equation

$$y' + p(x)y = q(x)$$

there is not a clever integrating factor formula that will always work to find the general solution of the second order linear differential equation

$$y'' + p(x)y' + q(x)y = f(x).$$

Rather, we will usually resort to vector space theory and algorithms based on clever guessing, as in the following example:

**Exercise 4)** Consider the homogeneous linear DE for $y(x)$

$$y'' - 2y' - 3y = 0$$

4a) Find two exponential functions $y_1(x) = e^{rx}, y_2(x) = e^{sx}$ that solve this DE.

\[y' + ay = 0\]
\[y' = -ay\]
\[y = Ce^{-ax}\]

exponentials worked for 1st order, const coeff homog. linear DE's, so why not try them for higher order.

\[y'' - 2y' - 3y = r^2e^{rx} - 2(re^{rx}) - 3e^{rx}\]
\[= e^{rx}[r^2 - 2r - 3]\]
\[= e^{rx}(r-3)(r+1)\]
\[r = 3, -1\] will work.

So $e^{3x}, e^{-x}$ are solns.

Since soln space is sub vector space, $c_1e^{3x}, c_2e^{-x}, c_1e^{3x} + c_2e^{-x}$ are solns.

**Monday:** explained.
4b) Show that every IVP

\[ y'' - 2y' - 3y = 0 \]
\[ y(0) = b_0 \]
\[ y'(0) = b_1 \]

can be solved with a unique linear combination

\[ y(x) = c_1 y_1(x) + c_2 y_2(x), \quad (\text{where } c_1, c_2 \text{ depend on } b_0, b_1). \]

\[ = c_1 e^{3x} + c_2 e^{-x} \]

Then use the uniqueness theorem to deduce that \( y_1, y_2 \) span the solution space to this homogeneous differential equation. Since these two functions are not constant multiples of each other, they are linearly independent and a basis for the 2-dimensional solution space!
Some important facts about spanning sets, independence, bases and dimension follow from one key fact, and then logic. We will want to use these facts going forward, in our study of differential equations.

**key fact:** If \( n \) vectors \( v_1, v_2, \ldots, v_n \) span a vector space \( W \) then any collection \( w_1, w_2, \ldots, w_N \) of vectors in \( W \) with \( N > n \) will always be linearly dependent. (This is explained on pages 254-255 of the text, and has to do with matrix facts that we already know.) Notice too that this fact fits our intuition based on what we know in the special cases that we've studied, in particular \( W = \mathbb{R}^n \).

Thus:

1) If a finite collection of vectors in \( W \) is linearly independent, then no collection with fewer vectors can span all of \( W \). (This is because if the smaller collection did span, the larger collection wouldn't have been linearly independent after all, by the **key fact**.)

2) Every basis of \( W \) has the same number of vectors, so the concept of dimension is well-defined and doesn't depend on choice of basis. (This is because if \( v_1, v_2, \ldots, v_n \) are a basis for \( W \) then every larger collection of vectors is dependent by the **key fact** and every smaller collection fails to span by (1), so only collections with exactly \( n \) vectors have a chance to be bases.)

3) Let the dimension of \( W \) be the number \( n \), i.e. there is some basis \( v_1, v_2, \ldots, v_n \) for \( W \). Then if vectors \( w_1, w_2, \ldots, w_n \) span \( W \) then they're automatically linearly independent and thus a basis. (If they were dependent we could delete one of the \( w_j \) that was a linear combination of the others and still have a spanning set. This would violate (1) since \( v_1, v_2, \ldots, v_n \) are linearly independent.)

4) Let the dimension of \( W \) be the number \( n \), i.e. there is some basis \( v_1, v_2, \ldots, v_n \) for \( W \). Then if \( w_1, w_2, \ldots, w_n \) are in \( W \) and are linearly independent, they automatically span \( W \) and thus are a basis. (If they didn't span \( W \) we could augment with a vector \( w_{n+1} \) not in their span and have a collection of \( n+1 \) still independent* vectors in \( W \), violating the **key fact**.

* Check: If \( w_1, w_2, \ldots, w_n \) are linearly independent, and \( w_{n+1} \not\in \text{span} \{ w_1, w_2, \ldots, w_n \} \), then \( w_1, w_2, \ldots, w_n, w_{n+1} \) are also linearly independent. This fact generalizes the ideas we used when we figured out all possible subspaces of \( \mathbb{R}^3 \). Here's how it goes:

To show the larger collection is still linearly independent study the equation

\[
    c_1w_1 + c_2w_2 + \ldots + c_nw_n + d\ w_{n+1} = \mathbf{0}.
\]

Since \( w \not\in \text{span} \{ w_1, w_2, \ldots, w_n \} \) it must be that \( d = 0 \) (since otherwise we could solve for \( w_{n+1} \) as a linear combination of \( w_1, w_2, \ldots, w_n \)). But once \( d = 0 \), we have

\[
    c_1w_1 + c_2w_2 + \ldots + c_nw_n = \mathbf{0}
\]

which implies \( c_1 = c_2 = \ldots = c_n = 0 \) by the independence of \( w_1, w_2, \ldots, w_n \).
**Theorem 3:** The solution space to the second order homogeneous linear differential equation

\[ y'' + p(x)y' + q(x)y = 0 \]

is 2-dimensional on any interval \( I \) for which the hypotheses of the existence-uniqueness theorem hold.

**proof:**

Pick any \( x_0 \in I \). Find solutions \( y_1(x), y_2(x) \) to initial value problems at \( x_0 \) so that the so-called **Wronskian matrix** for \( y_1, y_2 \) at \( x_0 \)

\[
W(y_1, y_2)(x_0) = \begin{bmatrix}
y_1(x_0) & y_2(x_0) \\
y_1'(x_0) & y_2'(x_0)
\end{bmatrix}
\]

is invertible (i.e. \( y_1(x_0), y_2(x_0), y_1'(x_0), y_2'(x_0) \) are a basis for \( \mathbb{R}^2 \), or equivalently so that the determinant of the Wronskian matrix (called just the **Wronskian**) is non-zero at \( x_0 \)).

- You may be able to find suitable \( y_1, y_2 \) by a method like we used in the previous example, but the existence-uniqueness theorem guarantees they exist even if you don’t know how to find formulas for them.

Under these conditions, the solutions \( y_1, y_2 \) are actually a basis for the solution space! Here’s why:

- **span:** the condition that the Wronskian matrix is invertible at \( x_0 \) means we can solve each IVP there with a linear combination \( y = c_1y_1 + c_2y_2 \): In that case, to solve the IVP

\[
y'' + p(x)y' + q(x)y = 0
\]

\[
y(x_0) = b_0
\]

\[
y'(x_0) = b_1
\]

we set

\[
y(x) = c_1y_1(x) + c_2y_2(x) . \]

At \( x_0 \) we wish to find \( c_1, c_2 \) so that

\[
\begin{align*}
c_1y_1(x_0) + c_2y_2(x_0) &= b_0 \\
c_1y_1'(x_0) + c_2y_2'(x_0) &= b_1
\end{align*}
\]

This system is equivalent to the the matrix equation

\[
\begin{bmatrix}
y_1(x_0) & y_2(x_0) \\
y_1'(x_0) & y_2'(x_0)
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= \begin{bmatrix}
b_0 \\
b_1
\end{bmatrix}
\]

When the Wronskian matrix at \( x_0 \) has an inverse, the unique solution \( [c_1, c_2]^T \) is given by
\[
\begin{bmatrix}
  c_1 \\
  c_2 
\end{bmatrix} = \begin{bmatrix}
  y_1(x_0) & y_2(x_0) \\
  y_1'(x_0) & y_2'(x_0)
\end{bmatrix}^{-1} \begin{bmatrix}
  b_0 \\
  b_1
\end{bmatrix}.
\]

Since the uniqueness theorem says each IVP has a unique solution, we've found it!

\[y(x) = c_1y_1(x) + c_2y_2(x).\]

- **Span**: Since each solution \(y(x)\) to the differential equation solves *some* initial value problem at \(x_0\), this gives all solutions, as we let \([b_0, b_1]^T\) vary freely in \(\mathbb{R}^2\). So each solution \(y(x)\) is a linear combination of \(y_1, y_2\). Thus \(\{y_1, y_2\}\) spans the solution space.

- **Linear independence**: If we have the identity
  
  \[c_1y_1(x) + c_2y_2(x) = 0\]

  then by differentiating each side with respect to \(x\) we also have
  
  \[c_1y_1'(x) + c_2y_2'(x) = 0.\]

  Evaluating at \(x = x_0\) this is the system

  \[
  \begin{align*}
  c_1y_1(x_0) + c_2y_2(x_0) &= 0 \\
  c_1y_1'(x_0) + c_2y_2'(x_0) &= 0
  \end{align*}
\]

\[
\begin{bmatrix}
  y_1(x_0) & y_2(x_0) \\
  y_1'(x_0) & y_2'(x_0)
\end{bmatrix} \begin{bmatrix}
  c_1 \\
  c_2
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}.
\]

so

\[
\begin{bmatrix}
  c_1 \\
  c_2
\end{bmatrix} = \begin{bmatrix}
  y_1(x_0) & y_2(x_0) \\
  y_1'(x_0) & y_2'(x_0)
\end{bmatrix}^{-1} \begin{bmatrix}
  0 \\
  0
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}.
\]
The theory for \( n^{th} \) order linear differential equations is conceptually the same as for second order...

**Definition:** An \( n^{th} \) order linear differential equation for a function \( y(x) \) is a differential equation that can be written in the form

\[
A_n(x)y^{(n)} + A_{n-1}(x)y^{(n-1)} + \ldots + A_1(x)y' + A_0(x)y = F(x).
\]

We search for solution functions \( y(x) \) defined on some specified interval \( I \) of the form \( a < x < b \), or \((a, \infty), (-\infty, a)\) or (usually) the entire real line \((-\infty, \infty)\). In this chapter we assume the function \( A_n(x) \neq 0 \) on \( I \), and divide by it in order to rewrite the differential equation in the standard form

\[
y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y' + a_0y = f.
\]

\((a_{n-1}, \ldots, a_1, a_0, f\) are all functions of \( x \), and the DE above means that equality holds for all value of \( x \) in the interval \( I \).)

**Theorem 1** (Existence-Uniqueness Theorem): Let \( a_{n-1}(x), a_{n-2}(x), \ldots, a_1(x), a_0(x), f(x) \) be specified continuous functions on the interval \( I \), and let \( x_0 \in I \). Then there is a unique solution \( y(x) \) to the initial value problem

\[
y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y' + a_0y = f
\]

\[
y^{(n)}(x_0) = b_n, \quad y^{(n-1)}(x_0) = b_{n-1}, \quad \ldots, \quad y'(x_0) = b_1, \quad y(x_0) = b_0
\]

and \( y(x) \) exists and is \( n \) times continuously differentiable on the entire interval \( I \).
The differential equation

\[ y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f \]

is called linear because the operator \( L \) defined by

\[ L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y \]

satisfies the so-called linearity properties

1. \( L(y_1 + y_2) = L(y_1) + L(y_2) \)
2. \( L(cy) = cL(y), c \in \mathbb{R} \).

• The proof that \( L \) satisfies the linearity properties is just the same as it was for the case when \( n = 2 \), which we checked.

The following two theorems only use the linearity properties of the operator \( L \). I've kept the same numbering we used for the case \( n = 2 \).

**Theorem 2:** The solution space to the homogeneous linear DE

\[ y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0 \]

is a subspace.

**Theorem 4:** The general solution to the nonhomogeneous \( n^{th} \) order linear DE

\[ y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f \]

is \( y = y_p + y_H \) where \( y_p \) is any single particular solution and \( y_H \) is the general solution to the homogeneous DE. \( y_H \) is called \( y_c \), for complementary solution, in the text.
**Theorem 3:** The solution space to the $n^{th}$ order homogeneous linear differential equation

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y \equiv 0$$

is $n$-dimensional. Thus, any $n$ independent solutions $y_1, y_2, \ldots, y_n$ will be a basis, and all homogeneous solutions will be uniquely expressible as linear combinations

$$y_H = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$  

**proof:** By the existence half of **Theorem 1**, we know that there are solutions for each possible initial value problem for this (homogeneous case) of the IVP for $n^{th}$ order linear DEs. So, pick solutions $y_1(x), y_2(x), \ldots, y_n(x)$ so that their vectors of initial values (which we'll call initial value vectors)

$$
\begin{bmatrix}
    y_1(x_0) \\
    y_1'(x_0) \\
    y_1''(x_0) \\
    \vdots \\
    y_1^{(n-1)}(x_0)
\end{bmatrix},
\begin{bmatrix}
    y_2(x_0) \\
    y_2'(x_0) \\
    y_2''(x_0) \\
    \vdots \\
    y_2^{(n-1)}(x_0)
\end{bmatrix},
\ldots,
\begin{bmatrix}
    y_n(x_0) \\
    y_n'(x_0) \\
    y_n''(x_0) \\
    \vdots \\
    y_n^{(n-1)}(x_0)
\end{bmatrix}
$$

are a basis for $\mathbb{R}^n$ (i.e. these $n$ vectors are linearly independent and span $\mathbb{R}^n$). (Well, you may not know how to "pick" such solutions, but you know they exist because of the existence theorem.)

**Claim:** In this case, the solutions $y_1, y_2, \ldots, y_n$ are a basis for the solution space. In particular, every solution to the homogeneous DE is a unique linear combination of these $n$ functions and the dimension of the solution space is $n$ .... discussion on next page.
• Check that \( y_1, y_2, \ldots, y_n \) span the solution space: Consider any solution \( y(x) \) to the DE. We can compute its vector of initial values

\[
\begin{bmatrix}
  y'(x_0) \\
  y''(x_0) \\
  y'''(x_0) \\
  \vdots \\
  y^{(n-1)}(x_0)
\end{bmatrix} =
\begin{bmatrix}
  b_0 \\
  b_1 \\
  b_2 \\
  \vdots \\
  b_{n-1}
\end{bmatrix}.
\]

Now consider a linear combination \( z = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \). Compute its initial value vector, and notice that you can write it as the product of the Wronskian matrix at \( x_0 \) times the vector of linear combination coefficients:

\[
\begin{bmatrix}
  z(x_0) \\
  z'(x_0) \\
  \vdots \\
  z^{(n-1)}(x_0)
\end{bmatrix} =
\begin{bmatrix}
  y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\
  y_1'(x_0) & y_2'(x_0) & \ldots & y_n'(x_0) \\
  \vdots & \vdots & \vdots & \vdots \\
  y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0)
\end{bmatrix}
\begin{bmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_n
\end{bmatrix}.
\]

We've chosen \( y_1, y_2, \ldots, y_n \) so that the Wronskian matrix at \( x_0 \) has an inverse, so the matrix equation

\[
\begin{bmatrix}
  y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\
  y_1'(x_0) & y_2'(x_0) & \ldots & y_n'(x_0) \\
  \vdots & \vdots & \vdots & \vdots \\
  y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0)
\end{bmatrix}
\begin{bmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_n
\end{bmatrix} =
\begin{bmatrix}
  b_0 \\
  b_1 \\
  \vdots \\
  b_{n-1}
\end{bmatrix}
\]

has a unique solution \( \mathbf{c} \). For this choice of linear combination coefficients, the solution \( c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \) has the same initial value vector at \( x_0 \) as the solution \( y(x) \). By the uniqueness half of the existence-uniqueness theorem, we conclude that

\[
y(x) = z(x) = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.
\]

Thus \( y_1, y_2, \ldots, y_n \) span the solution space.

- **Linear independence**: If a linear combination \( c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \equiv 0 \), then differentiate this identity \( n - 1 \) times, and then substitute \( x = x_0 \) into the resulting \( n \) equations. This yields the Wronskian matrix equation above, with \( [b_0, b_1, \ldots, b_{n-1}]^T = [0, 0, \ldots, 0]^T \). So the matrix equation above implies that \( [c_1, c_2, \ldots, c_n]^T = 0 \). So \( y_1, y_2, \ldots, y_n \) are also linearly independent.

Thus \( y_1, y_2, \ldots, y_n \) are a basis for the solution space and the general solution to the homogeneous DE can be written as

\[
y_H = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.
\]
Let's do some new exercises that tie these ideas together. (We may do these exercises while or before we wade through the general discussions on the previous pages!)

**Exercise 2** Consider the 3\textsuperscript{rd} order linear homogeneous DE for \( y(x) \):
\[
L(y) = y''' + 3y'' - y' - 3y = 0.
\]

Find a basis for the 3-dimensional solution space, and the general solution. Use the Wronskian matrix (or determinant) to verify you have a basis. Hint: try exponential functions.

Try:

\begin{align*}
y &= e^{rx} \\
y' &= re^{rx} \\
y'' &= r^2 e^{rx} \\
y''' &= r^3 e^{rx}
\end{align*}

\[
L(y) = r^3 e^{rx} + 3r^2 e^{rx} - re^{rx} - 3e^{rx} \equiv 0
\]

\[
= e^{rx} \left[ r^3 + 3r^2 - r - 3 \right]
\]

\[
= e^{rx} \left[ r^3 (r+3) - (r+3) \right]
\]

"Cheat"

\[
= e^{rx} \left[ (r+3)(r^2-1) \right] \equiv 0
\]

\[
(r+3)(r-1)(r+1) = 0
\]

\( r = 1, -1, -3 \)

\( e^x, e^{-x}, e^{-3x} \) are solns.

So
\[
y = c_1 e^x + c_2 e^{-x} + c_3 e^{-3x} \quad \text{and solns. these are all!}
\]