Wed March 14:
5.5: Finding y_P for non-homogeneous linear differential equations, ~~continued~~ starting Introduction of 5.6: forced oscillation problems, to get ready for lab. on Friday.

$$L(y) = f$$

(so that you can use the general solution $y = y_P + y_H$ to solve initial value problems).

Announcements:
- Use Tuesday's notes
- Quiz today

'til 10:47

Warm-up Exercise: Find a particular solution $y_p(x) = d_1 x + d_2$ to the inhomogeneous DE

$$L(y) = y'' + 4y' - 5y = 10x + 3$$

(Compute $L(y_p)$ and see what d_1 & d_2 must be)

$$y_p = d_1 x + d_2$$
$$y_p' = d_1$$
$$y_p'' = 0$$

$$L(y_p) = 0 + 4d_1 - 5(d_1 x + d_2) \overset{\text{want}}{=} 10x + 3$$

$$(-5d_1) \cdot x + (4d_1 - 5d_2) \overset{\text{want}}{=} 10 \cdot x + 3 \cdot 1$$

compare coeffs of "x" & "1" in this identity

$$\left.\begin{aligned} -5d_1 &= 10 \\ 4d_1 - 5d_2 &= 3 \end{aligned}\right\} \quad d_1 = -2, \quad d_2 = -\frac{11}{5}$$

$$-8 - 5d_2 = 3$$
$$-5d_2 = 11$$
$$d_2 = -\frac{11}{5}$$

$$\boxed{y_p(x) = -2x - \frac{11}{5}}$$

!

these are Tuesday's notes, covered on Wednesday

Section 5.5: Finding y_P for non-homogeneous linear differential equations

$$L(y) = f$$

(so that you can use the general solution $y = y_P + y_H$ to solve initial value problems). In this section the text switches back to $y = y(x)$, from $x = x(t)$.

There are two methods we will use:

- The method of undetermined coefficients uses guessing algorithms, and works for constant coefficient linear differential equations with certain classes of functions $f(x)$ for the non-homogeneous term. The method seems magic, but actually relies on vector space theory. We've already seen simple examples of this, where we seemed to pick particular solutions out of the air. This method is the main focus of section 5.5.

- The method of variation of parameters is more general, and yields an integral formula for a particular solution y_P, assuming you are already in possession of a basis for the homogeneous solution space. This method has the advantage that it works for any linear differential equation and any (continuous) function f. It has the disadvantage that the formulas can get computationally messy especially for differential equations of order $n > 2$. We'll study the case $n = 2$ only.

The easiest way to explain the method of undetermined coefficients is with examples.

Roughly speaking, you make a "guess" with free parameters (undetermined coefficients) that "looks like" the right side. AND, you need to include all possible terms in your guess that could arise when you apply L to the terms you know you want to include.

We'll make this more precise later in the notes.

Exercise 1) Find a particular solution $y_P(x)$ for the differential equation

$$L(y) := y'' + 4y' - 5y = 10x + 3 .$$

Hint: try $y_P(x) = d_1 x + d_2$ because L transforms such functions into ones of the form $b_1 x + b_2$. d_1, d_2 are your "undetermined coefficients", for the given right hand side coefficients $b_1 = 10, b_2 = 3$.

warm-up.

$y_P = d_1 x + d_2$

found $\boxed{y_P = -2x - \frac{11}{5}}$

Exercise 2) Use your work in 1 and your expertise with homogeneous linear differential equations to find the general solution to

$$y'' + 4y' - 5y = 10x + 3$$

Common error

Solve IVP with homog. soln. then add y_p to that result

$y = y_P + y_H$

$y'' + 4y' - 5y = 0$

$$\boxed{y = -2x - \frac{11}{5} + c_1 e^{-5x} + c_2 e^{x}}$$

$p(r) = r^2 + 4r - 5$

$= (r+5)(r-1) = 0$

$r = -5, 1$

$y_H = c_1 e^{-5x} + c_2 e^{x}$

(y_c) ↑ text

IVP $y(0) = 1$

$y'(0) = 3$

$y(0) = 1 = -\frac{11}{5} + c_1 + c_2$

$y'(0) = 3 = -2 - 5c_1 + c_2$

Exercise 3) Find a particular solution to

$$L(y) = y'' + 4y' - 5y = 14e^{2x}.$$

Hint: try $\boxed{y_P = d\,e^{2x}}$ because L transforms functions of that form into ones of the form $b\,e^{2x}$, i.e. $L(d\,e^{2x}) = b\,e^{2x}$. "d" is your "undetermined coefficient" for $b = 14$.

$$L(de^{2x}) = 4de^{2x} + 4(2de^{2x}) - 5de^{2x}$$

$$= de^{2x}[\underbrace{4+8-5}_{7}] \overset{\text{want}}{=} 14e^{2x}$$

$$7de^{2x} = 14e^{2x} \qquad \boxed{d = 2}$$

$$\boxed{y_P = 2e^{2x}}$$

for $y_p(x) = -2x - \frac{11}{5}$

$L(y_p) = 10x + 3$

for $y_q(x) = 2e^{2x}$

$L(y_q) = 14e^{2x}$

$L(y_1 + y_2) = L(y_1) + L(y_2)$

$L(cy) = cL(y)$

Exercise 4a) Use superposition (linearity of the operator L) and your work from the previous exercises to find the general solution to

$$L(y) = y'' + 4y' - 5y = 14e^{2x} - 20x - 6 .$$

4b) Solve (or at least set up the problem to solve) the initial value problem

$$y'' + 4y' - 5y = 14e^{2x} - 20x - 6$$
$$y(0) = 4$$
$$y'(0) = -4 .$$

try $y_r = -2y_p + y_q$

$$L(y_r) = -2L(y_p) + L(y_q)$$
$$= -2(10x+3) + 14e^{2x}$$
$$= 14e^{2x} - 20x - 6.$$

$$y_r = -2\left(-2x - \frac{11}{5}\right) + 2e^{2x}$$
$$= 4x + \frac{22}{5} + 2e^{2x}$$

general sol'n $y = y_r + y_H$

$$= 4x + \frac{22}{5} + 2e^{2x} + c_1 e^{-5x} + c_2 e^{x}$$

$$\begin{cases} y(0) = 4 = \frac{22}{5} + 2 + c_1 + c_2 \\ y'(0) = -4 = 4 + 4 - 5c_1 + c_2 \end{cases}$$

4c) Check your answer with technology.

y"(x)+4*y'(x)-5*y(x)=14*exp(2*x)-20*x-6, y(0)=4, y'(0)=-4

Web Apps   Examples   Random

Input:

$$\{y''(x) + 4\,y'(x) - 5\,y(x) = 14\exp(2\,x) - 20\,x - 6,\ y(0) = 4,\ y'(0) = -4\}$$

Open code

ODE classification:

second-order linear ordinary differential equation

Alternate forms:

More

$$\{y''(x) = -4\,y'(x) + 5\,y(x) - 20\,x + 14\,e^{2x} - 6,\ y(0) = 4,\ y'(0) = -4\}$$

$$\{5\,y(x) + 14\,e^{2x} = y''(x) + 4\,y'(x) + 20\,x + 6,\ y(0) = 4,\ y'(0) = -4\}$$

$$\{y''(x) + 4\,y'(x) - 5\,y(x) = 2\,(-10\,x + 7\,e^{2x} - 3),\ y(0) = 4,\ y'(0) = -4\}$$

Differential equation solution:

Approximate form   Step-by-step solution

$$y(x) = 4\,x + \underbrace{\frac{8\,e^{-5x}}{5} - 4\,e^{x}}_{y_H} + 2\,e^{2x} + \frac{22}{5}$$

y_P

Exercise 5) Find a particular solution to

$$L(y) := y'' + 4y' - 5y = 2\cos(3x).$$

Hint: To solve $L(y) = f$ we hope that f is in some finite dimensional subspace V that is preserved by L, i.e. $L : V \to V$.

- In Exercise 1 $V = span\{1, x\}$ and so we guessed $y_P = d_1 + d_2 x$.
- In Exercise 3 $V = span\{e^{2x}\}$ and so we guessed $y_P = d\, e^{2x}$.
- What's the smallest subspace V we can take in the current exercise? Can you see why $V = span\{\cos(3x)\}$ and a guess of $y_P = d\cos(3x)$ won't work?

$V = span\{\cos 3x, \sin 3x\}$ $\quad L: V \to V$

try
$$-5\left(y_P = d_1 \cos 3x + d_2 \sin 3x\right)$$
$$+4\left(y_P' = -3d_1 \sin 3x + 3d_2 \cos 3x\right)$$
$$1\left(y_P'' = -9d_1 \cos 3x - 9d_2 \sin 3x\right)$$

$$L(y_P) = \cos 3x\left(-5d_1 + 12d_2 - 9d_1\right) + \sin 3x\left(-5d_2 - 12d_1 - 9d_2\right) \overset{\text{want}}{=} 2\cos 3x + 0\sin 3x$$

$$-14d_1 + 12d_2 = 2$$
$$-12d_1 - 14d_2 = 0$$

$$-7d_1 + 6d_2 = 1$$
$$6d_1 + 7d_2 = 0$$

$$\begin{bmatrix} -7 & 6 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} -7 & 6 \\ 6 & 7 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \frac{1}{-85}\begin{bmatrix} 7 & -6 \\ -6 & -7 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -7/85 \\ 6/85 \end{bmatrix}$$

$$\boxed{y_P = -\frac{7}{85}\cos 3x + \frac{6}{85}\sin 3x}$$

(see next page!)

y"(x)+4*y'(x)-5*y(x)=2*cos(3*x)

Web Apps Examples Random

Input:

$$y''(x) + 4\, y'(x) - 5\, y(x) = 2 \cos(3\, x)$$

Open code

ODE classification:

second-order linear ordinary differential equation

Alternate forms: More

$$y''(x) = -4\, y'(x) + 5\, y(x) + 2 \cos(3\, x)$$

$$5\, y(x) + 2 \cos(3\, x) = y''(x) + 4\, y'(x)$$

$$y''(x) + 4\, y'(x) - 5\, y(x) = e^{-3 i x} + e^{3 i x}$$

Differential equation solution: Approximate form Step-by-step solution

$$y(x) = c_1 e^{-5 x} + c_2 e^{x} + \underbrace{\overset{A_2}{\frac{6}{85}} \sin(3\, x) - \overset{A_1}{\frac{7}{85}} \cos(3\, x)}$$

All of the previous exercises rely on:

Method of undetermined coefficients (base case): If you wish to find a particular solution y_P, i.e. $L(y_P) = f$ and if the non-homogeneous term f is in a finite dimensional subspace V with the properties that
(i) $L : V \to V$, i.e. L transforms functions in V into functions which are also in V; and
(ii) The only function $g \in V$ for which $L(g) = 0$ is $g = 0$.

Then there is always a unique $y_P \in V$ with $L(y_P) = f$.

why: We already know this fact is true for matrix transformations $L(\underline{\boldsymbol{x}}) = A_{n \times n}\underline{\boldsymbol{x}}$ with $L : \mathbb{R}^n \to \mathbb{R}^n$ (because if the only homogeneous solution is $\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$ then A reduces to the identity, so also each matrix equation $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ has a unique solution $\underline{\boldsymbol{x}}$. The theorem above is a generalization of that fact to general linear transformations. There is an "appendix" explaining this at the end of today's notes, for students who'd like to understand the details.

Exercise 6) Use the method of undetermined coefficients to guess the form for a particular solution $y_P(x)$ for a constant coefficient differential equation

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = f$$

(assuming the only such solution in your specified subspace that would solve the homogeneous DE is the zero solution):

6a) $L(y) = x^3 + 6x - 5$

$L : V \to V$

$y_P = d_1 x^3 + d_2 x^2 + d_3 x + d_4$

$V = \text{span}\{x^3, x^2, x, 1\}$

6b) $L(y) = 4e^{2x}\sin(3x)$

$y_P = d_1 e^{2x}\sin 3x + d_2 e^{2x}\cos 3x$

$V = \text{span}\{e^{2x}\sin 3x, e^{2x}\cos 3x\}.$

6c) $L(y) = x\cos(2x)$

$y_P = d_1 x\cos 2x + d_2 \cos 2x + d_3 x\sin 2x + d_4 \sin 2x$

$V = \text{span}\{x\cos 2x, \cos 2x, x\sin 2x, \sin 2x\}$

$$y''(x) + 4\cdot y'(x) - 5\cdot y(x) = x\cdot\cos(2\cdot x)$$

Differential equation solution:

Approximate form | Step-by-step solution

$$y(x) = c_1 e^{-5x} + c_2 e^{x} + \frac{1}{13} x\sin(x) + \frac{29\sin(x)}{338} - \frac{3}{26} x\cos(x) + \frac{\cos(x)}{169}$$