Theorem: Consider the autonomous differential equation

$$x'(t) = f(x)$$

with $f(x)$ and $\frac{\partial}{\partial x} f(x)$ continuous (so local existence and uniqueness theorems hold). Let $f(c) = 0$, i.e. $x(t) \equiv c$ is an equilibrium solution.

Suppose c is an *isolated zero* of f, i.e. there is an open interval containing c so that c is the only zero of f in that interval. The the stability of the equilibrium solution c can is completely determined by the local phase diagrams:

$$sign(f): \ ----0+++ \quad \Rightarrow \quad \leftarrow\leftarrow\leftarrow c \rightarrow\rightarrow\rightarrow \quad \Rightarrow \quad c \text{ is unstable}$$

$$sign(f): \ +++0---- \quad \Rightarrow \quad \rightarrow\rightarrow\rightarrow c \leftarrow\leftarrow\leftarrow \quad \Rightarrow \quad c \text{ is asymptotically stable}$$

$$sign(f): \ +++0+++ \quad \Rightarrow \quad \rightarrow\rightarrow\rightarrow c \rightarrow\rightarrow\rightarrow \quad \Rightarrow \quad c \text{ is unstable (half stable)}$$

$$sign(f): ----0---- \quad \Rightarrow \quad \leftarrow\leftarrow\leftarrow c \leftarrow\leftarrow\leftarrow \quad \Rightarrow \quad c \text{ is unstable (half stable)}$$

We can actually prove this Theorem with calculus!! (want to try?)

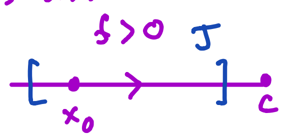
sppse $x'(t) \geq \delta > 0$, what's longest time it could take to travel L units ans $\frac{L}{\delta}$

const. speed. rate · time = distance (speed)

so at slowest speed $\delta \cdot T = L$

$T = \frac{L}{\delta}$

take half of one of pictures above.



claim: $\lim_{t \to \infty} x(t) = c$

$x_0 \in J$, closed interval on which $f(x) > 0$

by extreme value theorem from Calculus

$f(x) \geq \delta > 0$ (f attains its positive min. value.

length of $J = L$

after time $\frac{L}{\delta}$, $x(t)$ has left J

Pick right endpoint of J as close to c as I want (my δ get smaller, so it'll longer for $x(t)$ to leave.)

Shows $\lim_{t \to \infty} x(t) = c$

Constructing many solutions from one ....

Exercise 3) Use the chain rule to check that if $x(t)$ solves the autonomous DE

$$x'(t) = f(x) \qquad x'(t) = f(x(t)) \quad *$$

Then $X(t) := x(t-c)$ solves the same DE. What does this say about the geometry of representative solution graphs to autonomous DEs? Have we already noticed this?

Check

$$X'(t) = x'(t-c)\cdot\frac{d}{dt}(t-c) = x'(t-c)\cdot 1 \quad \text{Chain rule}$$
$$= f(x(t-c)) \quad *$$
$$= f(X(t))$$

horizontal translations of solns are solns

Exercise 4) Use the chain rule to check that if $x(t)$ solves the autonomous DE

$$x'(t) = f(x)$$

Then $X(t) := x(-t)$ solves

$$X'(t) = -f(X)$$

Chain rule: $\frac{d}{dt}X(t) = \frac{d}{dt}x(-t) = x'(-t)\cdot(-1)$

$$= -x'(-t)$$
$$= -f(x(-t))$$
$$= -f(X(t))$$

(related graphs are reflections across t-axis)

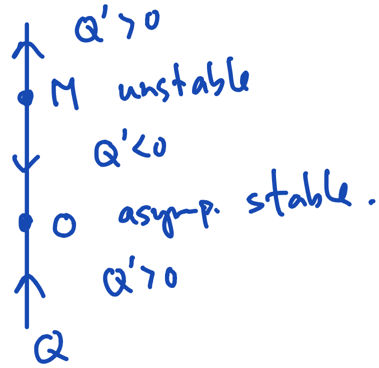
Application of Exercise 4: Understanding the "doomsday-extinction" model via the logistic model. With different hypotheses about fertility and mortality rates, one can arrive at a population model which looks like logistic, except the right hand side is the opposite of what it was in that case:

$$\begin{array}{lll} \text{Logistic:} & P'(t) = -a\,P^2 + b\,P & = kP(M-P) \\ \text{Doomsday-extinction:} & Q'(t) = a\,Q^2 - b\,Q & = kQ(Q-M) \end{array}$$

For example, suppose that the chances of procreation are proportional to population density (think alligators or insect plagues), i.e. the fertility rate $\beta = a\,Q(t)$, where $Q(t)$ is the population at time t. Suppose the morbidity rate is constant, $\delta = b$. With these assumptions the birth and death rates are $a\,Q^2$ and $-b\,Q$ .... which yields the DE above. In this case factor the right side:

$$\boxed{Q'(t) = a\,Q\left(Q - \frac{b}{a}\right) = k\,Q(Q-M).}$$

Exercise 5) Construct the phase diagram for the general doomsday-extinction model and discuss the stability of the equilbrium solutions.



If $P(t)$ solves the logistic differential equation

$$P'(t) = k\,P(M-P)$$

we deduce from exercise 4 that $Q(t) := P(-t)$ solves the doomsday-extinction differential equation

$$Q'(t) = k\,Q(Q-M)\ .$$

We can use this fact to recover a formula for solutions to doomsday-extinction IVPs. What does this say about how representative solution graphs are related, for the logistic and the doomsday-extinction models? The solution to the logistic IVP is

$$P(t) = \frac{MP_0}{\left(M - P_0\right)e^{-Mkt} + P_0}\ .$$

So for the doomsday-extinction IVP

$$Q'(t) = k\,Q(Q-M)$$
$$Q(0) = Q_0$$

the solution is

$$Q(t) = P(-t) = \frac{MQ_0}{\left(M - Q_0\right)e^{Mkt} + Q_0}$$

$k=1$ M

Example: We can use the formula on the previous page or work from scratch using partial fractions, to write down the solution to the doomsday-extinction IVP

$$x'(t) = x(x-1)$$
$$x(0) = 2 = Q_0$$

Does the solution exist for all $t > 0$? (Hint: no, there is a very bad doomsday at $t = \ln 2$.)

from previous page, $M=1$, $Q_0 = 2$, $k=1$

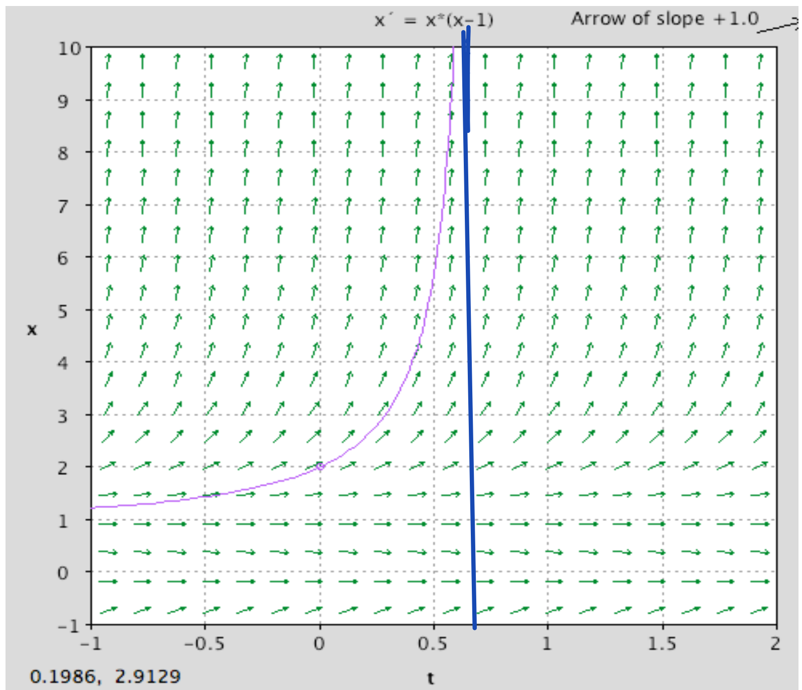
$$Q(t) = \frac{2}{(1-2)e^t + 2} = \frac{2}{2-e^t}$$

you also found this solution as the Wed. warm-up problem.

@ $t=0$, $Q(0) = \frac{2}{2-1} = 2$ ✓

@ $t = \ln 2$

"doomsday"



Math 2250-004
Friday Jan 26
2.2 - 2.3

- harvesting a logistic population (2.2)
- improved velocity models (2.3)

Announcements:

- On HW1 the solution key had the wrong initial conditions for 1.2 #26, so a lot of people lost points (26abc were worth 1 pt each.)
- There was a typo on soltns to w1.2b which may also have cost some people half a point

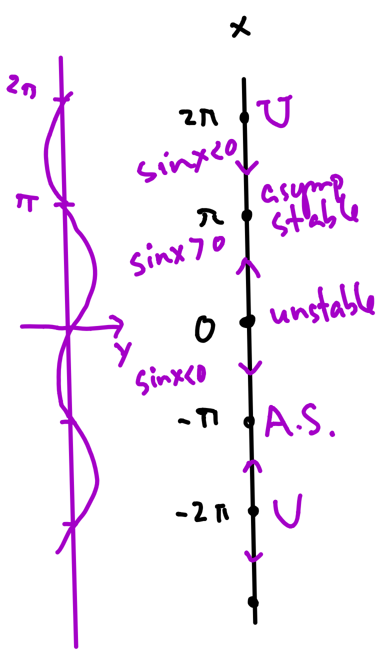
Using the updated soltns, please regrade those two problems. Then on the last page of HW3 indicate your old & corrected scores so our grader can update CANVAS

Warm-up Exercise: On Wed we noted that the equilibrium points for the autonomous DE

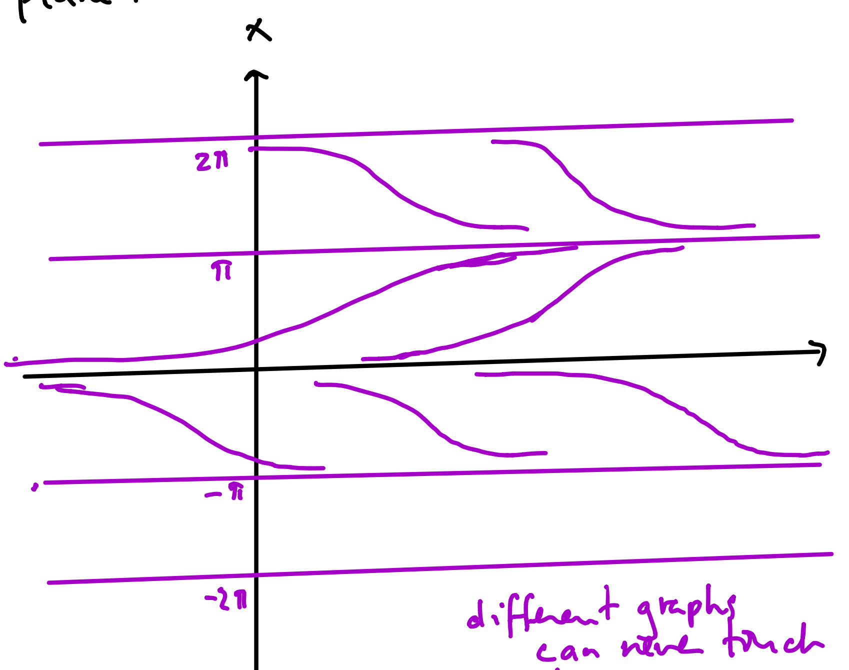
$$x'(t) = \sin x$$

'til 10:48

were $x = k\pi$, $k \in \mathbb{Z}$ (k an integer). Draw the phase diagram for this DE and determine the stability of each equilibrium solution. Also sketch representative solution graphs onto the t-x plane:



phase diagram. for $\boxed{x' = \sin x}$



solution graphs

different graphs can never touch because of existence-uniqueness theorem

2.2: <u>Further application of phase-portrait analysis:</u> harvesting a logistic population...text p.97 (or, why do fisheries sometimes seem to die out "suddenly"?) Consider the DE

see part c of lab #2.

$$P'(t) = a\,P - b\,P^2 - h\,.$$

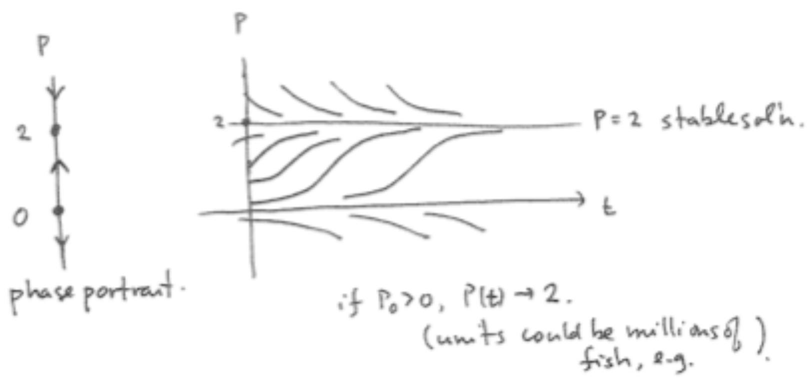
Notice that the first two terms represent a logistic rate of change, but we are now harvesting the population at a rate of h units per time. For simplicity we'll assume we're harvesting fish per year (or thousands of fish per year etc.) One could model different situations, e.g. constant "effort" harvesting, in which the effect on how fast the population was changing could be $h\,P$ instead of P.

For computational ease we will assume $a = 2$, $b = 1$. (One could actually change units of population and time to reduce to this case.)

for computational simplicity take $a = 2$, $b = 1$

Case 0 no harvesting

$$P'(t) = 2P - P^2 = P(2-P)$$



with harvesting:

$$P'(t) = 2P - P^2 - h = -(P^2 - 2P + h) = -(P-P_1)(P-P_2)$$

$$P_1, P_2 = \frac{2 \pm \sqrt{4-4h}}{2} = 1 \pm \sqrt{1-h}$$

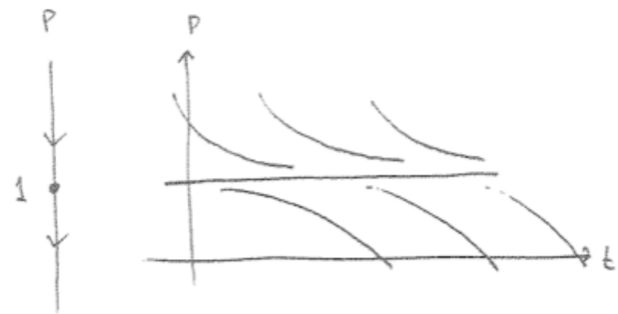
Case 1: subcritical harvesting

$0 < h < 1$

Case 2. Critical harvesting

$h = 1$

$P'(t) = -(P-1)^2$

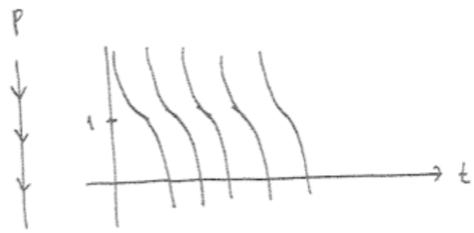


Case 3 Over harvesting

$h > 1$

complex roots.

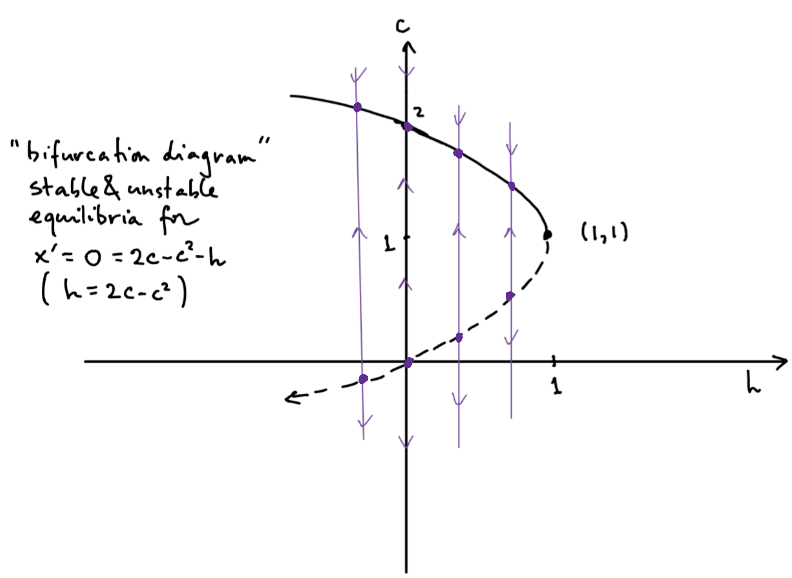
$$P'(t) = -(P^2 - 2P + h)$$
$$= -\left[(P-1)^2 + (h-1)\right]$$
$$< 0.$$



This model gives a plausible explanation for why many fisheries have "unexpectedly" collapsed in modern history. If $h < 1$ but near 1 and something perturbs the system a little bit (a bad winter, or a slight increase in fishing pressure), then the population and/or model could suddenly shift so that $P(t) \to 0$ very quickly.

google Cod fishery collapse Grand Banks

Here's one picture that summarizes all the cases - you can think of it as collection of the phase diagrams for different fishing pressures h. The upper half of the parabola represents the stable equilibria, and the lower half represents the unstable equilibria. Diagrams like this are called "bifurcation diagrams". In the sketch below, the point on the h- axis should be labeled $h = 1$, not h. What's shown is the parabola of equilibrium solutions, $c = 1 \pm \sqrt{1-h}$, i.e. $2c - c^2 - h = 0$, i.e. $h = c(2-c)$.



2.3 Improved velocity models: velocity-dependent drag forces

For particle motion along a line, with

position $x(t)$ (or $y(t)$),
velocity $x'(t) = v(t)$, and
acceleration $x''(t) = v'(t) = a(t)$

We have Newton's 2^{nd} law

$$m\, v'(t) = F$$

where F is the net force.

- We're very familiar with constant force $F = m\,\alpha$, where α is a constant:

$$v'(t) = \alpha$$
$$v(t) = \alpha\, t + v_0$$
$$x(t) = \frac{1}{2}\alpha\, t^2 + v_0\, t + x_0 .$$

Examples we've seen a lot of:

- $\alpha = -g$ near the surface of the earth, if up is the positive direction, or $\alpha = g$ if down is the positive direction.
- boats or cars or "particles" subject to constant acceleration or deceleration.

New today !!! Combine a constant force with a velocity-dependent drag force, at the same time. The text calls this a "resistance" force:

$$m\, v'(t) = m\,\alpha + F_R$$

Empirically/mathematically the resistance forces F_R depend on velocity, in such a way that their magnitude is

$$|F_R| \approx k\,|v|^p \ , 1 \le p \le 2 .$$

- $p = 1$ (linear model, drag proportional to velocity):

$$m\,v'(t) = m\,\alpha - k\,v$$

This linear model makes sense for "slow" velocities, as a linearization of the frictional force function, assuming that the force function is differentiable with respect to velocity...recall Taylor series for how the velocity resistance force might depend on velocity:

$$F_R(v) = F_R(0) + F_R{}'(0)\,v + \frac{1}{2!}F_R{}''(0)v^2 + \ldots$$

$F_R(0) = 0$ and for small enough v the higher order terms might be negligable compared to the linear term, so

$$F_R(v) \approx F_R{}'(0)\,v \approx -k\,v\ .$$

We write $-k\,v$ with $k > 0$, since the frictional force opposes the direction of motion, so sign opposite of the velocity's.

http://en.wikipedia.org/wiki/Drag_(physics)#Very_low_Reynolds_numbers:_Stokes.27_drag

Exercise 1: Rewrite the linear drag model as

$$v'(t) = \alpha - \rho\,v$$

our favorite 1st order DE!

where the $\rho = \dfrac{k}{m}$. Construct the phase diagram for v . Notice that $v(t)$ has exactly one constant (equilibrium) solution, and find it. Its value is called the *terminal velocity*. Explain why *terminal velocity* is an appropriate term of art, based on your phase diagram for velocity.

(This is a constant coefficient linear DE, so it's easy to solve. We'll do that in detail on Monday, and then we'll antidifferentiate $v(t)$ to find the position function $x(t)$. The text also works this out in detail.)