Math 2250-004 Week 2 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we will cover. These notes cover material in 1.3-1.5. They include some material from last Friday's notes that we did not get to then.

Tues Jan 16
- 1.3 slope fields and existence-uniqueness theorem for initial value problems
- 1.4 separable differential equations

Announcements:

- HW 1 due tomorrow, we'll cover remaining concepts from §1.3 today at start of class; stapled.
- Quiz tomorrow (1.3 & 1.4)
- Prof. K Office hours in JWB 240 downstairs 4:30-6:00 p.m.
  Dihai Dai T: 2-3 JWB 332.
- CANVAS

Warm-up Exercise:

a) Find all solutions $y(x)$ to $\frac{dy}{dx} = 2x(y-1)^2$
b) Solve the IVP, with $y(0)=1$

a) separable. $\frac{dy}{(y-1)^2} = 2x\,dx$ for $y \neq 1$. Note $y(x) \equiv 1$ is a solution, since for this $y(x)$, $y' = 0 = 2x(y-1)^2$.

$$\int \frac{dy}{(y-1)^2} = \int 2x\,dx$$

$$-\frac{1}{y-1} = x^2 + C$$

$$-\frac{1}{x^2+C} = y-1$$

$$\boxed{y = 1 - \frac{1}{x^2+C}} \quad \& \quad \boxed{y = 1}$$

$\boxed{\text{b) IVP with } y(0)=1\text{: the solution is } y(x) \equiv 1.}$

(the other soln's won't work: $y(0) = 1 - \frac{1}{C} \neq 1$ for any C)

....

Recall that on Friday we were talking about our intuition that initial value problems for first order differential equations always have unique solutions. At the end of class though, we'd just run into an example where this "fact" failed. Let's review and finish that example, and then discuss the existence-uniqueness theorem that gives conditions on the "slope function" for a first order DE that do guarantee the existence and uniqueness of solutions, as long as the graph stays inside an appropriately-sized rectangle containing the initial point.

Exercise 2a) Use separation of variables to solve the IVP

$$\begin{cases} \dfrac{dy}{dx} = y^{\left(\frac{2}{3}\right)} \\ y(0) = 0 \end{cases}$$

2b) But there are actually a lot more solutions to this IVP! (Solutions which don't arise from the separation of variables algorithm are called singular solutions.) Once we find these solutions, we can figure out why separation of variables missed them.

2c) Sketch some of these singular solutions onto the slope field below.

2 a) $\dfrac{dy}{dx} = y^{2/3}$

$y \neq 0$ $\dfrac{dy}{y^{2/3}} = 1 \cdot dx$

$\displaystyle\int \frac{dy}{y^{2/3}} = \int dx$

$\displaystyle\int y^{-2/3}\,dy$

$3y^{1/3} = x + C$

$y^{1/3} = \dfrac{x+c}{3}$

$\boxed{y = \left(\dfrac{x+c}{3}\right)^3}$

$y(0) = 0 = \left(\dfrac{c}{3}\right)^3 \Rightarrow c = 0$

$y(x) = \left(\dfrac{x}{3}\right)^3 = \dfrac{x^3}{27}$

$y(3) = 1$

2b) $y(x) \equiv 0$

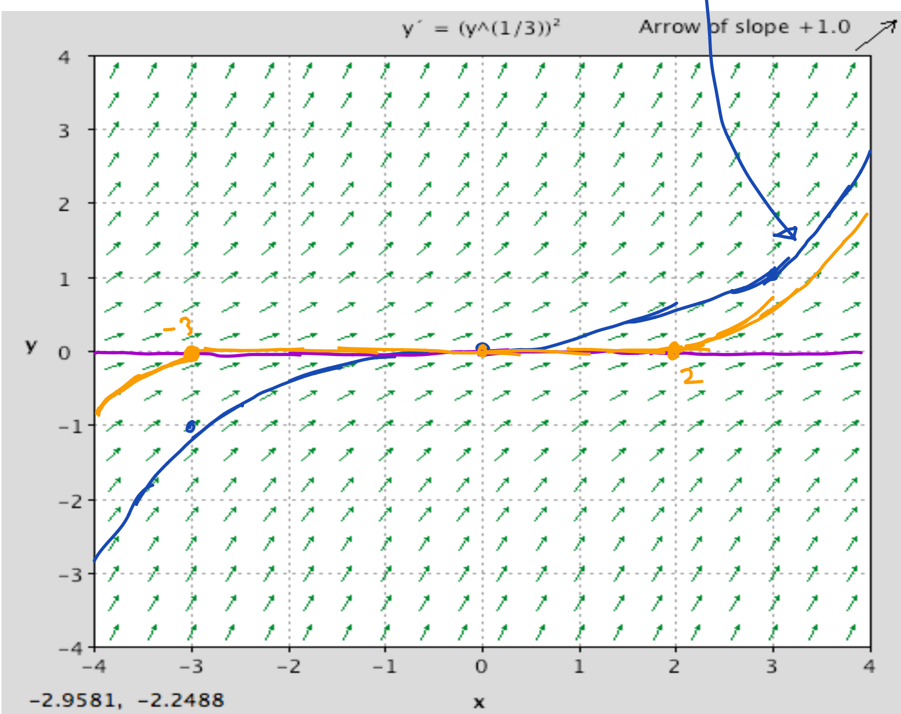
LHS $y' = 0$

RHS $y^{2/3} = 0$

2c) ∞'ly many soltns!!

$$y(x) = \begin{cases} \left(\dfrac{x+3}{3}\right)^3 & x \leq -3 \\ 0 & -3 \leq x \leq 2 \\ \left(\dfrac{x-2}{3}\right)^3 & x \geq 2 \end{cases}$$

(-3, 2 were arbitrary!)



Here's what's going on (stated in 1.3 page 24 of text; partly proven in Appendix A.)

Existence - uniqueness theorem for the initial value problem

Consider the IVP

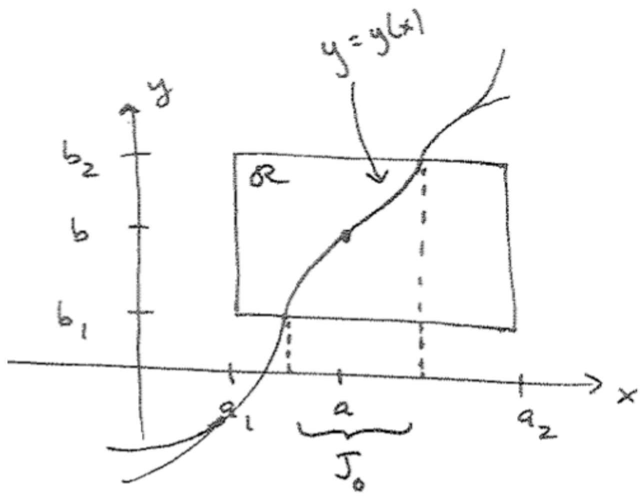
$$\frac{dy}{dx} = f(x, y)$$
$$y(a) = b$$

• Let the point (a, b) be interior to a coordinate rectangle $\mathcal{R}: a_1 \leq x \leq a_2, b_1 \leq y \leq b_2$ in the $x-y$ plane.

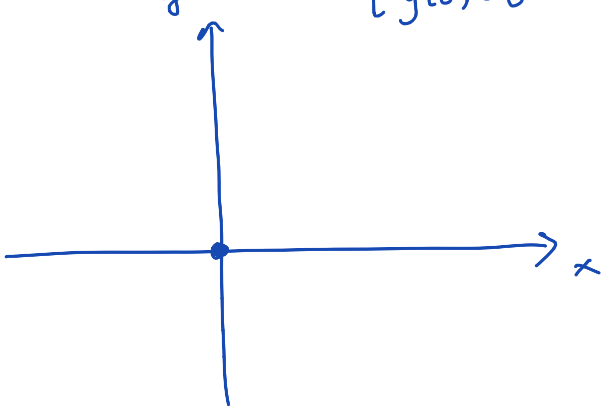
• Existence: If $f(x, y)$ is continuous in $\mathcal{R}$ (i.e. if two points in $\mathcal{R}$ are close enough, then the values of f at those two points are as close as we want). Then there exists a solution to the IVP, defined on some subinterval $J \subseteq [a_1, a_2]$.

• Uniqueness: If the partial derivative function $\frac{\partial}{\partial y} f(x, y)$ is also continuous in $\mathcal{R}$, then for any subinterval $a \in J_0 \subseteq J$ of x values for which the graph $y = y(x)$ lies in the rectangle, the solution is unique!

See figure below. The intuition for existence is that if the slope field $f(x, y)$ is continuous, one can follow it from the initial point to reconstruct the graph. The condition on the y-partial derivative of $f(x, y)$ turns out to prevent multiple graphs from being able to peel off.



Exercise 0 : Revisit $\begin{cases} \frac{dy}{dx} = y^{2/3} \\ y(0) = 0 \end{cases}$ on previous page



(1) $f(x,y) = y^{2/3}$ is continuous on $\mathbb{R}^2$, so solutions exist

(2) $\frac{\partial f}{\partial y} = \frac{2}{3} y^{-1/3} = \frac{2}{3} \frac{1}{\sqrt[3]{y}}$ is continuous except along the x-axis, $y=0$. In particular, it's not continuous at $(0,0)$, so we can't find any rectangle $\mathcal{R}$

with (0,0) in its interior, so that $\frac{\partial f}{\partial y}$ is continuous in R.
So can't conclude solutions are unique

Exercise 1 In the following examples that we've discussed earlier, check to see that our results are consistent with the existence-uniqueness theorem.

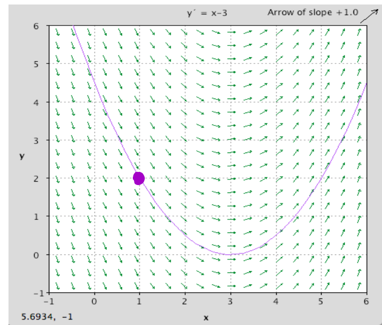
1a) discussed Friday:

$$\frac{dy}{dx} = x - 3$$
$$y(1) = 2\,.$$

We found the solution $y(x) = \frac{x^2}{2} - 3\,x + \frac{9}{2} = \frac{(x-3)^2}{2}$ via antidifferentiation. Is this consistent with the existence-uniqueness theorem?

$\exists$? ① $f(x,y) = x-3$ is continuous on $\mathbb{R}^2$
existence ② $\frac{\partial f}{\partial y} = 0$ is also cont.

① soltns exist
② there is only 1



1b) discussed Friday: On Wednesday's quiz you solved

$$\frac{dy}{dx} = y - x$$
$$y(0) = 0$$

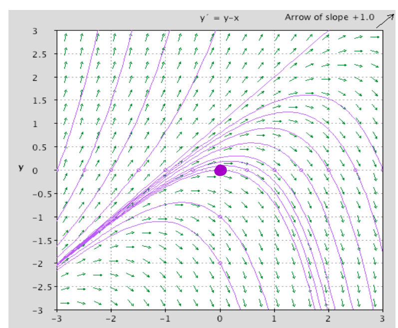
from a family of solutions that was given to you, and found a solution

$$y(x) = x + 1 - e^x\,.$$

Is this the only possible solution? Hint: use the existence-uniqueness theorem.

① $f(x,y) = y - x$ cont on $\mathbb{R}^2$
② $\frac{\partial f}{\partial y} = 1$ cont on $\mathbb{R}^2$

① soltns to IVP exist
② they're unique



1c) discussed Friday. We showed that the general solutions $y(x)$ to

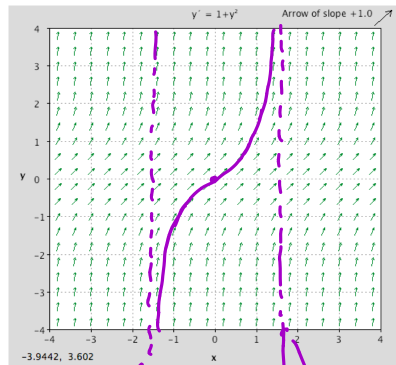
$$\frac{dy}{dx} = y^2 + 1$$

are

$$y(x) = \tan(x + C).$$

So for the initial value problem with $y(0) = 0$, the solution is $y(x) = \tan(x)$. Is this consistent with the existence-uniqueness theorem? What is the largest interval on which the solution exists (as a differentiable function)?

$y(x) = \tan x$



$-\pi/2$ $\quad$ $\pi/2$

① $f(x,y) = y^2 + 1$ is cont on $\mathbb{R}^2$

② $\frac{\partial f}{\partial y} = 2y$ is cont on $\mathbb{R}^2$

① sol'ns exist

② unique $\rightarrow$ as long as the graph stays in $\mathbb{R}^2$

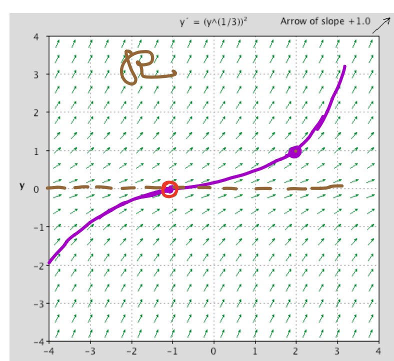
i.e. $-\pi/2 < x < \pi/2$

1d) Using the differential equation for $y(x)$, from Friday's cliff-hanger, but with a different initial condition:

$$\frac{dy}{dx} = y^{2/3}$$

$$y(2) = 1 .$$

Find the largest interval on which this IVP has a unique solution. (Recall, that for the differential equation we have solutions $y(x) \equiv 0$, and $y(x) = \left(\frac{x + C}{3}\right)^3$, and differentiable piecewise-defined solutions made out of these.)



① $f(x,y) = y^{2/3}$ cont on $\mathbb{R}^2$

sol exists.

$y(x) = \left(\frac{x+C}{3}\right)^3$

$y(2) = 1 = \left(\frac{2+C}{3}\right)^3 \Rightarrow C = 1$

$y(x) = \left(\frac{x+1}{3}\right)^3$ is a sol'n.

② $\frac{\partial f}{\partial y} = \frac{2}{3} y^{-1/3}$ not cont along $y = 0$ (x-axis)

$= \frac{2}{3} \frac{1}{\sqrt[3]{y}}$

so until graph leaves $\mathcal{R}$ sol'n is unique

$\boxed{\text{Let } \mathcal{R} \text{ be the region } -\infty < x < \infty,\ y > 0}$

Most first order differential equation solutions don't have "elementary" expressions, even when the existence-uniqueness theorem guarantees that the solutions exist. In these cases one uses "numerical solutions", generated by algorithms that use the slope field in sophisticated ways. Our text discusses some of these algorithms in sections 2.4-2.6, which we will cover during week 4.

Exercise 2: Do the initial value problems below have unique solutions? Can you find them? (Notice these are NOT separable differential equations.) Can Wolfram alpha find formulas for the solutions?

a)

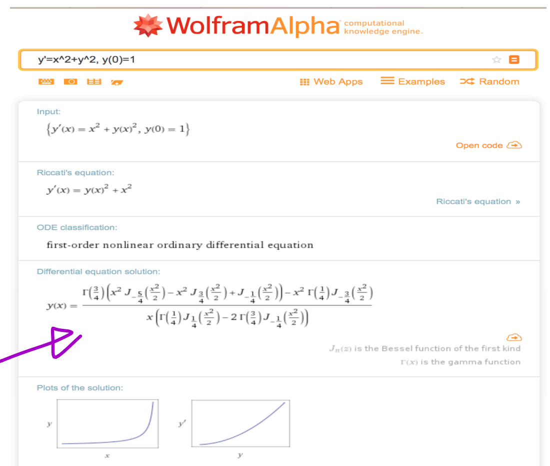
$$y' = x^2 + y^2$$
$$y(0) = 1$$

① $f(x,y)$ cont in $\mathbb{R}^2$

② $\frac{\partial f}{\partial y} = 2y$ cont in $\mathbb{R}^2$

IVP has a unique sol'n

amazingly complicated solution formula



b)

$$y' = x^4 + y^4$$
$$y(0) = 0$$

① $f(x,y) = x^4 + y^4$ cont in $\mathbb{R}^2$

② $\frac{\partial f}{\partial y} = 4y^3$ also cont. in $\mathbb{R}^2$

$\Rightarrow$ IVP has a unique sol'n

but there is no formula for the exact solution!