Tues March 13:

5.4-5.5 Finish Monday's notes on 5.4, Then begin 5.5: Finding y_P for non-homogeneous linear differential equations

$$L(y) = f$$

(so that you can use the general solution $y = y_P + y_H$ to solve initial value problems). We're back to $y = y(x)$ in this section...

(w9.3a is still to be handed in)

Announcements:
- w9-3 bcde on HW is now optional. You'll have a lab problem this week along these lines.
- My office hours today are canceled
- We'll finish Monday's notes today.

'til 10:47

Warm-up Exercise: Solve for $x(t)$:

§5.4

"critically damped"

$$\begin{cases} x'' + 6x' + 9x = 0 \\ x(0) = 1 \\ x'(0) = 3/2 \end{cases}$$

$$\boxed{x(t) = e^{-3t} + \tfrac{9}{2} t e^{-3t}}$$

$p(r) = r^2 + 6r + 9 = (r+3)^2$ $\quad r = -3$, double root

recipe: $x(t) = c_1 e^{-3t} + c_2 t e^{-3t}$

$\Rightarrow x'(t) = -3c_1 e^{-3t} + c_2\left[e^{-3t} - 3te^{-3t}\right]$

$x(0) = 1 = c_1 + c_2 \cdot 0 \Rightarrow c_1 = 1$

$x'(0) = 3/2 = -3c_1 + c_2 = -3 + c_2$

$3 + 1.5 = c_2$

$4.5 = c_2$ ✓

The motion exhibited by the solutions

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

to the undamped oscillator DE

$$x'' + \omega_0^{\,2} x = 0$$

is called <u>simple harmonic motion.</u> The reason for this name is that $x(t)$ can be rewritten in "amplitude-phase form" as

$$x(t) = C\cos(\omega_0 t - \alpha) = C\cos(\omega_0 (t - \delta))$$

in terms of an <u>amplitude</u> $C > 0$ and a <u>phase angle</u> α (or in terms of a <u>time delay</u> δ).

To see why this is so, equate the two forms and see how the coefficients A, B, C and phase angle α must be related:

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \alpha).$$

<u>Exercise 2)</u> Use the addition angle formula $\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$ to show that the two expressions above are equal provided

$$A = C\cos\alpha$$
$$B = C\sin\alpha\ .$$

So if C, α are given, the formulas above determine A, B. Conversely, if A, B are given then

$$C = \sqrt{A^2 + B^2}$$
$$\frac{A}{C} = \cos(\alpha),\ \frac{B}{C} = \sin(\alpha)$$

determine C, α. These correspondences are best remembered using a diagram in the $A - B$ plane:

We considered the possible identity

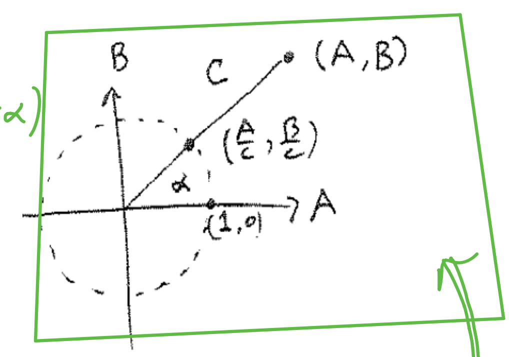
$$A\cos\omega_0 t + B\sin\omega_0 t = C\cos(\omega_0 t - \alpha)$$

The cos addition angle formula

$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$
$$* \quad \cos(a-b) = \cos a\cos b + \sin a\sin b$$

lets us expand the right side

$$\boxed{A}\cos\omega_0 t + B\sin\omega_0 t = C(\cos\omega_0 t\cos\alpha + \sin\omega_0 t\sin\alpha)$$
$$= \boxed{C\cos\alpha}\cos\omega_0 t + (C\sin\alpha)\sin\omega_0 t$$



picture

$$\boxed{\begin{array}{l} \text{So } A = C\cos\alpha \\ \quad\ B = C\sin\alpha \\ \text{(how to get } A, B \text{ from } C, \alpha) \end{array}}$$

$$\Longrightarrow A^2 + B^2 = C^2(\cos^2\alpha + \sin^2\alpha)$$
$$A^2 + B^2 = C^2$$

$\Longrightarrow$ get C, α from A, B

$$\boxed{\begin{array}{l} C = \sqrt{A^2 + B^2} \\ \cos\alpha = \frac{A}{C} \\ \sin\alpha = B/C \quad (\&\ \tan\alpha = B/A) \end{array}}$$

It is important to understand the behavior of the functions

$$A\cos(\omega_0\, t) + B\sin(\omega_0\, t) = C\cos(\omega_0 t - \alpha) = C\cos(\omega_0 (t - \delta))$$

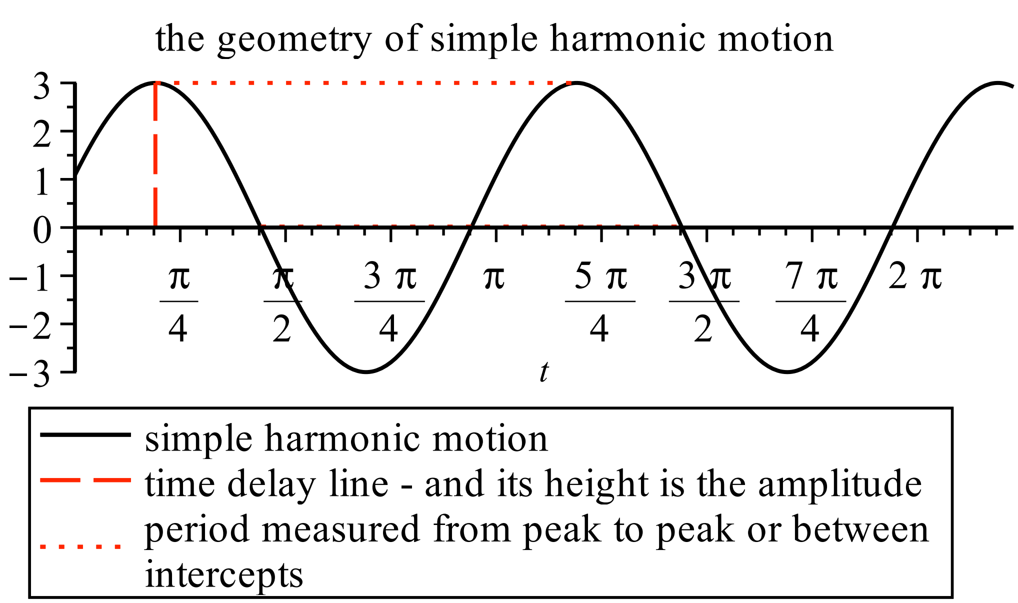
and the standard terminology:

The <u>amplitude</u> C is the maximum absolute value of $x(t)$. The *phase angle* α is the radians of $\omega_0 t$ on the unit circle, so that $\cos(\omega_0 t - \alpha)$ evaluates to 1. The time delay δ is how much the graph of $C\cos(\omega_0 t)$ is shifted to the right along the t-axis in order to obtain the graph of $x(t)$. Note that

$$\omega_0 = \text{angular velocity} \quad \text{units: radians/time}$$

$$f = \text{frequency} = \frac{\omega_0}{2\pi} \quad \text{units: cycles/time}$$

$$T = \text{period} = \frac{2\pi}{\omega_0} \quad \text{units: time/cycle.}$$



Exercise 3) A mass of 2 kg oscillates without damping on a spring with Hooke's constant $k = 18\ \frac{N}{m}$. It is initially stretched 1 m from equilibrium, and released with a velocity of $\frac{3}{2}\ \frac{m}{s}$.

$c = 0$

3a) Show that the mass' motion is described by $x(t)$ solving the initial value problem

$$x'' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

$x(0) = 1$ $\quad$ $x'(0) = 3/2.$

3b) Solve the IVP in a, and convert $x(t)$ into amplitude-phase and amplitude-time delay form. Sketch the solution, indicating amplitude, period, and time delay. Check your work with the Wolfram alpha output on the next page

3a) $mx'' + cx' + kx = 0$

$2x'' + 18x = 0$

$x'' + 9x = 0$

$p(r) = r^2 + 9 = 0$
$r^2 = -9$
$r = \pm 3i$
complex solns e^{3it}, e^{-3it}

$x'' + \omega_0^2 x = 0$

3b). $x(t) = A\cos 3t + B\sin 3t$

$x'(t) = -3A\sin 3t + 3B\cos 3t$

$(\cos 3t)'' = (-3\sin 3t)' = -9\cos 3t$

$(\cos 3t)'' + 9\cos 3t = 0$ ✓

$\cos 0 = 1$
$\sin 0 = 0$

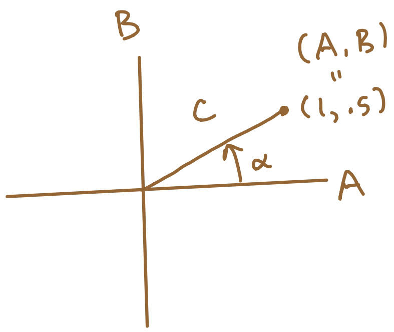
$x(0) = \boxed{1 = A}$

$x'(0) = 3/2 = 3B \Rightarrow \boxed{B = 1/2}$

$$\boxed{x(t) = \cos 3t + .5\sin 3t}$$

$\omega_0 = 3$ rad/sec.
$f = \frac{\omega_0}{2\pi} = \frac{3}{2\pi}$ cycles/sec
$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{3}$ sec/cycle.

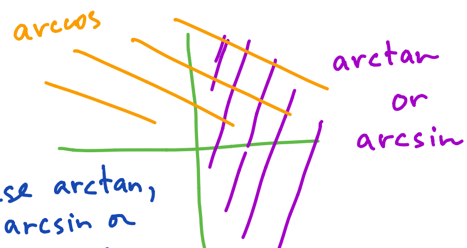
$= C\cos(3t - \alpha) = C\cos(3(t - \delta))$ $\quad$ $\delta = \alpha/3$



$C = \sqrt{A^2 + B^2}$
$\cos\alpha = \frac{A}{C}$
$\sin\alpha = \frac{B}{C}$
$\tan\alpha = B/A$

$C = \sqrt{1.25} = 1.12$

$\tan\alpha = .5$ $\quad$ $\alpha = \arctan(.5) = .464$
α in 1st quad. arct
radians.



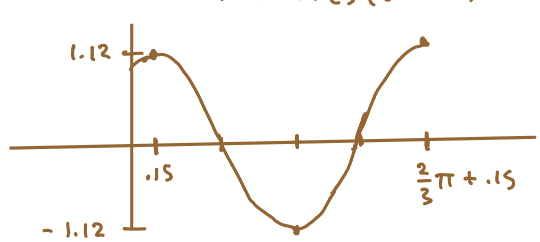
use arctan, arcsin or arccos, but add π to your answer.

$\cos\alpha = \frac{1}{\sqrt{1.25}}$

$\alpha = \arccos\left(\frac{1}{\sqrt{1.25}}\right) = .464$

$x(t) = 1.12\cos(3t - .464)$
$= 1.12\cos(3(t - .15))$

period $= \frac{2}{3}\pi \approx 2$



x''(t)+9*x(t)=0, x(0)=1, x'(0)=3/2

Web Apps | Examples | Random

Input:

$$\left\{x''(t) + 9\,x(t) = 0,\ x(0) = 1,\ x'(0) = \frac{3}{2}\right\}$$

Open code

Autonomous equation:

$$x''(t) = -9\,x(t)$$

Autonomous equation »

ODE classification:

second-order linear ordinary differential equation

Alternate forms:

$$\left\{x''(t) = -9\,x(t),\ x(0) = 1,\ x'(0) = \frac{3}{2}\right\}$$

$$\{x''(t) + 9\,x(t) = 0,\ x(0) = 1,\ 2\,x'(0) = 3\}$$

Differential equation solution:

Step-by-step solution

$$x(t) = \frac{1}{2}\sin(3\,t) + \cos(3\,t)$$ ← our answer

WolframAlpha computational knowledge engine.

plot cos(3*t)+.5*sin(3*t),t=0..Pi

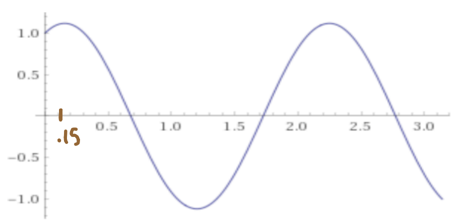
Web Apps | Examples | Random

Input interpretation:

plot	$\cos(3\,t) + 0.5\sin(3\,t)$	$t = 0$ to π

Open code

Plot:



Case 2: Unforced mass-spring system with damping: (This discussion is in Monday's notes but we'll likely need the start of Tuesday to finish it.)

- 3 possibilities that arise when the damping coefficient $c > 0$. There are three cases, depending on the roots of the characteristic polynomial:

$$m\,x'' + c\,x' + k\,x = 0$$

$$x'' + \frac{c}{m}\,x' + \frac{k}{m}\,x = 0 \qquad \bullet$$

rewrite as

$$x'' + 2\,p\,x' + \omega_0^2\,x = 0. \qquad \bullet \quad \text{for quadratic formula.}$$

$\left(p = \frac{c}{2\,m},\ \omega_0^2 = \frac{k}{m}\right)$. The characteristic polynomial is

$$r^2 + 2\,p\,r + \omega_0^2 = 0 \qquad ar^2 + br + c = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which has roots

$$r = \frac{-2\,p \pm \sqrt{4\,p^2 - 4\omega_0^2}}{2} = -p \pm \sqrt{p^2 - \omega_0^2}. \qquad \sqrt{4\,p^2 - 4\omega_0^2} = 2\sqrt{p^2 - \omega_0^2}$$

Case 2a) ($p^2 < \omega_0^2$, or $c^2 < 4\,m\,k$) underdamped. Complex roots

undamped $r = \pm\sqrt{-\omega_0^2} = \pm i\omega_0$

$$r = -p \pm \sqrt{p^2 - \omega_0^2} = -p \pm i\,\omega_1$$

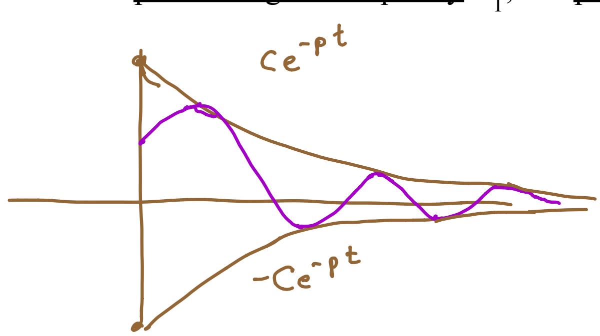
$r = a \pm ib$

$e^{at}\cos bt,\ e^{at}\sin bt$

with $\omega_1 = \sqrt{\omega_0^2 - p^2} < \omega_0$, the undamped angular frequency.

$$x(t) = e^{-p\,t}\left(A\cos(\omega_1 t) + B\sin(\omega_1 t)\right) = e^{-p\,t}C\cos(\omega_1 t - \alpha_1).$$

- solution decays exponentially to zero, but oscillates infinitely often, with exponentially decaying pseudo-amplitude $e^{-p\,t}C$ and pseudo-angular frequency ω_1, and pseudo-phase angle α_1.



$$r^2 + 2pr + \omega_0^2 = 0$$

which has roots

$$r = -\frac{2p \pm \sqrt{4p^2 - 4\omega_0^2}}{2} = -p \pm \sqrt{p^2 - \omega_0^2}\,.$$

Case 2b) ($p^2 > \omega_0^2$, or $c^2 > 4mk$). overdamped. In this case we have two negative real roots

$$r_1 = -p - \sqrt{p^2 - \omega_0^2} < 0$$

$$r_1 < r_2 = -p + \sqrt{p^2 - \omega_0^2} < 0$$

and

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = e^{r_2 t}\left(c_1 e^{\overbrace{(r_1 - r_2)}^{<0} t} + c_2\right).$$

- solution converges to zero exponentially fast; solution passes through equilibrium location $x = 0$ at most once.

Case 2c) ($p^2 = \omega_0^2$, or $c^2 = 4mk$) critically damped. Double real root $r_1 = r_2 = -p = -\dfrac{c}{2m}$.

$$x(t) = e^{-pt}\left(c_1 + c_2 t\right).$$

- solution converges to zero exponentially fast, passing through $x = 0$ at most once, just like in the overdamped case. The critically damped case is the transition between overdamped and underdamped:

Exercise 4) Classify (underdamped, overdamped, critically damped), by finding the roots of the characteristic polynomial. Write down the general solution. Could you solve the IVP for $x(t)$?

4a)

$$x'' + 6x' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

warmup $\quad p(r) = r^2 + 6r + 9 = \boxed{(r+3)^2} = 0$

$r = -3$

$$x(t) = c_1 e^{-3t} + c_2 t e^{-3t}$$

critically damped.

(a double negative root)

4b)

$$x'' + 10x' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

$$p(r) = r^2 + 10r + 9 = (r+1)(r+9)$$

$$x(t) = c_1 e^{-t} + c_2 e^{-9t}$$

overdamped

(2 neg. real roots)

4c)

$$x'' + 2x' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

$$p(r) = r^2 + 2r + 9 = (r+1)^2 + 8 = 0$$

$$(r+1)^2 = -8$$

$$r + 1 = \pm i\, 2\sqrt{2}$$

$$r = -1 \pm i\, 2\sqrt{2}$$

$$x(t) = c_1 e^{-t} \cos(2\sqrt{2}\, t) + c_2 e^{-t} \sin(2\sqrt{2}\, t)$$

underdamped

(complex root)

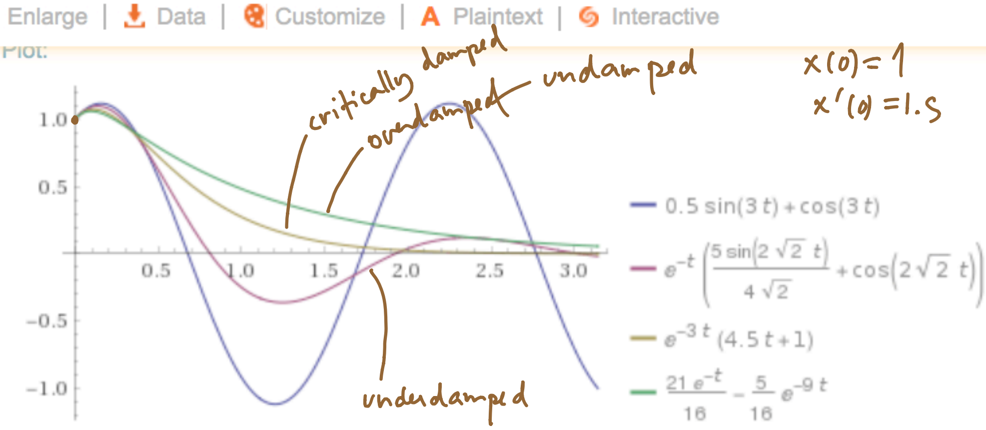
The same initial displacement and velocity, and mass and spring constant - for an undamped, underdamped, overdamped, and critically damped mass-spring problem: (Courtesy Wolfram alpha).

Input interpretation:

plot	$\cos(3t) + 0.5\sin(3t)$	$t = 0$ to π
	$\exp(-t)\left(\frac{5}{8}\sqrt{2}\sin\left(2\sqrt{2}\,t\right) + \cos\left(2\sqrt{2}\,t\right)\right)$	
	$\exp(-3t)(1 + 4.5t)$	
	$\frac{21}{16}\exp(-t) - \frac{5}{16}\exp(-9t)$	

Enlarge | Data | Customize | Plaintext | Interactive

Plot:



$x(0) = 1$

$x'(0) = 1.5$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

$i(\alpha+\beta)$

$i\alpha + i\beta$

$\rightarrow \;\|\,?$

$$e^{i\alpha}\, e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$