Math 2250-004
Week 11: March 26-28
5.6 applications of linear differential equations, and solution techniques for non-homogeneous problems.
(This material is on our exam, this Friday)
10.1-10.2 introduction to Laplace transforms (This material will not be on the midterm exam.)

Mon March 26:
5.6 Forced oscillations: undamped case.

maybe most important section of course, for engineers

Announcements:
- Exam 2 Friday
  - posted two practice exams & topics review notes (labs this week will be exam review)
  - thru 5.6 (from where last exam left off, 3.6 defs?)
  - HW due in class on Wed., or until 6:00 pm @ LCB 204

'til 10:46

Warm-up Exercise: What's your guess for $x_p(t)$? (undetermined coefficients)

Mond.

a) $x'' + 4x = 2\cos 3t$ $\qquad x_p(t) = A\cos 3t + \cancel{B\sin 3t}$

only has even derivs

resonance

b) $x'' + 4x = 3\underbrace{\cos 2t}_{\text{is homog. soln.}}$ $\qquad x_p(t) = A t\cos 2t + B t\sin 2t$ (Case 2)

Tues. $\rightarrow$ c) $x'' + 5x' + 4x = 3\cos 2t$

$x_p(t) = A\cos 2t + B\sin 2t$

Section 5.6: forced oscillations in mechanical (and electrical) systems. We will continue to discuss section 5.6 on the Monday after spring break. Today we'll discuss what happens when there is no damping - $c = 0$. We'll deal with the damped case after spring break.

But here is an Overview for all cases:

Forced oscillator

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega\, t)$$

using section 5.5 undetermined coefficients algorithms.

sinusoidal force

F_0 = amplitude of forcing Fcn

- undamped $(c = 0)$:

In this case the complementary homogeneous differential equation for $x(t)$ is

$$m\,x'' + k\,x = 0$$
$$x'' + \frac{k}{m}x = 0$$
$$x'' + \omega_0^2\, x = 0$$

$x_H = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$

which has simple harmonic motion solutions $\boxed{x_H(t) = C\cos(\omega_0 t - \alpha)}$. So for the non-homogeneous DE the method of undetermined coefficients implies we can find particular and general solutions as follows:

$m x'' + kx = F_0 \cos\omega t$

$x'' + \omega_0^2 x = \frac{F_0}{m} \cos \omega t$

- $\omega \ne \omega_0 := \sqrt{\dfrac{k}{m}}$ $\Rightarrow x_P = A\cos(\omega\, t)$ because only even derivatives, we don't need $\sin(\omega\, t)$ terms !!

$$\Rightarrow x = x_P + x_H = A\cos(\omega\, t) + C_0 \cos(\omega_0 t - \alpha_0)\,.$$

- $\omega \ne \omega_0$ but $\omega \approx \omega_0$, $C \approx C_0$ Beating!

- $\omega = \omega_0$ $\Rightarrow x_P = t\left(A\cos(\omega_0\, t) + B\sin(\omega_0\, t)\right)$ (Case II of undetermined coefficients.)

$= \omega_0$

$$\Rightarrow x = x_P + x_H = C\,t\cos(\omega_0\, t - \alpha) + C_0\cos(\omega_0 t - \alpha_0)\,.$$

("pure" resonance!)

....

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega\, t)$$

- damped $(c > 0)$: in all cases $x_P = A\cos(\omega\, t) + B\sin(\omega\, t) = C\cos(\omega\, t - \alpha)$ (because the roots of the characteristic polynomial are never $\pm i\,\omega$ when $c > 0$).
  - underdamped: $x = x_P + x_H = C\cos(\omega\, t - \alpha) + e^{-p\,t} C_1 \cos(\omega_1 t - \alpha_1)\,.$
  - critically-damped: $x = x_P + x_H = C\cos(\omega\, t - \alpha) + e^{-p\,t}(c_1 t + c_2)\,.$
  - over-damped: $x = x_P + x_H = C\cos(\omega\, t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}\,.$

- in all three <u>damped</u> cases on the previous page, $x_H(t) \to 0$ exponentially and is called the <u>transient solution</u> $x_{tr}(t)$ (because it disappears as $t \to \infty$. And in <u>these damped cases</u> $x_P(t)$ as above is called the <u>steady periodic solution</u> $x_{sp}(t)$ (because it is what persists as $t \to \infty$, and because it's periodic).

- if c is small enough and $\omega \approx \omega_0$ then the amplitude C of $x_{sp}(t)$ can be large relative to $\frac{F_0}{m}$, and the system can exhibit <u>practical resonance.</u> This can be an important phenomenon in electrical circuits, where amplifying signals is important.

<u>forced undamped oscillations</u>:

<u>Exercise 1a)</u> Solve the initial value problem for $x(t)$:

$$x'' + 9x = 80\cos(5t)$$
$$x(0) = 0$$
$$x'(0) = 0 .$$

1b) This superposition of two sinusoidal functions <u>is</u> periodic because there is a common multiple of their (shortest) periods. What is this (common) period?

1c) Compare your solution and reasoning with the display at the bottom of this page.

$x'' + 9x = 0$ $\qquad \omega_0^2 = 9$

$x_H(t) = c_1 \cos 3t + c_2 \sin 3t$ $\qquad \omega_0 = 3.$

$$\left.\begin{aligned} x_p(t) &= A\cos 5t \\ x_p' &= -5A\sin 5t \\ x_p'' &= -25A\cos 5t \end{aligned}\right\} \Rightarrow x''+9x = -25A\cos 5t + 9A\cos 5t = -16A\cos 5t = 80\cos 5t \Rightarrow \boxed{A=-5}$$

$x(t) = x_p + x_H$

$x = -5\cos 5t + c_1\cos 3t + c_2 \sin 3t$

$x(0) = 0 = -5 + c_1 \Rightarrow c_1 = 5$

$x'(0) = 0 = 3c_2 \Rightarrow c_2 = 0.$

$\boxed{x(t) = -5\cos 5t + 5\cos 3t}$

period $T = \frac{2\pi}{5}$ $\qquad T = \frac{2\pi}{3}$

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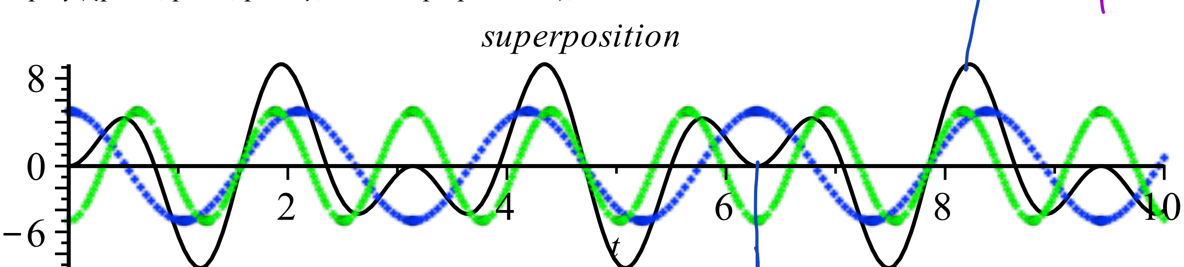
> with(plots):
> plot1 := plot(-5·cos(5·t), t = 0..10, color = green, style = point):
  plot2 := plot(5·cos(3·t), t = 0..10, color = blue, style = point):
  plot3 := plot(-5·cos(5·t) + 5·cos(3·t), t = 0..10, color = black):
  display({plot1, plot2, plot3}, title='superposition');
```

• x_p • x_H



period 2π

Least common multiple of $\frac{2\pi}{5}, \frac{2\pi}{3}$ is 2π

In general: Use the method of undetermined coefficients to solve the initial value problem for $x(t)$, in the case $\omega \neq \omega_0 = \sqrt{\frac{k}{m}}$:

$m x'' + kx = F_0 \cos \omega t$

$x_H(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$

$$x''(t) + \overbrace{\frac{k}{m}}^{=\omega_0^2} x(t) = \frac{F_0}{m}\cos(\omega t)$$

$$x(0) = x_0$$

$$x'(0) = v_0$$

$\omega_0^2 \; (\; x_p(t) = A\cos\omega t \;)$

$+\,0 \; (\; x_p'(t) = -A\omega \sin\omega t \;)$

$+\,1 \; (\; x_p''(t) = -A\omega^2 \cos\omega t \;)$

$x_p'' + \omega_0^2 x_p = \cos\omega t \left[\omega_0^2 A - A\omega^2\right] = \left[\frac{F_0}{m}\right]\cos\omega t$

$A(\omega_0^2 - \omega^2) = \frac{F_0}{m} \Longrightarrow A = \frac{F_0}{m(\omega_0^2-\omega^2)}$

$x(t) = x_p + x_H$

$x = \frac{F_0}{m(\omega_0^2-\omega^2)}\cos\omega t + c_1\cos\omega_0 t + c_2 \sin\omega_0 t$

$x(0) = x_0 = \frac{F_0}{m(\omega_0^2-\omega^2)} + c_1 \qquad c_1 = x_0 - \frac{F_0}{m(\omega_0^2-\omega^2)}$

$x'(0) = v_0 = 0 + 0 + c_2\omega_0 \qquad c_2 = \frac{v_0}{\omega_0}$

$$x(t) = \frac{F_0}{m(\omega_0^2-\omega^2)}\cos\omega t + \left(x_0 - \frac{F_0}{m(\omega_0^2-\omega^2)}\right)\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$$

$$x(t) = \frac{F_0}{m(\omega^2-\omega_0^2)}\left(\cos\omega_0 t - \cos\omega t\right) + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$$

Solution:

$$x(t) = \frac{F_0}{m\left(\omega^2 - \omega_0^2\right)} \underbrace{\left(\cos\left(\omega_0\, t\right) - \cos\left(\omega\, t\right)\right)} + x_0 \cos\left(\omega_0\, t\right) + \frac{v_0}{\omega_0} \sin\left(\omega_0 t\right)$$

There is an interesting <u>beating</u> phenomenon for $\omega \approx \omega_0$ (but still with $\omega \neq \omega_0$). This is explained analytically via trig identities, and is familiar to musicians in the context of superposed sound waves (which satisfy the homogeneous linear "wave equation" partial differential equation):

$$\left.\begin{aligned} \cos(\underbrace{\alpha - \beta}_{\omega_0 t}) - \cos(\underbrace{\alpha + \beta}_{\omega t}) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \\ &\quad - \left(\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\right) \\ &= 2\sin(\alpha)\sin(\beta)\ . \end{aligned}\right\}$$

Set $\alpha = \frac{1}{2}\left(\omega + \omega_0\right)t$, $\beta = \frac{1}{2}\left(\omega - \omega_0\right)t$ in the identity above, to rewrite the first term in $x(t)$ as a product rather than a difference:

$$x(t) = \frac{F_0}{m\left(\omega^2 - \omega_0^2\right)} 2\sin\left(\underbrace{\frac{1}{2}\left(\omega + \omega_0\right)t}_{\substack{\approx \omega_0 \\ (\text{if } \omega \approx \omega_0)}}\right)\sin\left(\underbrace{\frac{1}{2}\left(\omega - \omega_0\right)t}\right) + x_0 \cos\left(\omega_0 t\right) + \frac{v_0}{\omega_0}\sin\left(\omega_0 t\right).$$

$$T = \frac{2\pi}{\frac{1}{2}(\omega - \omega_0)} = \frac{4\pi}{|\omega - \omega_0|}$$

In this product of sinusoidal functions, the first one has angular frequency and period close to the original angular frequencies and periods of the original sum. But the second sinusoidal function has small angular frequency and long period, given by

$$\text{angular frequency: } \frac{1}{2}\left(\omega - \omega_0\right), \qquad \text{period: } \frac{4\pi}{\left|\omega - \omega_0\right|}\ .$$

We will call <u>half</u> that period the <u>beating period</u>, as explained by the next exercise:

$$\text{beating period: } \frac{2\pi}{\left|\omega - \omega_0\right|}, \qquad \text{beating amplitude: } \frac{2F_0}{m\left|\omega^2 - \omega_0^2\right|}\ .$$

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)}(\cos\omega_0 t - \cos\omega t) + x_0 \cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$$

$$x(t) = \frac{F_0\, 2}{m(\omega^2 - \omega_0^2)} \sin\left(\frac{\omega+\omega_0}{2} t\right) \sin\left(\frac{\omega-\omega_0}{2} t\right) + x_0 \cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$$

We will call half that period the beating period, as explained by the next exercise:

$$\text{beating period: } \frac{2\pi}{|\omega - \omega_0|}, \qquad \text{beating amplitude: } \frac{2F_0}{m|\omega^2 - \omega_0^2|}.$$

Exercise 2a) Use one of the formulas on the previous page to write down the IVP solution $x(t)$ to

$$x'' + 9x = 80\cos(3.1\,t)$$
$$x(0) = 0$$
$$x'(0) = 0.$$

$\bullet$ $F_0 = 8$ $\quad \omega_0 = 3$ $\quad \omega = 3.1$ $\quad m = 1$

2b) Compute the beating period and amplitude. Compare to the graph shown below.

$3.1^2 - 3^2 = ((3.1)-3)(3.1+3)$

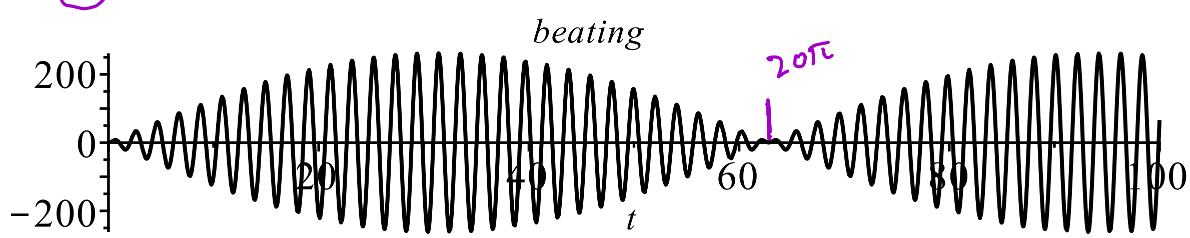
$$x(t) = \frac{80\cdot 2}{1\cdot(.61)} \underbrace{\sin(3.05t)\sin(.05t)}$$
$$= \frac{160}{.61} \ldots$$

262.3

$T = \frac{2\pi}{.05} = 40\pi$

beating period $= 20\pi$

> plot(262.3 $\cdot$ sin(3.05 $\cdot$ t)sin(.05 $\cdot$ t), t = 0 ..100, color = black, title = `beating`);



Resonance:

Resonance! $\omega = \omega_0$ (and the limit as $\omega \to \omega_0$)

$$\begin{cases} x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega_0 t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

using §5.5, guess

$$\begin{aligned}
+\omega_0^2 \big(\quad & x_p = t\,(A\cos\omega_0 t + B\sin\omega_0 t) & \big) \\
0 \big(\quad & x_p' = t\,(-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t) + A\cos\omega_0 t + B\sin\omega_0 t & \big) \\
+1 \big(\quad & x_p'' = t\,(-A\omega_0^2\cos\omega_0 t - B\omega_0^2\sin\omega_0 t) + \left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right]2 & \big)
\end{aligned}$$

$$L(x_p) = t\,(0) + 2\left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right] \overset{\text{want}}{=} \frac{F_0}{m}\cos\omega_0 t$$

Deduce $A = 0$, $B = \frac{F_0}{2m\omega_0}$

$$x_p(t) = \underbrace{\frac{F_0}{2m\omega_0}\,t\sin\omega_0 t}_{\text{sats } x(0)=0,\ x'(0)=0}$$

, so IVP soln is

$$\boxed{x(t) = \frac{F_0}{2m\omega_0}\,t\sin\omega_0 t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

You can also get this solution by letting $\omega \to \omega_0$ in the beating formula. We will probably do it that way in class, on the next page.

in the case $\omega \neq \omega_0 = \sqrt{\frac{k}{m}}$ we copy the IVP solution in both forms, from previous page

$$x''(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\cos(\omega t)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)}(\cos(\omega_0 t) - \cos(\omega t)) + x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)$$

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)} 2\sin\left(\frac{1}{2}(\omega + \omega_0)t\right)\underbrace{\sin\left(\frac{1}{2}(\omega - \omega_0)t\right)}_{\text{as } \omega \to \omega_0} + x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t).$$

$\sin\theta \approx \theta$ $\quad \sin\theta = \theta \pm \frac{\theta^3}{3!}$

If we let $\omega \to \omega_0$ this solution will converge to the $\underbrace{\text{resonance IVP solution}}_{\frac{1}{2}(\omega-\omega_0)t \ + \text{ error.}}$ on the previous page....

$$\frac{F_0}{m\cancel{(\omega-\omega_0)}(\omega+\omega_0)} \cancel{2}\sin\left(\frac{\omega+\omega_0}{2}t\right)\cancel{\tfrac{1}{2}}\cancel{(\omega-\omega_0)}t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$$

$$x(t) = \frac{F_0}{2m\omega_0}t\sin\omega_0 t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$$

$$x(t) = \frac{F_0}{2\, m\, \omega_0}\, t \sin\left(\omega_0 t\right) \;\; + x_0 \cos\left(\omega_0\, t\right) \;\; + \frac{v_0}{\omega_0} \sin\left(\omega_0 t\,\right)$$

Exercise 3a) Solve the IVP

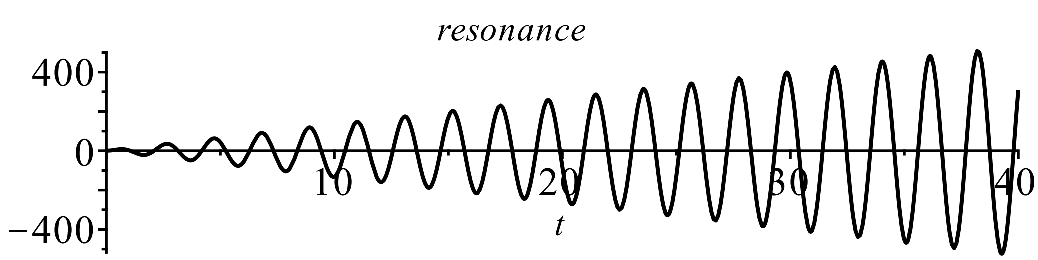
$$x'' + 9\, x = 80 \cos(3\, t)$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

First just use the general solution formula above this exercise and substitute in the appropriate values for the various terms.

$$x(t) = \frac{80}{2\cdot 1\cdot 3}\; t \sin 3t$$
$$= \frac{40}{3}\; t \sin 3t$$

3b) Compare the solution graph below with the beating graph in exercise 2.

> $plot\left(\frac{40}{3} t\cdot \sin(3\cdot\; t),\; t = 0\,..40,\; color = black,\; title = \text{`resonance`}\right);$



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- After finishing the discussion of undamped forced oscillations, we will discuss the physics and mathematics of damped forced oscillations

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega\,t) \ .$$

Here are some links which address how these phenomena arise, also in more complicated real-world applications in which the dynamical systems are more complex and have more components. Our baseline cases are the starting points for understanding these more complicated systems. We'll also address some of these more complicated applications when we move on to systems of differential equations, in a few weeks.

http://en.wikipedia.org/wiki/Mechanical_resonance (wikipedia page with links)
http://www.nset.org.np/nset/php/pubaware_shaketable.php (shake tables for earthquake modeling)
http://www.youtube.com/watch?v=M_x2jOKAhZM (an engineering class demo shake table)
http://www.youtube.com/watch?v=j-zczJXSxnw (Tacoma narrows bridge)
http://en.wikipedia.org/wiki/Electrical_resonance (wikipedia page with links)
http://en.wikipedia.org/wiki/Crystal_oscillator (crystal oscillators)