Math 2250-004
Fri Jan 19
- 1.5: linear differential equations

Announcements:

- one more example of linear DE
- input-output modeling (important application)

Warm-up Exercise: Let $a, b$ be constants. Solve this IVP for $x(t)$.
Note: The DE is linear (& separable).

$$\text{IVP}\ \begin{cases} x'(t) + \underbrace{a}\, x(t) = \underbrace{b} \\ x(0) = x_0 \end{cases}$$

① $\int P(t)\,dt = \int a\,dt = at$

$e^{at}(x' + ax) = be^{at}$

② $\frac{d}{dt}(e^{at}x) = be^{at}$

③ $e^{at}x(t) = \int be^{at}\,dt$

$e^{at}x = \frac{b}{a}e^{at} + C$

④ $x = \dfrac{\frac{b}{a}e^{at} + C}{e^{at}}$

$$\boxed{x = \frac{b}{a} + Ce^{-at}}$$

⑤ $x(0) = x_0 = \frac{b}{a} + C \Rightarrow C = x_0 - \frac{b}{a}$

$$\boxed{x(t) = \frac{b}{a} + \left(x_0 - \frac{b}{a}\right)e^{-at}}$$

$x'(t) + \underbrace{P(t)}\,x(t) = \underbrace{Q(t)}$

① $e^{\int P(t)dt}(x' + Px) = e^{\int P(t)dt}Q$

② $\frac{d}{dt}\left(e^{\int P(t)dt}x\right) = e^{\int P(t)dt}Q(t) := g(t)$

③ $e^{\int P(t)dt}x(t) = \int g(t)\,dt + C$

④ solve for $x(t)$

⑤ solve for $C$, for IVP

includes

- $b = 0$: exp growth/decay
- also Newton's law of cooling
- also, the easiest input/output

Algorithm for solving linear DEs is a method to use the differentiation product rule backwards:

$$y' + P(x)y = Q(x)$$

Let $\int P(x)\,dx$ be any antiderivative of $P$. Multiply both sides of the DE by its exponential to yield an equivalent DE:

$$② \qquad e^{\int P(x)dx}(y' + P(x)y) = e^{\int P(x)dx}Q(x)$$

because the left side is a derivative (product rule):

$$① \qquad \frac{d}{dx}\left(e^{\int P(x)dx}y\right) = e^{\int P(x)dx}Q(x)\ .$$

So you can antidifferentiate both sides with respect to $x$:

$$e^{\int P(x)dx}y = \int e^{\int P(x)dx}Q(x)\,dx \qquad + C.$$

Dividing by the positive function $e^{\int P(x)dx}$ yields a formula for $y(x)$. Notice, if you look carefully at this formula for the solution, that if $P(x)$, $Q(x)$ are defined and continuous on any interval *I*, then the resulting formula for $y(x)$ can be used to find the solution to any IVP with initial point in that interval, defined on the entire interval. This is in contrast to what can happen with separable differential equations.

$$① \quad \frac{d}{dx}\left(e^{\int P(x)dx}y(x)\right) = e^{\int P(x)dx}P(x)y + e^{\int P(x)dx}y'$$

$$(fg)' = f'g + fg' \qquad = e^{\int P(x)dx}(P(x)y + y') \quad ②$$

Exercise 3: Find all solutions to the linear (and also separable) DE

$$y'(x) = \frac{6x - 3xy}{x^2 + 1}$$

$y'(x) + P(x)y = Q(x)$

Hint: as you can verify with Wolfram alpha, the general solution is $y(x) = 2 + C\left(x^2+1\right)^{-\frac{3}{2}}$. If we have time we'll work this problem once using the linear DE algorithm, and once using the separable DE algorithm, for practice. Applications Monday!

$$y'(x) + \underbrace{\frac{3x}{x^2+1}}_{P(x)} y(x) = \frac{6x}{x^2+1}$$

$$\int P(x)\,dx = \int \frac{3x}{x^2+1}\,dx$$

$u = x^2+1$
$du = 2x\,dx$
$\frac{du}{2} = x\,dx$

$$= \int \frac{\frac{3}{2}\,du}{u}$$

$$= \frac{3}{2}\ln u = \frac{3}{2}\ln(x^2+1)$$

$$e^{\int P(x)dx} = e^{\frac{3}{2}\ln(x^2+1)} = \left(e^{\ln(x^2+1)}\right)^{3/2} = (x^2+1)^{3/2} \quad \text{I.F.}$$

① $e^{\int P(x)dx}\left(y' + Py\right) = e^{\int P(x)dx} Q$

② $\frac{d}{dx}\left(e^{\int P(x)dx} y\right) = e^{\int P(x)} Q(x)$

③ $e^{\int P(x)dx} y = \int \quad dx$

① $(x^2+1)^{3/2}\left(y' + \frac{3x}{x^2+1} y\right) = \frac{6x}{x^2+1}(x^2+1)^{3/2}$

② $\frac{d}{dx}\left[(x^2+1)^{3/2} y\right] = 6x(x^2+1)^{1/2}$

$$(x^2+1)^{3/2} y = \int 6x(x^2+1)^{1/2}\,dx = \int 3u^{1/2}\,du = 3\cdot\frac{2}{3}u^{3/2} + C$$

$u = x^2+1$
$du = 2x\,dx$

$$= 2(x^2+1)^{3/2} + C$$

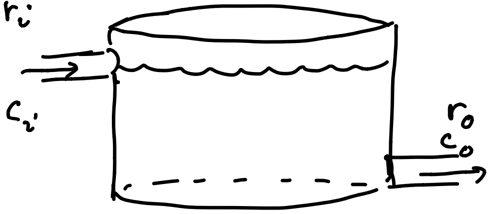
③ $(x^2+1)^{3/2} y = 2(x^2+1)^{3/2} + C$

④ $\boxed{y = 2 + C(x^2+1)^{-3/2}}$

# Input-output modeling

volume rate incoming $\frac{\text{Vol}}{\text{time}}$ $r_i$

solute concentration incoming $\frac{\text{mass}}{\text{volume}}$ $c_i$



$r_o$ $\frac{\text{Vol}}{\text{time}}$ rate out

$c_o$ concentration out $\frac{\text{mass}}{\text{volume}}$

- $V(t)$ = volume , $V(0) = V_0$
- $x(t)$ = solute mass , $x(0) = x_0$

Volume: short time interval $\Delta t$

estimate $\Delta V \cong r_i \, \Delta t - r_o \, \Delta t$

$\frac{\text{Vol}}{\text{time}} \cdot \text{time}$ ✓

$$\frac{\Delta V}{\Delta t} \cong r_i - r_o$$

$$\Rightarrow \begin{cases} V'(t) = r_i(t) - r_o(t) \\ V(0) = V_0 \end{cases}$$

§ 1.2 DE ; just antidiff for $V(t)$

$x(t)$ : short time interval $\Delta t$

estimate $\Delta x \cong \Delta x_{in} + \Delta x_{out} + \cancel{\Delta x}_{\text{created}}$ in our tank

but it is there in retirement act.

$\Delta x_{in} \cong (\Delta V_{in}) c_i$

$\Delta x_{out} \cong -\Delta V_{out} c_o$

$\Delta x \cong r_i \, \Delta t \, c_i - r_o \, \Delta t \, c_o$

$\frac{\cancel{\text{vol}}}{\cancel{\text{time}}} \cdot \cancel{\text{time}} \cdot \frac{\text{mass}}{\cancel{\text{vol}}}$ ✓

$$\frac{\Delta x}{\Delta t} \cong r_i c_i - r_o c_o$$

$$\Rightarrow \quad x'(t) = r_i c_i - r_o \underbrace{c_o}$$

well-mix assumption

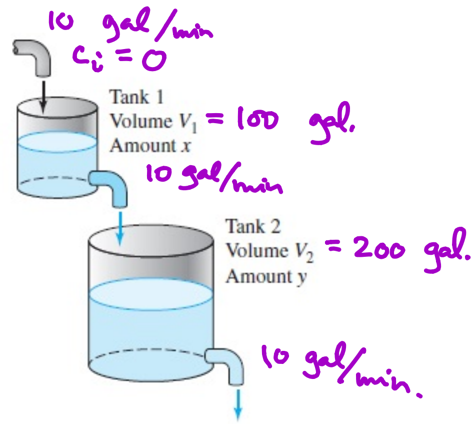
$c_o$ = average concentration $= \frac{x(t)}{V(t)}$

$$x'(t) = r_i c_i - r_o \frac{x(t)}{V(t)}$$

$$\boxed{\begin{aligned} & x' + \frac{r_o}{V} x = r_i c_i \\ & x(0) = x_0 \end{aligned}}$$

linear DE.

4. **Mixture Problem:** Consider the *cascade of* two tanks shown in the following figure with $V_1 = 100$ (gal) and $V_2 = 200$ (gal) the volumes of brine in the two tanks. Each tank also initially contains 50 lb of salt. The three flow rates indicated in the figure are each 10 gal/min, with pure water flowing into tank 1.




A cascade of two tanks.

(a) Find the amount $x(t)$ of salt in tank 1 at time $t$.

(b) Suppose that $y(t)$ is the amount of salt in tank 2 at time $t$. Show first that

$$\frac{dy}{dt} = \frac{x}{10} - \frac{y}{20}$$

and then solve for $y(t)$, using the function $x(t)$ found in part (a).

(c) Finally, find the maximum amount of salt ever in tank 2.

(a) $x(0) = 50$ lb.

$$x'(t) = r_i c_i - r_o c_o$$

$$x'(t) = 0 - 10\,\frac{x(t)}{100} = -.1\,x(t)$$

$$\begin{cases} x'(t) = -.1\,x(t) \\ x(0) = 50 \end{cases}$$

$$x(t) = 50e^{-.1t}$$

(exp. decay)