Wednesday Jan 24
2.2: Autonomous Differential Equations.

Announcements:

- HW returns → scores on back. I post grading rubric on public page & solutions on CANVAS
- in class quiz today.

Warm-up Exercise: Solve this IVP, which is an example of the "Doomsday-extinction" population model we'll mention in class today

'til ~~10:48~~ 10:50

$x = x(t)$

$$\begin{cases} \frac{dx}{dt} = x(x-1) = x^2 - x \\ x(0) = 2 \end{cases}$$

$\cancel{x'(t) + P(t)x = Q}$

$$\frac{dx}{x(x-1)} = dt \qquad \text{also } x \equiv 0,\ x \equiv 1$$

$$\frac{1}{x(x-1)}\,dx = dt$$

$$\left(\frac{A}{x} + \frac{B}{x-1}\right)dx = dt$$

$$\frac{1}{1}\left(\frac{1}{x-1} - \frac{1}{x}\right) \quad \text{or} \quad \frac{1}{-1}\left(\frac{1}{x} - \frac{1}{x-1}\right)$$

$$\frac{x-(x-1)}{(x-1)(x)} \qquad \frac{x-1-x}{x(x-1)}$$

$$\frac{1}{(x-1)(x)} \qquad \frac{-1}{x(x-1)}$$

cont

$$\int\left(\frac{1}{x-1} - \frac{1}{x}\right)dx = \int dt$$

$$\ln|x-1| - \ln|x| = t + C$$

$$\ln\left|\frac{x-1}{x}\right| = t + C_1$$

$$e \qquad e$$

$$\left|\frac{x-1}{x}\right| = e^t e^{C_1}$$

$$\frac{x-1}{x} = Ce^t, \quad @\, t=0 \quad \frac{2-1}{2} = C \quad \frac{1}{2} = C$$

$$x - 1 = \frac{1}{2}e^t x$$

$$x - \frac{1}{2}e^t x = 1$$

$$x\left(1 - \frac{1}{2}e^t\right) = 1$$

$$\boxed{x(t) = \frac{1}{1 - \frac{1}{2}e^t}}$$

2.2: Recall, that a first order DE for $x = x(t)$ is written as

$$x' = f(t, x)\ ,$$

which is shorthand for

$$x'(t) = f(t, x(t)).$$

Definition: If the slope function f only depends on the value of $x(t)$, and not on t itself, then we call the first order differential equation *autonomous*:

$$x' = f(x)\ . \quad *$$

if $x(t) \equiv c$ solves $*$
LHS = RHS
$0 = f(c)$

Example: The logistic DE, $P' = k\,P(M - P)$ is an autonomous differential equation for $P(t)$, because how fast the population is changing only depends on the value of the population.

Definition: Constant solutions $x(t) \equiv c$ to autonomous differential equations $x' = f(x)$ are called *equilibrium solutions*. Since the derivative of a constant function $x(t) \equiv c$ is zero, the values c of equilibrium solutions are exactly the roots c to $f(c) = 0$.

Example: The functions $P(t) \equiv 0$ and $P(t) \equiv M$ are the equilibrium solutions for the logistic DE.

$P' = kP(M-P) \qquad P = P(t)$

Exercise 1: Find the equilibrium solutions of

constant fcns have values c which are roots of slope fcn

1a) $x'(t) = 3x - x^2$

$= x(3-x)$

$x(t) \equiv 0, \quad x(t) \equiv 3$

1b) $x'(t) = x^3 + 2x^2 + x \;=\; x(x^2+2x+1) = x(x+1)^2$

eq. solfns $x \equiv 0, \quad x \equiv -1.$

1c) $x'(t) = \sin(x)$. $\quad \ldots -2\pi, -\pi, 0, \pi, 2\pi, \ldots \quad$ i.e. $k\pi$ where $k \in \mathbb{Z}$

integers

1d) $x'(t) = x(x-1) \qquad x(t) \equiv 0, 1$.

Def: Let $x(t) \equiv c$ be an equilibrium solution for an autonomous DE. Then

· c is a *stable* equilibrium solution if solutions with initial values close enough to c stay close to c. $t > 0$

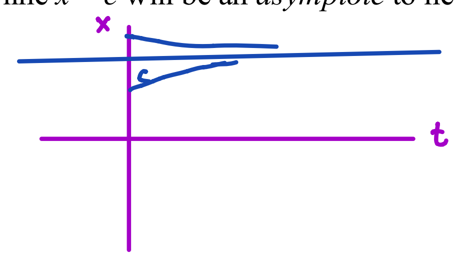
There is a precise way to say this, but it requires quantifiers: For every $\varepsilon > 0$ there exists a $\delta > 0$ so that for solutions with $\underbrace{|x(0) - c| < \delta}_{\text{how close to start}}$, we have $\underbrace{|x(t) - c| < \varepsilon}_{\text{if you want to stay this close}}$ for all $t > 0$.

· c is an *unstable* equilibrium if it is not stable.

· c is an *asymptotically stable* equilibrium solution if it's stable and in addition, if $x(0)$ is close enough to c, then $\lim_{t \to \infty} x(t) = c$.

Precisely there exists a $\delta > 0$ so that if $|x(0) - c| < \delta$ then $\lim_{t \to \infty} x(t) = c$.

Notice that if c is asymptotically stable, then the horizontal line $x = c$ will be an *asymptote* to nearby solution graphs $x = x(t)$)



asymptotically stable.

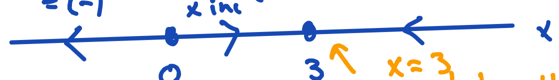
Exercise 2: Use phase diagram analysis to guess the stability of the equilibrium solutions in Exercise 1. For (a) you've worked out a solution formula already, so you'll know you're right. For (b), (c), use the Theorem on the next page to justify your answers.

2a) $x'(t) = 3x - x^2$
$= x(3-x)$

$x<0$: $x' = (-)(+) = (-)$

$0<x<3$: $x' = (+)(+) = (+)$, x inc

$x>3$: $x' = x(3-x) = (+)(-) = (-)$, $x(t)$ dec



$x=0$ unstable

$x=3$ asymptotically stable

2b) $x'(t) = x^3 + 2x^2 + x$
$x(x+1)^2$

$x<-1$: $x' = (-)(+) = (-)$, x dec

$-1<x<0$: $x' = (-)(+) = (-)$, x dec

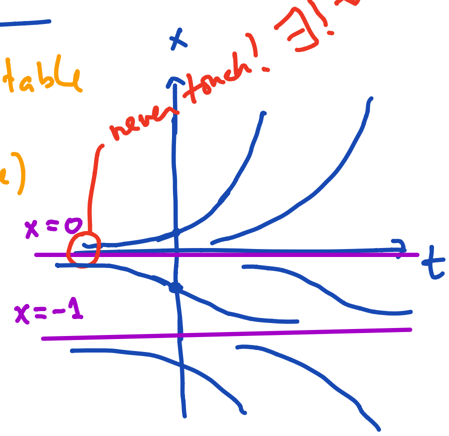
$x>0$, $x' = (+)(+) = (+)$, x inc

$x=0$ unstable

$x=-1$ unstable (semistable, or half stable)

2c) $x'(t) = \sin(x)$.

It's a mystery



Theorem: Consider the autonomous differential equation

$$x'(t) = f(x)$$

with $f(x)$ and $\dfrac{\partial}{\partial x} f(x)$ continuous (so local existence and uniqueness theorems hold). Let $f(c) = 0$, i.e. $x(t) \equiv c$ is an equilibrium solution.

Suppose c is an *isolated zero* of f, i.e. there is an open interval containing c so that c is the only zero of f in that interval. The the stability of the equilibrium solution c can is completely determined by the local phase diagrams:

$sign(f)$: $----0+++ \quad \Rightarrow \quad \leftarrow\leftarrow\leftarrow c \rightarrow\rightarrow\rightarrow \quad \Rightarrow$ c is unstable

$sign(f)$: $+++0---- \quad \Rightarrow \quad \rightarrow\rightarrow\rightarrow c \leftarrow\leftarrow\leftarrow \quad \Rightarrow$ c is asymptotically stable

$sign(f)$: $+++0+++ \quad \Rightarrow \quad \rightarrow\rightarrow\rightarrow c \rightarrow\rightarrow\rightarrow \quad \Rightarrow$ c is unstable (half stable)

$sign(f)$: $----0---- \quad \Rightarrow \quad \leftarrow\leftarrow\leftarrow c \leftarrow\leftarrow\leftarrow \quad \Rightarrow$ c is unstable (half stable)

We can actually prove this Theorem with calculus!! (want to try?)