Tues Jan 23
2.1 Improved population models

Announcements:
- Chapter 2.1-2.3 text hw pages are posted, along with EP 3.7 for electrical circuits
- HW due tomorrow
- Quiz tomorrow
- Office hours 5:00-6:00 today, JWB 240

Warm-up Exercise: Find the partial fraction decomposition:

$$\frac{1}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1}$$

(This would be useful if you wanted to solve

$$\frac{dx}{dt} = x^2-3x-4$$ via separation of variables)

'til 10:48

long way:

$$\frac{1}{(x-4)(x-1)} = \frac{A}{x-4} + \frac{B}{x+1}$$

$$\frac{1}{(x-4)(x-1)} = \frac{A(x+1)+B(x-4)}{(x-4)(x+1)}$$

numerators:

$$1 = A(x+1) + B(x-4)$$

@ $x=-1$: $1 = -5B \Rightarrow B = -\frac{1}{5}$

@ $x=4$: $1 = 5A \Rightarrow A = \frac{1}{5}$

So

$$\frac{1}{(x-4)(x-1)} = \frac{1}{5}\frac{1}{x-4} - \frac{1}{5}\frac{1}{x+1}$$

$$\boxed{= \frac{1}{5}\left(\frac{1}{x-4} - \frac{1}{x+1}\right)}$$

shortcut: for $a \neq b$

$\frac{1}{(x-a)(x-b)}$ is always a multiple of $\left(\frac{1}{x-a} - \frac{1}{x-b}\right)$.

For us,

$$\frac{1}{x-4} - \frac{1}{x+1} = \frac{(x+1)-(x-4)}{(x-4)(x+1)}$$

$$= \frac{5}{(x-4)(x+1)}$$

divide by 5, to get

$$\boxed{\frac{1}{5}\left(\frac{1}{x-4} - \frac{1}{x+1}\right) = \frac{1}{(x-4)(x+1)}}$$

2.1: Let $P(t)$ be a population at time t. Let's call them "people", although they could be other biological organisms, decaying radioactive elements, accumulating dollars, or even molecules of solute dissolved in a liquid at time t (2.1.23). Consider:

$B(t)$, birth rate (e.g. $\frac{people}{year}$);

$\beta(t) := \frac{B(t)}{P(t)}$, fertility rate ($\frac{people}{year}$ per *person*)

$B(t) = \beta(t)\, P(t)$
$D(t) = \delta(t)\, P(t)$

$\mathrm{D}(t)$, death rate (e.g. $\frac{people}{year}$);

$\delta(t) := \frac{\mathrm{D}(t)}{P(t)}$, mortality rate ($\frac{people}{year}$ per *person*)

Then in a closed system (i.e. no migration in or out) we can write the governing DE two equivalent ways:

$$P'(t) = B(t) - \mathrm{D}(t) \quad \bullet$$
$$P'(t) = \big(\beta(t) - \delta(t)\big)P(t)\,. \quad \bullet$$

<u>Model 1:</u> constant fertility and mortality rates, $\beta(t) \equiv \beta_0 \ge 0,\ \delta(t) \equiv \delta_0 \ge 0$, constants.

$$\Rightarrow P' = \big(\beta_0 - \delta_0\big)P = k\,P\,. \quad \bullet$$

This is our familiar exponential growth/decay model, depending on whether $k > 0$ or $k < 0$.

<u>Model 2:</u> population fertility and mortality rates only depend on population P, but they are not constant:

$$\beta = \beta_0 + \beta_1 P \quad \bullet$$
$$\delta = \delta_0 + \delta_1 P \quad \bullet$$

with $\beta_0, \beta_1, \delta_0, \delta_1$ constants. This implies

$$P' = (\beta - \delta)P = \big(\big(\beta_0 + \beta_1 P\big) - \big(\delta_0 + \delta_1 P\big)\big)P$$
$$= \big(\underbrace{\big(\beta_0 - \delta_0\big)}_{b>0} + \underbrace{\big(\beta_1 - \delta_1\big)}_{<0,\ =-a}P\big)P\,.$$

For viable populations, $\beta_0 > \delta_0$. For a sophisticated (e.g. human) population we might also expect $\beta_1 < 0$, and resource limitations might imply $\delta_1 > 0$. With these assumptions, and writing $\beta_1 - \delta_1 = -a < 0$, $\beta_0 - \delta_0 = b > 0$ one obtains the <u>logistic differential equation:</u>

$$* \quad P' = (b - a\,P)P \quad \bullet$$
$$P' = b\,P - a\,P^2\text{, or equivalently} \quad \bullet$$
$$* \quad P' = a\,P\left(\frac{b}{a} - P\right) = k\,P(M - P)\,. \quad \bullet$$

$k = a > 0$, $M = \frac{b}{a} > 0$. (One can consider other cases as well.)

$P'(t) = kP(M-P)$ <u>logistic DE</u>

Exercise 1a): Discuss qualitative features of the slope field for the logistic differential equation for $P = P(t)$. Notice that the "isoclines" (curves where the slope function is constant) are horizontal lines

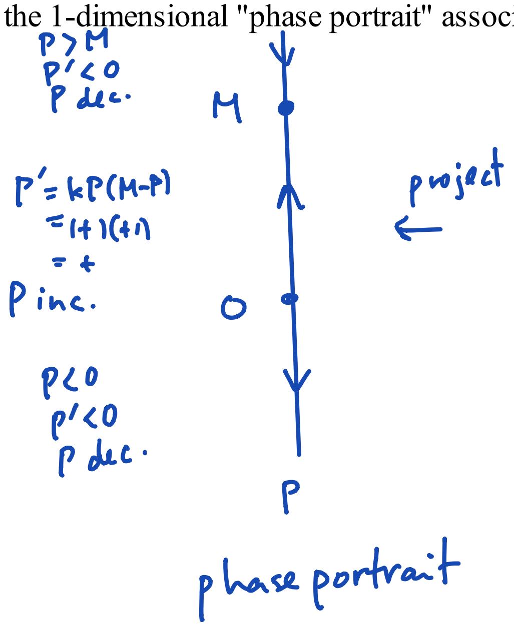
$$\frac{dP}{dt} = P' = k\,P(M - P)$$

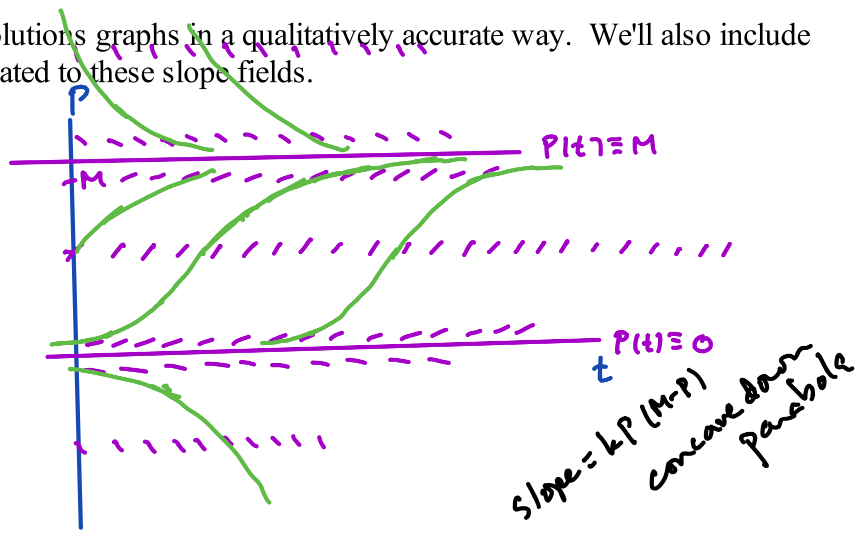
Also note that there are two constant ("equilibrium") solutions. What are they?

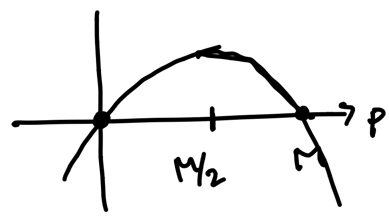
$P(t) \equiv 0$: LHS $= P'(t) = 0 =$ RHS $= k \cdot 0 \cdot (M-0)$.

$P(t) \equiv M$

b) Sketch the slope field and apparent solutions graphs in a qualitatively accurate way. We'll also include the 1-dimensional "phase portrait" associated to these slope fields.







c) When discussing the logistic equation, the value M is called the "carrying capacity" of the (ecological or other) system. Discuss why this is a good way to describe M. Hint: if $P(0) = P_0 > 0$, and $P(t)$ solves the logistic equation, what is the apparent value of $\lim_{t \to \infty} P(t)$? Note that by the existence-uniqueness theorem, different solution graphs may never touch each other, so the time-varying solution graphs never touch the horizontal graph asymptotes.

$P_0 > 0$, $\lim_{t \to \infty} P(t) = M$.

Exercise 2: Solve the logistic DE IVP

$$P' = k\,P(M-P)$$
$$P(0) = P_0$$

via separation of variables. Verify that the solution formula is consistent with the slope field and phase diagram discussion from exercise 1. Hint: You should find that

$$P(t) = \frac{MP_0}{(M-P_0)e^{-Mkt} + P_0}.$$

Solution (we will work this out step by step in class, using the fact that the logistic DE is separable. It is not linear!!): (in HW, similar).

$$\frac{dP}{dt} = kP(M-P)$$

$$\frac{dP}{P(P-M)} = -k\,dt \qquad (P \neq 0, M)$$

$$\frac{1}{P(P-M)}\,dP = -k\,dt$$

$$\frac{1}{M}\Big(\underbrace{\frac{1}{P-M} - \frac{1}{P}}_{\frac{M}{P(P-M)}}\Big)\,dP = -k\,dt$$

$$\int \Big(\frac{1}{P-M} - \frac{1}{P}\Big)\,dP = \int -Mk\,dt$$

$$\ln|P-M| - \ln|P| = -Mkt + C$$

$$e^{\ln\left|\frac{P-M}{P}\right|} = e^{-Mkt + C_1}$$

$$\left|\frac{P-M}{P}\right| = e^{C_1}e^{-Mkt}$$

$$\frac{P-M}{P} = Ce^{-Mkt} \qquad C = \pm e^{C_1}$$

$$\left(@\ t=0:\ \frac{P_0-M}{P_0} = C\right)$$

$$P - M = PCe^{-Mkt}$$

$$P - PCe^{-Mkt} = M$$

$$P(t)\left[1 - Ce^{-Mkt}\right] = M$$

$$P(t) = \frac{M}{1 - \left(\frac{P_0-M}{P_0}\right)e^{-Mkt}} = \frac{MP_0}{P_0 - (P_0-M)e^{-Mkt}}$$

$$\boxed{P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-Mkt}}}$$

$$P(t) = \frac{MP_0}{P_0 + \left(M - P_0\right)e^{-Mkt}} .$$

Notice that because $\lim_{t \to \infty} e^{-Mkt} = 0$,

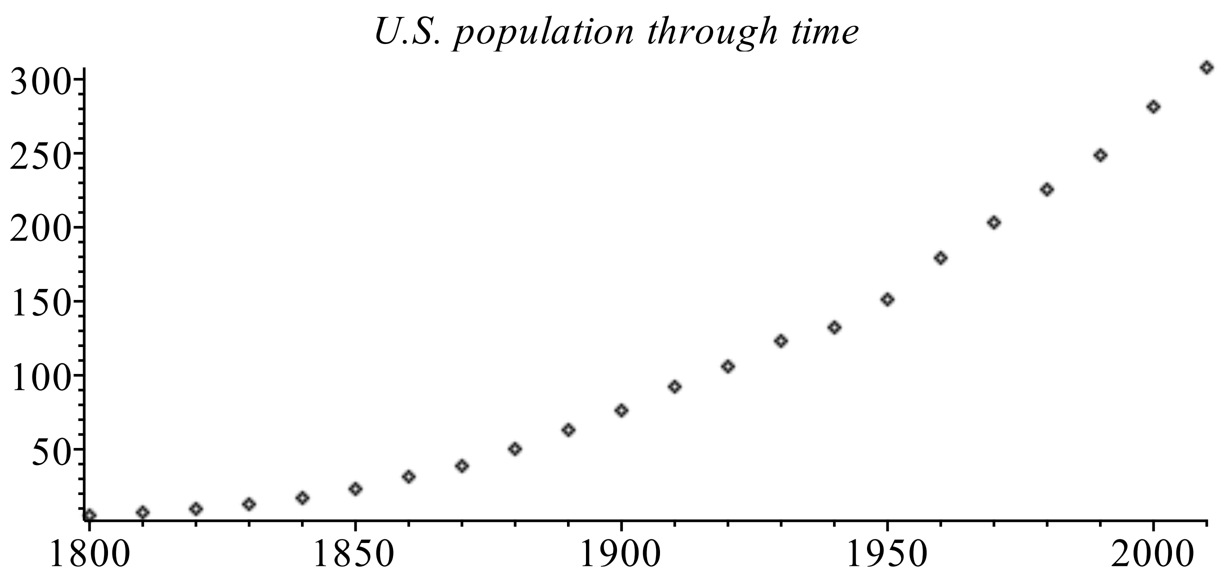
$$\boxed{\lim_{t \to \infty} P(t) = \frac{MP_0}{P_0} = M \text{ as expected.}}$$

<u>Note:</u> If $P_0 > 0$ the denominator stays positive for $t \geq 0$, so we know that the formula for $P(t)$ is a differentiable function for all $t > 0$. (If the denominator became zero, the function would blow up at the corresponding vertical asymptote.) To check that the denominator stays positive check that (i) if $P_0 < M$ then the denominator is a sum of two positive terms; if $P_0 = M$ the separation algorithm actually fails because you divided by 0 to get started but the formula actually recovers the constant equilibrium solution $P(t) \equiv M$; and if $P_0 > M$ then $\left|M - P_0\right| < P_0$ so the second term in the denominator can never be negative enough to cancel out the positive P_0, for $t > 0$.)

Application!

The Belgian demographer P.F. Verhulst introduced the logistic model around 1840, as a tool for studying human population growth. Our text demonstrates its superiority to the simple exponential growth model, and also illustrates why mathematical modelers must always exercise care, by comparing the two models to actual U.S. population data.

```
> restart : # clear memory
  Digits := 5 : #work with 5 significant digits
> pops := [[1800, 5.3], [1810, 7.2], [1820, 9.6], [1830, 12.9],
      [1840, 17.1], [1850, 23.2], [1860, 31.4], [1870, 38.6],
      [1880, 50.2], [1890, 63.0], [1900, 76.2], [1910, 92.2],
      [1920, 106.0], [1930, 123.2], [1940, 132.2], [1950, 151.3],
      [1960, 179.3], [1970, 203.3], [1980, 225.6], [1990, 248.7],
      [2000, 281.4], [2010, 308.]] : #I added 2010 - between 306-313
      # I used shift-enter to enter more than one line of information
      # before executing the command.
> with(plots) : # plotting library of commands
  pointplot(pops, title = `U.S. population through time`);
```




```
>
```

Unlike Verhulst, the book uses data from 1800, 1850 and 1900 to get constants in our two models. We let t=0 correspond to 1800.

Exponential Model: For the exponential growth model $P(t) = P_0\, e^{r\,t}$ we use the 1800 and 1900 data to get values for P_0 and r :

> $P0 := 5.308;$
> $solve(P0 \cdot \exp(r \cdot 100) = 76.212, r);$

$$P0 := 5.308$$

$$0.026643 \quad \textbf{(1)}$$

> $P1 := t \rightarrow 5.308 \cdot \exp(.02664 \cdot t);$ *#exponential model -eqtn (9) page 83*

$$P1 := t \rightarrow 5.308\, e^{0.02664\, t} \quad \textbf{(2)}$$

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Logistic Model: We get P_0 from 1800, and use the 1850 and 1900 data to find k and M :

> $P2 := t \rightarrow M \cdot P0 / (P0 + (M - P0) \cdot \exp(-M \cdot k \cdot t));$ *# logistic solution we worked out*

$$P2 := t \rightarrow \frac{M\, P0}{P0 + (M - P0)\, e^{-M\,k\,t}} \quad \textbf{(3)}$$

> $solve(\{P2(50) = 23.192, P2(100) = 76.212\}, \{M, k\});$

$$\{M = 188.12, k = 0.00016772\} \quad \textbf{(4)}$$

> $M := 188.12;$
> $k := .16772\text{e-}3;$
> $P2(t);$ *#should be our logistic model function,*
> *#equation (11) page 84.*

$$M := 188.12$$

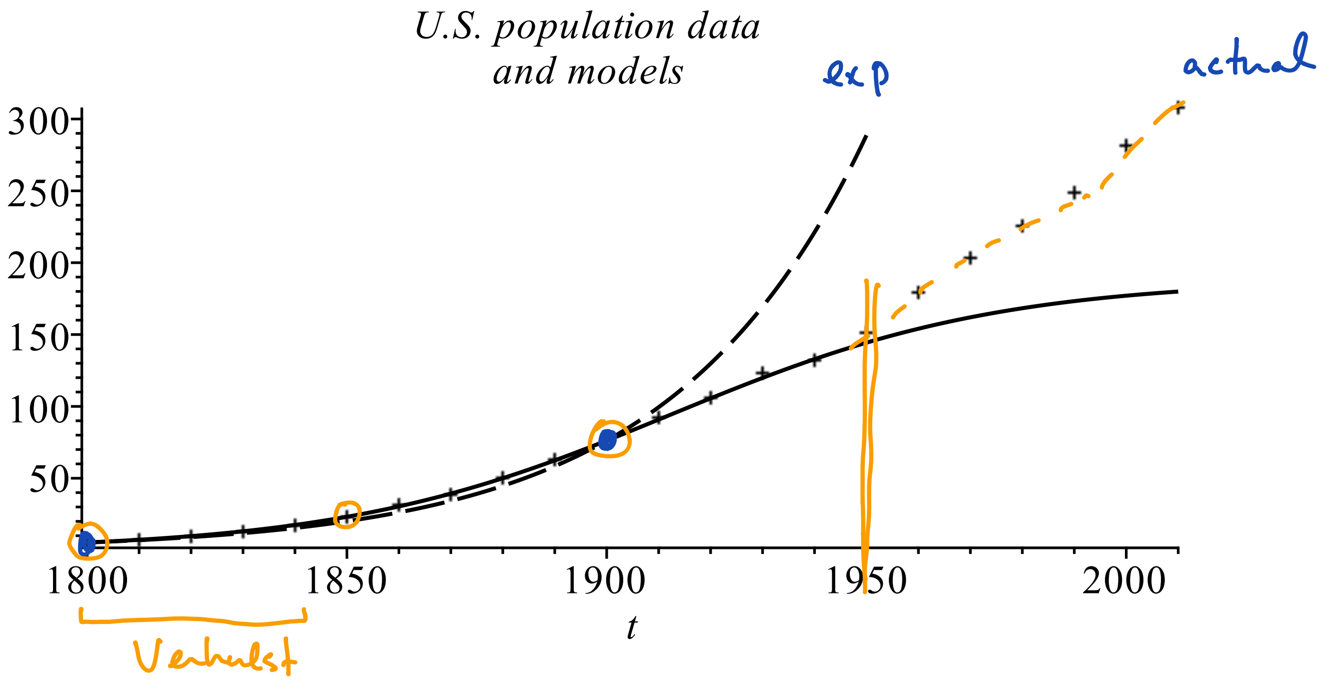
$$k := 0.00016772$$

$$\frac{998.54}{5.308 + 182.81\, e^{-0.031551\, t}} \quad \textbf{(5)}$$

>

Now compare the two models with the real data, and discuss. The exponential model takes no account of the fact that the U.S. has only finite resources.

```
> plot1 := plot(P1(t-1800), t = 1800 ..1950, color = black, linestyle = 3) :
   #this linestyle gives dashes for the exponential curve
  plot2 := plot(P2(t-1800), t = 1800 ..2010, color = black) :
  plot3 := pointplot(pops, symbol = cross) :
  display({plot1, plot2, plot3}, title = `U.S. population data
  and models`);
```

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```

Any ideas on why the logistic model begins to fail (with our parameters) around 1950?