Math 2250-004 Week 3 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes include material from 1.5, EP3.7, 2.1-2.2 and an introduction to 2.3.

posted on HW page.

Mon Jan 22
1.5, EP3.7 linear DEs, and applications.

Announcements:
- I'll post #2 HW.
- posting HW solutions & quizzes (& solutions) on CANVAS.
- start Chptr 2 tomorrow

Warm-up Exercise: Put into standard form & solve:

$$t\,x'(t) = x + 3 \qquad \underline{\underline{(t>0)}}$$

hint:

$$* \quad x' + P(t)x = Q(t)$$

$$\frac{dx}{dt} = f(x)g(t)$$

'til 10:49

e.g. linear

$$t\,x'(t) - x(t) = 3$$

$$t\,x' - x = 3$$

$\div t$

$$x' \underbrace{- \frac{1}{t}}\, x = \frac{3}{t} \quad *$$

I.F. $$e^{\int -\frac{1}{t}dt} = e^{-\ln t} = \left(e^{\ln t}\right)^{-1} = \frac{1}{t}$$

$$\frac{1}{t}\left(x' - \frac{1}{t}x\right) = \frac{3}{t^2}$$

$$\frac{d}{dt}\left(\frac{1}{t}x\right) = \frac{3}{t^2} \quad \checkmark$$

$$\frac{1}{t}x = \int \frac{3}{t^2}dt = -\frac{3}{t} + C.$$

$$\Rightarrow \boxed{x = -3 + Ct}$$

Input-out models often lead to the linear (and separable) differential equation IVP for functions $x(t)$

$$x'(t) + a\,x(t) = b$$
$$x(0) = x_0$$

where a, b are constants. As our warm-up exercise on Friday we found that the solution to this IVP is

$$x(t) = \frac{b}{a} + \left(x_o - \frac{b}{a}\right)e^{-a\,t}.$$

Exercise 1: Use the result above to solve a pollution problem IVP and answer the following question (p. 55-56 text): Lake Huron typically has a constant concentration for a certain pollutant. Due to an industrial accident, Lake Erie has suddenly obtained a concentration five times as large. Lake Erie has a volume of $480\ km^3$, and water flows into and out of Lake Erie at a rate of $350\ km^3$ per year. Essentially all of the in-flow is from Lake Huron and the outflow goes to Lake Ontario (see map). We expect that as time goes by, the water from Lake Huron will flush out Lake Erie. Assuming that the pollutant concentration is roughly the same everywhere in Lake Erie, about how long will it be until this concentration is only twice the background concentration from Lake Huron?

google "toledo river burned"

Huron → Erie → Ontario

a) Set up the initial value problem. ~~Maybe use~~ symbols c for the background concentration (in Huron),

$V = 480\ km^3$

$r = 350\ \dfrac{km^3}{y}$

@ $t=0$ Erie concentration $= 5c$

$V'(t) = r_i - r_o = 0,\ V = 480\ km^3$

$x(t)$ = pollutant amount in Erie

$$x'(t) = \underbrace{r_i c_i}_{\frac{\cancel{vol}}{time}\cdot\frac{mass}{\cancel{vol}}} - r_o c_o = rc - \frac{r\,x(t)}{V}$$

$$\begin{cases} x'(t) + \frac{r}{V}x(t) = rc \\ x(0) = 5cV \end{cases}$$

Q: When will $x(t) = 2cV$?

http://www.enchantedlearning.com/usa/statesbw/greatlakesbw.GIF

b) Solve the IVP, and then answer the question posed in the exercise.

we could use your previous work, but let's solve it again ☺

$P = \frac{r}{V} \qquad \int P\,dt = \frac{r}{V}t$

$x' + P(t)x = Q(t)$

I.F. $e^{\frac{r}{V}t}$

$$e^{\frac{r}{V}t}\left(x' + \frac{r}{V}x\right) = e^{\frac{r}{V}t}rc$$

$$\frac{d}{dt}\left[e^{\frac{r}{V}t}x\right] = rc\,e^{\frac{r}{V}t}$$

$$e^{\frac{r}{V}t}x = \int rc\,e^{\frac{r}{V}t}dt = rc\frac{V}{r}e^{\frac{r}{V}t} + D = cVe^{\frac{r}{V}t} + D$$

$\div$ IF: $x(t) = cV + De^{-\frac{r}{V}t}$

$x(0) = 5cV = cV + D \Rightarrow D = 4cV$

$$\boxed{x(t) = cV + 4cVe^{-\frac{r}{V}t}}$$

(makes sense as $t \to \infty$)

? $2cV = cV + 4cVe^{-\frac{r}{V}t}$

$2 = 1 + 4e^{-r/V\,t}$

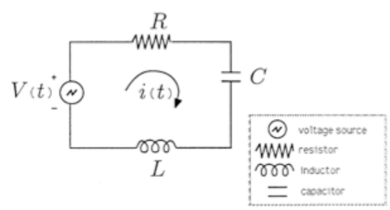
$.25 = e^{-\frac{r}{V}t}$

$\ln(.25) = -\frac{r}{V}t$

$t = -\frac{V}{r}\ln(.25) = 1.9$ years

EP 3.7 This is a supplementary section. I've posted a .pdf on our homework page.

Often the same DE can arise in completely different-looking situations. For example, first order linear DE's also arise (as special cases of second order linear DE's) in simple *RLC* circuit modeling.



circuit element	voltage drop	units
inductor	$L\,I'(t)$	L Henries (H)
resistor	$R\,I(t)$	R Ohms (Ω)
capacitor	$\frac{1}{C}Q(t)$	C Farads (F)

http://cnx.org/content/m21475/latest/pic012.png

Charge $Q(t)$ coulombs accumulates on the capacitor, at a rate $I(t)$ ($i(t)$ in the diagram above) amperes (coulombs/sec), i.e $Q'(t) = I(t)$.

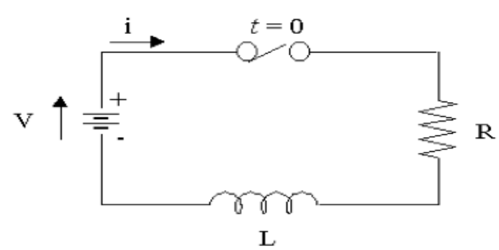
Kirchoff's Law: The sum of the voltage drops around any closed circuit loop equals the applied voltage $V(t)$ (volts). The units of voltage are energy units - Kirchoff's Law says that a test particle traversing any closed loop returns with the same potential energy level it started with:

For $Q(t)$: $\quad L\,Q''(t) + R\,Q'(t) + \frac{1}{C}Q(t) = V(t)$

For $I(t)$: $\quad L\,I''(t) + R\,I'(t) + \frac{1}{C}I(t) = V'(t)$

if no inductor, or if no capacitor, then Kirchoff's Law yields 1^{st} order linear DE's, as below:

Exercise 2: Consider the $R - L$ circuit below, in which a switch is thrown at time $t = 0$. Assume the voltage V is constant, and $I(0) = 0$. Find $I(t)$. Interpret your results.



http://www.intmath.com/differential-equations/5-rl-circuits.php

Kirchoff:

$$R I + L I'(t) = V$$

$$L I' + R I = V$$

$\div L$ $\quad \begin{cases} I' + \frac{R}{L} I = \frac{V}{L} \\ I(0) = 0 \end{cases}$ $\quad I(t)$

$$\begin{cases} x'(t) + a\,x(t) = b \\ x(0) = x_0 \end{cases}$$

$$x(t) = \frac{b}{a} + \left(x_0 - \frac{b}{a}\right) e^{-at}$$

$a = \frac{R}{L}$ $\quad \frac{b}{a} = \frac{V}{R}$

$b = \frac{V}{L}$ $\quad x_0 = 0$

$$I(t) = \frac{V}{R} + \left(0 - \frac{V}{R}\right) e^{-\frac{R}{L}t}$$

$$\boxed{I(t) = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)}$$

$$\lim_{t \to \infty} I(t) = \frac{V}{R}$$