Start here Friday, in Wed. notes

BUT LOOK OUT

Exercise 2a) Find a particular solution to

$$L(y) = y'' + 4y' - 5y = 4e^x .$$

Hint: since $y_H = c_1 e^x + c_2 e^{-5x}$, a guess of $y_P = a\,e^x$ will not work (and $span\{e^x\}$ does not satisfy the "base case" conditions for undetermined coefficients). Instead try

$$y_P = d\,x\,e^x$$

and factor $L = D^2 + 4D - 5 = [D + 5] \circ [D - 1]$.

$-5\,(y_P = dxe^x)$

$+4\,[y_P' = d(e^x + xe^x)]$

$1\,(y_P'' = d(e^x + e^x + xe^x))$

$y_P'' = d(2e^x + xe^x)$

$L(y_P) = xe^x(\cancel{-5d+4d+d}) \to 0$

$+ e^x(4d + 2d)$ same

want $= 4e^x$

$6d = 4$

$d = \frac{2}{3}$

$$\boxed{y_P = \frac{2}{3}xe^x}$$

Why

$L(y) = (D^2 + 4D - 5)y.$

$D^2 + 4D - 5$

$= (D+5)\circ(D-1)$

$\underbrace{(D-1)e^x = 0 \quad (D+5)e^{-5x} = 0}_{\text{reason that } e^x,\, e^{-5x} \text{ were a basis for } y_H}$

$(D+5)\circ\underbrace{(D-1)Axe^x}$

$(D+5)Ae^x$

$= Ae^x + 5Ae^x$

$= 6Ae^x$

$(D-b)\,xe^{bx} = e^{bx}$

$(D-b)\,x^2e^{bx} = 2e^{bx}$

$(D-b)\,f(x)e^{bx}$

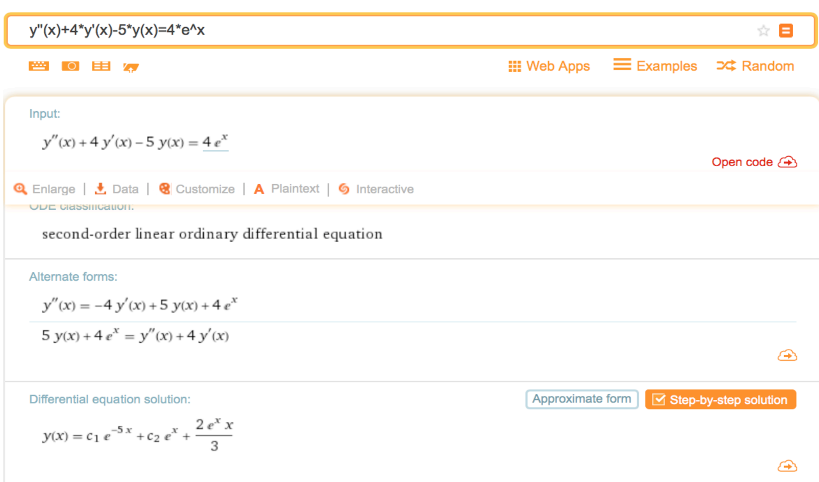
$= f'(x)e^{bx}$

$(D-b)(f(x)e^{bx})$

$= f'(x)e^{bx} + \cancel{fbe^{bx}}$

$\cancel{-bfe^{bx}}$

2b) check work with technology



A vector space theorem like the one for the base case, except for $L : V \rightarrow W$, combined with our understanding of how to factor constant coefficient differential operators (as in lab you're working on this week for homogeneous DE's) leads to an extension of the method of undetermined coefficients, for right hand sides which can be written as sums of functions having the indicated forms below. See the discussion in pages 341-346 of the text, and the table on page 346, reproduced here.

<u>Method of undetermined coefficients (extended case)</u>: Finding $y_P$ for non-homogeneous linear differential equations

$$L(y) = f$$

If $L$ has a factor $(\mathrm{D} - r)^s$ and $e^{rx}$ is also associated with (a portion of) the right hand side $f(x)$ then the corresponding guesses you would have made in the "base case" need to be multiplied by $x^s$ , as in <u>Exercise 2</u>. (This is like your current lab problem, and if you understand that, you have an inkling of why this recipe works. On the other hand, if you don't understand that problem, there's another one this week so you get a second chance. :-) ) You may also need to use superposition, as in our earlier exercises, if different portions of $f(x)$ are associated with different exponential functions.

Extended case of undetermined coefficients

| $f(x)$ | $y_P$ | $s > 0$ when $p(r)$ has these roots: |
|---|---|---|
| $P_m(x) = b_0 + b_1 + \ldots + b_m x^m$ | $x^s\left(c_0 + c_1 x + c_2 x^2 + \ldots + c_m x^m\right)$ | $r = 0$ $(r-0)^s$ |
| $b_1 \cos(\omega x) + b_2 \sin(\omega x)$ | $x^s\left(c_1 \cos(\omega x) + c_2 \sin(\omega x)\right)$ | $r = \pm i\omega$ |
| $e^{ax}\left(b_1 \cos(\omega x) + b_2 \sin(\omega x)\right)$ | $x^s e^{ax}\left(c_1 \cos(\omega x) + c_2 \sin(\omega x)\right)$ | $r = a \pm i\omega$ |
| $b_0 e^{ax}$ | $x^s c_0 e^{ax}$ | $r = a$ |
| $\left(b_0 + b_1 + \ldots + b_m x^m\right) e^{ax}$ | $x^s\left(c_0 + c_1 x + c_2 x^2 + \ldots + c_m x^m\right) e^{ax}$ | $r = a$ |

our warm-up example:

$$y'' + 4y' - 5y = 4e^x \qquad p(r) = (r+5)(r-1)^1$$

guess: $y_P = x^1(Ae^x)$

Exercise 3) Set up the undetermined coefficients particular solutions for the examples below. When necessary use the extended case to modify the undetermined coefficients form for $y_P$. Use technology to check if your "guess" form was right.

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$

3a) $y''' + 2y'' = x^2 + 6x$

(So the characteristic polynomial for $L(y) = 0$ is $\underbrace{r^3 + 2r^2} = \underbrace{r^2(r+2)} = \underbrace{(r-0)^2(r+2)}$.)

$$y_p = x^2\left(d_1 x^2 + d_2 x + d_3\right)$$

Differential equation solution:

$$y(x) = c_3 x^2 + c_2 x + c_1 + \frac{x^5}{180} + \frac{x^4}{12}$$

3b) $y'' - 4y' + 13y = 4e^{2x}\sin(3x)$

(So the characteristic polynomial for $L(y) = 0$ is

$(r^2 - 4r + 13) = (r-2)^2 + 9 = \underbrace{(r-2+3i)}\underbrace{(r-2-3i)}$.)

$r = 2 \pm 3i$

$y_H$: $e^{2x}\cos 3x$, $e^{2x}\sin 3x$

would tried

$y = Ae^{2x}\cos 3x + Be^{2x}\sin 3x$

instead: $y = x\left(Ae^{2x}\cos 3x + Be^{2x}\sin 3x\right)$

Differential equation solution:

$$y(x) = c_1 e^{2x}\sin(3x) + c_2 e^{2x}\cos(3x) + \cancel{\frac{4}{37}e^x\sin(3x) + \frac{24}{37}e^x\cos(3x)}$$

$-\frac{2}{3} x e^{2x}\cos 3x$

I typed in the wrong right hand side the first time corrected DE and solution...

WolframAlpha computational knowledge engine

y"(x)-4*y'(x)+13*y(x)=4*e^(2*x)*sin(3*x)

Web Apps   Examples   Random

Input:

$$y''(x) - 4\,y'(x) + 13\,y(x) = 4\,e^{2x}\sin(3\,x)$$

Open code

ODE classification:

second-order linear ordinary differential equation

Alternate forms:

More

$$y''(x) = 4\,y'(x) - 13\,y(x) + 4\,e^{2x}\sin(3\,x)$$

$$4\left(y'(x) + e^{2x}\sin(3\,x)\right) = y''(x) + 13\,y(x)$$

$$y''(x) - 4\,y'(x) + 13\,y(x) = 4\,e^{2x}\sin(x)\,(2\cos(2\,x) + 1)$$

Differential equation solution:

Approximate form   Step-by-step solution

$$y(x) = c_1\,e^{2x}\sin(3\,x) + c_2\,e^{2x}\cos(3\,x) - \frac{2}{3}\,e^{2x}\,x\cos(3\,x)$$

3c) $y'' + 5y' + 4y = 5\cos(2x) + 4e^{x} + 5e^{-x}$.

(So the characteristic polynomial for $L(y) = 0$ is $p(r) = r^2 + 5r + 4 = (r+4)(r+1)$.)

Differential equation solution:

$$y(x) = c_1 e^{-4x} + c_2 e^{-x} + \frac{5e^{-x}x}{3} + \frac{2e^{x}}{5} + \sin(x)\cos(x)$$

Variation of Parameters: This is an alternate method for finding particular solutions. Its advantage is that is always provides a particular solution, even for non-homogeneous problems in which the right-hand side doesn't fit into a nice finite dimensional subspace preserved by $L$, and even if the linear operator $L$ is not constant-coefficient. The formula for the particular solutions can be somewhat messy to work with, however, once you start computing.

Here's the formula: Let $y_1(x), y_2(x),...y_n(x)$ be a basis of solutions to the homogeneous DE

$$L(y) := y^{(n)} + p_{n-1}(x)y^{(n-1)} + ... + p_1(x)y' + p_0(x)y = 0 .$$

Then $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + ... + u_n(x)y_n(x)$ is a particular solution to

$$L(y) = f$$

provided the coefficient functions (aka "varying parameters") $u_1(x), u_2(x),...u_n(x)$ have derivatives satisfying the matrix equation

$$\begin{bmatrix} u_1' \\ u_2' \\ : \\ u_n' \end{bmatrix} = \left[ W(y_1, y_2,..., y_n) \right]^{-1} \begin{bmatrix} 0 \\ 0 \\ : \\ f \end{bmatrix}$$

where $\left[ W(y_1, y_2,..., y_n) \right]$ is the Wronskian matrix.

Here's how to check this fact when $n = 2$: Write

$$y_p = y = u_1 y_1 + u_2 y_2 .$$

Thus

$$y' = u_1 y_1' + u_2 y_2' + \left( u_1' y_1 + u_2' y_2 \right) .$$

Set

$$\left( u_1' y_1 + u_2' y_2 \right) = 0.$$

Then

$$y'' = u_1 y_1'' + u_2 y_2'' + \left( u_1' y_1' + u_2' y_2' \right) .$$

Set

$$\left( u_1' y_1' + u_2' y_2' \right) = f.$$

Notice that the two $( ... )$ equations are equivalent to the matrix equation

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

which is equivalent to the $n = 2$ version of the claimed condition for $y_p$. Under these conditions we compute

$$\begin{aligned} & p_0 \left[ y = u_1 y_1 + u_2 y_2 \right] \\ & + p_1 \left[ y' = u_1 y_1' + u_2 y_2' \right] \\ & + 1 \left[ y'' = u_1 y_1'' + u_2 y_2'' + f \right] \\ & L(y) = u_1 L(y_1) + u_2 L(y_2) + f \\ & L(y) = 0 + 0 + f = f \end{aligned}$$

Appendix: The following two theorems justify the method of undetermined coefficients, in both the "base case" and the "extended case." We will not discuss these in class, but I'll be happy to chat about the arguments with anyone who's interested, outside of class. They only use ideas we've talked about already, although they are abstract.

Theorem 0:

- Let $V$ and $W$ be vector spaces. Let $V$ have dimension $n < \infty$ and let $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, ..., \underline{\mathbf{v}}_n\}$ be a basis for $V$.
- Let $L : V \rightarrow W$ be a linear transformation, i.e. $L(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = L(\underline{\boldsymbol{u}}) + L(\underline{\boldsymbol{v}})$ and $L(c\,\underline{\boldsymbol{u}}) = c\,L(\underline{\boldsymbol{u}})$ holds $\forall\ \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in V, c \in \mathbb{R}$.) Consider the range of $L$, i.e.

$$Range(L) := \{L(d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + ... + d_n\underline{\mathbf{v}}_n) \in W, \text{such that each } d_j \in \mathbb{R}\}$$
$$= \{d_1L(\underline{\boldsymbol{v}}_1) + d_2L(\underline{\boldsymbol{v}}_2) + ... + d_nL(\underline{\mathbf{v}}_n) \in W, \text{such that each } d_j \in \mathbb{R}\}$$
$$= span\{L(\underline{\boldsymbol{v}}_1), L(\underline{\boldsymbol{v}}_2), ... L(\underline{\boldsymbol{v}}_n)\}.$$

Then $Range(L)$ is $n - dimensional$ if and only if the only solution to $L(\underline{\boldsymbol{v}}) = \underline{\mathbf{0}}$ is $\underline{\boldsymbol{v}} = \underline{\mathbf{0}}$.

*proof:*

(i) $\Leftarrow$: The only solution to $L(\underline{\boldsymbol{v}}) = \underline{\mathbf{0}}$ is $\underline{\boldsymbol{v}} = \underline{\mathbf{0}}$ implies $Range(L)$ is $n - dimensional$:

If we can show $L(\underline{\boldsymbol{v}}_1), L(\underline{\boldsymbol{v}}_2), ... L(\underline{\boldsymbol{v}}_n)$ are linearly independent, then they will be a basis for $Range(L)$ and this subspace will have dimension $n$. So, consider the dependency equation:

$$d_1L(\underline{\boldsymbol{v}}_1) + d_2L(\underline{\boldsymbol{v}}_2) + ... + d_nL(\underline{\mathbf{v}}_n) = \underline{\mathbf{0}}.$$

Because $L$ is a linear transformation, we can rewrite this equation as

$$L(d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + ... + d_n\underline{\mathbf{v}}_n) = \underline{\mathbf{0}}.$$

Under our assumption that the only homogeneous solution is the zero vector, we deduce

$$d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + ... + d_n\underline{\mathbf{v}}_n = \underline{\mathbf{0}}.$$

Since $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, ..., \underline{\mathbf{v}}_n$ are a basis they are linearly independent, so $d_1 = d_2 = ... = d_n = 0$.

□

(ii) $\Rightarrow$: $Range(L)$ is $n - dimensional$ implies the only solution to $L(\underline{\boldsymbol{v}}) = \underline{\mathbf{0}}$ is $\underline{\boldsymbol{v}} = \underline{\mathbf{0}}$: Since the range of $L$ is $n - dimensional$, $L(\underline{\boldsymbol{v}}_1), L(\underline{\boldsymbol{v}}_2), ... L(\underline{\boldsymbol{v}}_n)$ must be linearly independent. Now, let $\underline{\boldsymbol{v}} = d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + ... + d_n\underline{\mathbf{v}}_n$ be a homogeneous solution, $L(\underline{\boldsymbol{v}}) = \underline{\mathbf{0}}$. In other words,

$$L(d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + ... + d_n\underline{\mathbf{v}}_n) = \underline{\mathbf{0}}$$
$$\Rightarrow d_1L(\underline{\boldsymbol{v}}_1) + d_2L(\underline{\boldsymbol{v}}_2) + ... + d_nL(\underline{\mathbf{v}}_n) = \underline{\mathbf{0}}$$
$$\Rightarrow d_1 = d_2 = ... = d_n = 0 \Rightarrow \underline{\boldsymbol{v}} = \underline{\mathbf{0}}.$$

□

Theorem 1 Let Let $V$ and $W$ be vector spaces, both with the same dimension $n < \infty$. Let $L : V \rightarrow W$ be a linear transformation. Let the only solution to $L(\underline{\boldsymbol{v}}) = \underline{\mathbf{0}}$ be $\underline{\boldsymbol{v}} = \underline{\mathbf{0}}$. Then for each $\underline{\boldsymbol{w}} \in W$ there is a unique $\underline{\boldsymbol{v}} \in V$ with $L(\underline{\boldsymbol{v}}) = \underline{\boldsymbol{w}}$.

*proof*: By Theorem 0, the dimension of $Range(L)$ is $n - dimensional$. Therefore it must be all of $W$. So for each $\underline{\boldsymbol{w}} \in W$ there is at least one $\boldsymbol{v}_P \in V$ with $L(\underline{\boldsymbol{v}}_P) = \underline{\boldsymbol{w}}$. But the general solution to $L(\underline{\boldsymbol{v}}) = \underline{\boldsymbol{w}}$ is $\underline{\boldsymbol{v}} = \underline{\boldsymbol{v}}_P + \underline{\boldsymbol{v}}_H$, where $\underline{\boldsymbol{v}}_H$ is the general solution to the homogeneous equation. By assumption, $\underline{\boldsymbol{v}}_H = \underline{\mathbf{0}}$, so the particular solution is unique.

□

Remark: In the base case of undetermined coefficients, $W = V$. In the extended case, $W$ is the space in which $f$ lies, and $V = x^s W$, i.e. the space of all functions which are obtained from ones in $W$ by multiplying them by $x^s$. This is because if $L$ factors as

$$L = (D - r_1 I)^{k_1} \circ (D - r_2 I)^{k_2} \circ \ldots \circ (D - r_m I)^{k_m}$$

and if $f$ is in a subspace $W$ associated with the characteristic polynomial root $r_m$, then for $s = k_m$ the factor $(D - r_m I)^{k_m}$ of $L$ will transform the space $V = x^s W$ back into $W$, and not transform any non-zero function in $V$ into the zero function. And the other factors of $L$ will then preserve $W$, also without transforming any non-zero elements to the zero function.

Math 2250-004
Friday Mar 16
- Section 5.6: Forced mechanical vibrations.

Announcements:
- HW for Wed March 28 is posted (sections 5.5-5.6)
- We'll begin §5.6 today. It's using §5.5 methods to understand
$$m x'' + c x' + k x = F_0 \cos \omega t$$
- Exam March 30 covers thru §5.6.

'til 10:47

Warm-up Exercise: Try to find a particular solution $y_P(x)$ to the differential equation
$$L(y) = y'' + 4y' - 5y = 4e^x$$

$$\boxed{\begin{aligned} y &= \tfrac{4}{3} x e^x \,? \\ &= \tfrac{2}{3} x e^x \,? \end{aligned}}$$

you would've tried
$$y_P = Ae^x$$
$$y_P'' + 4y_P' - 5y_P$$
$$= Ae^x + 4Ae^x - 5Ae^x$$
$$= e^x A(1+4-5) = 0$$

note: for $L(y) = 0$
$$p(r) = r^2 + 4r - 5 = (r+5)(r-1)$$
$$y_H = c_1 e^{-5x} + c_2 e^x$$

our guess was a homogeneous solution.

Section 5.6: forced oscillations in mechanical (and electrical) systems. We will continue to discuss section 5.6 on the Monday after spring break. Today we'll discuss what happens when there is no damping - $c = 0$. We'll deal with the damped case after spring break.

But here is an Overview for all cases:

Forced oscillator

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega\, t)$$

using section 5.5 undetermined coefficients algorithms.

sinusoidal force

$F_0$ = amplitude of forcing Fcn

- undamped $(c = 0)$:

  In this case the complementary homogeneous differential equation for $x(t)$ is

$$m\,x'' + k\,x = 0$$
$$x'' + \frac{k}{m}x = 0$$
$$x'' + \omega_0^2\, x = 0$$

$x_H = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$

which has simple harmonic motion solutions $\boxed{x_H(t) = C\cos(\omega_0 t - \alpha)}$. So for the non-homogeneous DE the method of undetermined coefficients implies we can find particular and general solutions as follows:

$m x'' + kx = F_0 \cos \omega t$

$x'' + \omega_0^2 x = \frac{F_0}{m} \cos \omega t$

- $\omega \neq \omega_0 := \sqrt{\frac{k}{m}}$ $\Rightarrow x_P = A\cos(\omega\, t)$ because only even derivatives, we don't need $\sin(\omega\, t)$ terms !!

$$\Rightarrow x = x_P + x_H = A\cos(\omega\, t) + C_0 \cos(\omega_0 t - \alpha_0)\ .$$

- $\omega \neq \omega_0$ but $\omega \approx \omega_0$, $C \approx C_0$ Beating!

- $\omega = \omega_0$ $\Rightarrow x_P = t\big(A\cos(\omega_0\, t) + B\sin(\omega_0\, t)\big)$ (Case II of undetermined coefficients.)

$= \omega_0$

$$\Rightarrow x = x_P + x_H = C\, t\cos(\omega_0\, t - \alpha) + C_0 \cos(\omega_0 t - \alpha_0)\ .$$

("pure" resonance!)

........................................................................................

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega\, t)$$

- damped $(c > 0)$: in all cases $x_P = A\cos(\omega\, t) + B\sin(\omega\, t) = C\cos(\omega\, t - \alpha)$ (because the roots of the characteristic polynomial are never $\pm i\,\omega$ when $c > 0$).
  - underdamped: $x = x_P + x_H = C\cos(\omega\, t - \alpha) + e^{-p\,t} C_1 \cos(\omega_1 t - \alpha_1)$ .
  - critically-damped: $x = x_P + x_H = C\cos(\omega\, t - \alpha) + e^{-p\,t}(c_1 t + c_2)$ .
  - over-damped: $x = x_P + x_H = C\cos(\omega\, t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$ .

- in all three damped cases on the previous page, $x_H(t) \to 0$ exponentially and is called the transient solution $x_{tr}(t)$ (because it disappears as $t \to \infty$. And in these damped cases $x_P(t)$ as above is called the steady periodic solution $x_{sp}(t)$ (because it is what persists as $t \to \infty$, and because it's periodic).

- if $c$ is small enough and $\omega \approx \omega_0$ then the amplitude $C$ of $x_{sp}(t)$ can be large relative to $\frac{F_0}{m}$, and the system can exhibit practical resonance. This can be an important phenomenon in electrical circuits, where amplifying signals is important.

forced undamped oscillations:

Exercise 1a) Solve the initial value problem for $x(t)$:

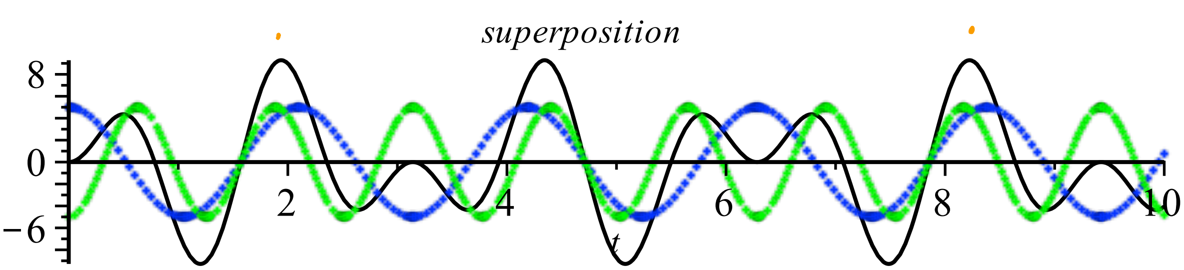
$$x'' + 9x = 80\cos(5t)$$
$$x(0) = 0$$
$$x'(0) = 0 .$$

1b) This superposition of two sinusoidal functions is periodic because there is a common multiple of their (shortest) periods. What is this (common) period?

1c) Compare your solution and reasoning with the display at the bottom of this page.

$c = 0, \ \omega \neq \omega_0$

```
> with(plots) :
> plot1 := plot(−5·cos(5·t), t = 0 ..10, color = green, style = point) :
  plot2 := plot(5·cos(3·t), t = 0 ..10, color = blue, style = point) :
  plot3 := plot(−5·cos(5·t) + 5·cos(3·t), t = 0 ..10, color = black) :
  display({plot1, plot2, plot3}, title='superposition');
```



In general: Use the method of undetermined coefficients to solve the initial value problem for $x(t)$, in the case $\omega \neq \omega_0 = \sqrt{\frac{k}{m}}$ :

$$x''(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\cos(\omega t)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

Solution:

$$x(t) = \frac{F_0}{m\left(\omega^2 - \omega_0^2\right)}\left(\cos\left(\omega_0 t\right) - \cos(\omega t)\right) + x_0\cos\left(\omega_0 t\right) + \frac{v_0}{\omega_0}\sin\left(\omega_0 t\right)$$

There is an interesting beating phenomenon for $\omega \approx \omega_0$ (but still with $\omega \neq \omega_0$). This is explained analytically via trig identities, and is familiar to musicians in the context of superposed sound waves (which satisfy the homogeneous linear "wave equation" partial differential equation):

$$\begin{aligned}\cos(\alpha - \beta) - \cos(\alpha + \beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)\\ &\quad - (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta))\\ &= 2\sin(\alpha)\sin(\beta)\ .\end{aligned}$$

Set $\alpha = \frac{1}{2}\left(\omega + \omega_0\right)t$, $\beta = \frac{1}{2}\left(\omega - \omega_0\right)t$ in the identity above, to rewrite the first term in $x(t)$ as a product rather than a difference:

$$x(t) = \frac{F_0}{m\left(\omega^2 - \omega_0^2\right)}2\sin\left(\frac{1}{2}\left(\omega + \omega_0\right)t\right)\sin\left(\frac{1}{2}\left(\omega - \omega_0\right)t\right) + x_0\cos\left(\omega_0 t\right) + \frac{v_0}{\omega_0}\sin\left(\omega_0 t\right)\ .$$

In this product of sinusoidal functions, the first one has angular frequency and period close to the original angular frequencies and periods of the original sum. But the second sinusoidal function has small angular frequency and long period, given by

$$\text{angular frequency: } \frac{1}{2}\left(\omega - \omega_0\right), \qquad \text{period: } \frac{4\pi}{\left|\omega - \omega_0\right|}\ .$$

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)}(\cos\omega_0 t - \cos\omega t) + x_0 \cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$$

$$x(t) = \frac{F_0\, 2}{m(\omega^2-\omega_0^2)} \sin\left(\frac{\omega+\omega_0}{2} t\right)\sin\left(\frac{\omega-\omega_0}{2} t\right) + x_0 \cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$$

We will call half that period the beating period, as explained by the next exercise:

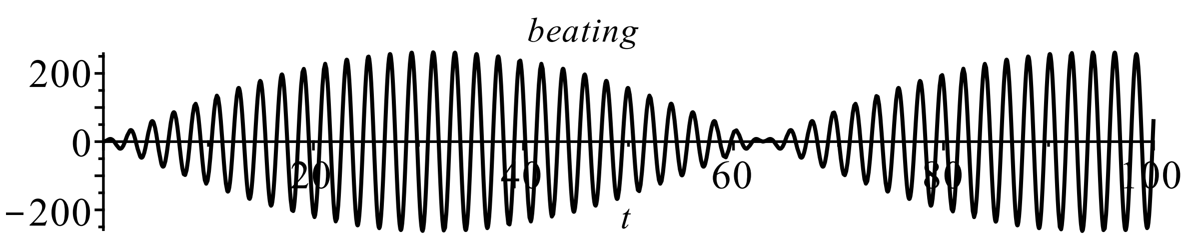
$$\text{beating period: } \frac{2\pi}{|\omega - \omega_0|}, \qquad \text{beating amplitude: } \frac{2 F_0}{m|\omega^2 - \omega_0^2|}.$$

Exercise 2a) Use one of the formulas on the previous page to write down the IVP solution $x(t)$ to

$$x'' + 9x = 80\cos(3.1\, t)$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

2b) Compute the beating period and amplitude. Compare to the graph shown below.

```
> plot(262.3·sin(3.05·t)sin(.05·t), t=0..100, color=black, title=`beating`);
```



Resonance:

Resonance! $\omega = \omega_0$ (and the limit as $\omega \to \omega_0$)

$$\begin{cases} x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega_0 t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

using § 5.5, guess

$$\begin{aligned}
+\omega_0^2 \big( \quad & x_p = t\,(A\cos\omega_0 t + B\sin\omega_0 t) && \big) \\
0 \big( \quad & x_p' = t\,(-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t) + A\cos\omega_0 t + B\sin\omega_0 t && \big) \\
+1 \big( \quad & x_p'' = t\,(-A\omega_0^2\cos\omega_0 t - B\omega_0^2\sin\omega_0 t) + \left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right]2 && \big)
\end{aligned}$$

$$L(x_p) = t\,(0) + 2\left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right] \overset{\text{want}}{=} \frac{F_0}{m}\cos\omega_0 t$$

Deduce $A = 0$, $B = \frac{F_0}{2m\omega_0}$

$$x_p(t) = \underbrace{\frac{F_0}{2m\omega_0}\,t\sin\omega_0 t}_{\text{sats } x(0)=0,\ x'(0)=0}$$

, so IVP soln is

$$\boxed{x(t) = \frac{F_0}{2m\omega_0}\,t\sin\omega_0 t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

You can also get this solution by letting $\omega \to \omega_0$ in the beating formula. We will probably do it that way in class, on the next page.

in the case $\omega \neq \omega_0 = \sqrt{\frac{k}{m}}$ we copy the IVP solution in both forms, from previous page

$$x''(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\cos(\omega t)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

$$x(t) = \frac{F_0}{m\left(\omega^2 - \omega_0^2\right)}\left(\cos\left(\omega_0 t\right) - \cos(\omega t)\right) + x_0\cos\left(\omega_0 t\right) + \frac{v_0}{\omega_0}\sin\left(\omega_0 t\right)$$

$$x(t) = \frac{F_0}{m\left(\omega^2 - \omega_0^2\right)} 2\sin\left(\frac{1}{2}\left(\omega + \omega_0\right)t\right)\sin\left(\frac{1}{2}\left(\omega - \omega_0\right)t\right) + x_0\cos\left(\omega_0 t\right) + \frac{v_0}{\omega_0}\sin\left(\omega_0 t\right).$$

If we let $\omega \rightarrow \omega_0$ this solution will converge to the resonance IVP solution on the previous page....

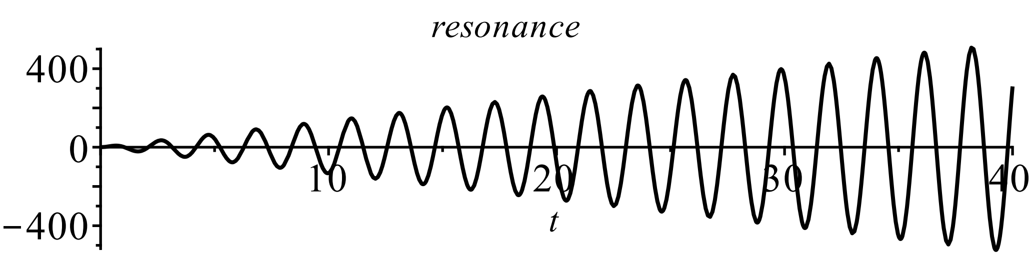
Exercise 3a) Solve the IVP

$$x'' + 9x = 80\cos(3t)$$
$$x(0) = 0$$
$$x'(0) = 0 .$$

First just use the general solution formula above this exercise and substitute in the appropriate values for the various terms. Then, if time, use variation of parameters (see the last pages of today's notes), to check a particular solution and to illustrate this alternate method for finding particular solutions.

3b) Compare the solution graph below with the beating graph in exercise 2.

> $plot\left(\frac{40}{3} t\cdot\sin(3\cdot t),\ t = 0..40,\ color = black,\ title = `resonance`\right);$




>

• After finishing the discussion of undamped forced oscillations, we will discuss the physics and mathematics of damped forced oscillations

$$m x'' + c x' + k x = F_0 \cos(\omega t) \ .$$

Here are some links which address how these phenomena arise, also in more complicated real-world applications in which the dynamical systems are more complex and have more components. Our baseline cases are the starting points for understanding these more complicated systems. We'll also address some of these more complicated applications when we move on to systems of differential equations, in a few weeks.

http://en.wikipedia.org/wiki/Mechanical_resonance (wikipedia page with links)
http://www.nset.org.np/nset/php/pubaware_shaketable.php (shake tables for earthquake modeling)
http://www.youtube.com/watch?v=M_x2jOKAhZM (an engineering class demo shake table)
http://www.youtube.com/watch?v=j-zczJXSxnw (Tacoma narrows bridge)
http://en.wikipedia.org/wiki/Electrical_resonance (wikipedia page with links)
http://en.wikipedia.org/wiki/Crystal_oscillator (crystal oscillators)