Math 2250-004
Wed Jan 17

- 1.4: separable DEs, with applications.

Announcements:

- Hand in HW in folders at front - alphabetize by last name
- Quiz today ☺
- for labs: Office hours Jose Yanez M,W 12:00-1:00 WEB 1622
- do part of Friday's notes today §1.5 Dihan Dai T 2-3, W: 5:00-6:00 JWB 332

Warm-up Exercise: Find the solution to the exponential growth/decay model for a quantity $\boxed{x(t):}$

$$\begin{cases} x'(t) = -kx \\ x(0) = x_0 \end{cases} \qquad (k>0)$$

'til 10:50

$$\frac{dx}{dt} = -kx$$

$$\frac{1}{x}\,dx = -k\,dt \qquad (x \neq 0) \leftarrow \text{if } x(t) \equiv 0\text{, that's a soln too}$$

$$\int \frac{1}{x}\,dx = \int -k\,dt$$

$$e^{\ln|x|} = e^{-kt+C}$$

$$|x| = e^{-kt+C} = e^{C}e^{-kt}$$

$$x = Ae^{-kt} \qquad (A = e^{C}, \text{ or } -e^{C})$$

$$x(t) = Ae^{-kt} \qquad (A \in \mathbb{R})$$

IVP $\quad x(0) = x_0 = A\cdot e^{0} = A$

$$\boxed{x(t) = x_0 e^{-kt}}$$

For your section 1.4 homework in the upcoming week I've assigned a selection of separable DE's - some applications will be familiar, e.g. exponential growth/decay and Newton's Law of cooling. As an example of these sorts of applications:

Exercise 1) When relatively small amounts of sugar (or any other solute) are dissolving in continuously mixed water, the amount A that remains undissolved after t minutes is well-modeled by a differential equation

$$\frac{dA}{dt} = -k\,A.$$

(Because how fast the sugar dissolves is proportional to the amount that remains undissolved.) Suppose 25 % of the sugar has dissolved after 1 minute.

1a) How long does it take for half of the sugar to dissolve? (This is the "half life" of the undissolved sugar.)

from warm-up,

$$A(t) = A_0 e^{-kt}$$

use to find k

$$A(1) = .75\cancel{A_0} = \cancel{A_0}e^{-k\cdot 1}$$

t in minutes

$$.75 = e^{-k}$$

$$\ln(.75) = -k,$$

$$k = -\ln(.75) \approxeq .288$$

$$\boxed{A(t) = A_0 e^{-.288t}}$$

1a) Question: $A(t) = \frac{1}{2}A_0$ solve for t

half life

$$\tfrac{1}{2}\cancel{A_0} = \cancel{A_0}e^{-kt}$$

$$\ln(.5) = -kt \Rightarrow \boxed{t = \frac{\ln(.5)}{-k} = \frac{\ln 2}{k}}$$

$$\boxed{\approxeq 2.41 \text{ min}}$$

1b) How long does it take until 90 % of the sugar has dissolved? Solve $A(t) = .1A_0$ for t

Section 1.5, linear differential equations:

A first order linear DE for $y(x)$ is one that can be written as

$$y' + P(x)y = Q(x) \qquad y = y(x)$$

$$x' + P(t)x = Q(t) \qquad x = x(t)$$

y' & y appear in a sum, with power 1

Exercise 1: Classify the differential equations below as linear, separable, both, or neither. Justify your answers.

a) $y'(x) = -2y + 4x^2$ linear: $\underset{P(x)}{y' + 2y} = \underset{Q(x)}{4x^2}$ not separable RHS not $f(x)g(y)$.

b) $y'(x) = x - y^2 + 1$ not linear not separable

c) $y'(x) = x^2 - x^2 y + 1$ linear $y' + \underbrace{x^2}_{P(x)} y = \underbrace{x^2+1}_{Q(x)}$ not sep

d) $y'(x) = \dfrac{6x - 3xy}{x^2 + 1}$ linear $y' + \dfrac{3x}{x^2+1} y = \dfrac{6x}{x^2+1}$ separable $\dfrac{dy}{dx} = \dfrac{x}{x^2+1}(6-3y)$!

e) $y'(x) = x^2 + y^2$ not linear not separable

f) $y'(x) = x^2 e^{x^3}$. linear $y' + \underbrace{0}_{P(x)}y = \underbrace{x^2 e^{x^3}}_{Q(x)}$ separable $y' = (x^2 e^{x^3})(1)$ (§1.2 antidiff)

g) for $x(t)$ $x'(t) = kx$ $x'(t) + P(t)x = Q(t)$

$x' \underbrace{-k}_{P(t)} x = \underbrace{0}_{Q(t)}$ linear also separable

Algorithm for solving linear DEs is a method to use the differentiation product rule backwards:

$$y' + P(x)y = Q(x)$$

Let $\int P(x)\,dx$ be any antiderivative of P. Multiply both sides of the DE by its exponential to yield an equivalent DE:

$$② \qquad e^{\int P(x)dx}(y' + P(x)y) = e^{\int P(x)dx}Q(x)$$

because the left side is a derivative (product rule):

$$① \qquad \frac{d}{dx}\left(e^{\int P(x)dx}y\right) = e^{\int P(x)dx}Q(x)\ .$$

So you can antidifferentiate both sides with respect to x:

$$e^{\int P(x)dx}y = \int e^{\int P(x)dx}Q(x)\,dx \qquad + C.$$

Dividing by the positive function $e^{\int P(x)dx}$ yields a formula for $y(x)$. Notice, if you look carefully at this formula for the solution, that if $P(x)$, $Q(x)$ are defined and continuous on any interval *I*, then the resulting formula for $y(x)$ can be used to find the solution to any IVP with initial point in that interval, defined on the entire interval. This is in contrast to what can happen with separable differential equations.

$$① \quad \frac{d}{dx}\left(e^{\int P(x)dx}y(x)\right) = e^{\int P(x)dx}P(x)y + e^{\int P(x)dx}y'$$

$$(fg)' = f'g + fg' \qquad\qquad = e^{\int P(x)dx}(P(x)y + y')$$

②

Exercise 2: Find all solutions to the differential equation

$$y'(x) = -2y + 4x^2,$$

and compare your solutions to the Wolfram alpha check.

$$y' + \underbrace{2y}_{P(x)} = \underbrace{\cancel{4x^2}}_{Q(x)}\ 4$$

$$e^{\int P(x)dx} = e^{2x} \quad \text{I.F.}$$

$$e^{2x}(y' + 2y) = e^{2x}4$$

$$\frac{d}{dx}\left(e^{2x}y(x)\right) = 4e^{2x}$$

$$\frac{d}{dx}\left(e^{2x}y\right) = 2e^{2x}y + e^{2x}y' \quad ✓$$

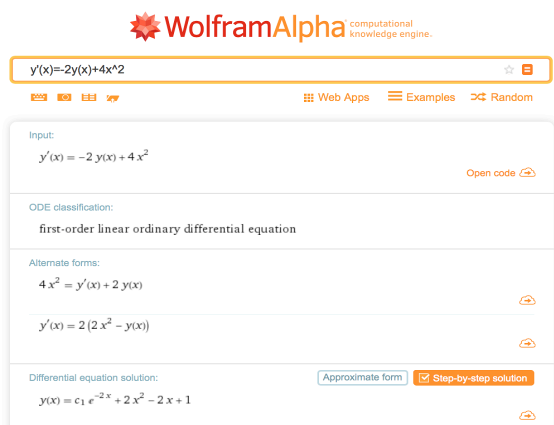
$$e^{2x}y = \int 4e^{2x}\,dx$$

$$e^{2x}y = 2e^{2x} + C$$

$\div e^{2x}$:

$$\boxed{y = 2 + Ce^{-2x}}$$

$$\left(\frac{2e^{2x} + C}{e^{2x}}\right)$$



practice 1.5 1, 7, 13

before lab