Math 2250-004
Week 13-14 concepts and homework
sections 6.1-6.2, 7.1-7.3 Due Wednesday April 18 at the start of class

Chapter 6.1-6.2:

6.1: finding eigenvalues and eigenvectors (or a basis for the eigenspace if the eigenspace is more than one-dimensional).
6.1: 7, 17, 19, 25

6.2: diagonalizability: recognizing matrices for which there is an eigenbasis, and those for which there is not one. Writing the eigenbasis equation as AP = PD where P is the matrix of eigenvector columns, and D is the diagonal matrix of eigenvalues.
6.2: 3, 9, 21, 23.

w13.1) Find eigenvalues and eigenvectors (or a basis for the eigenspace if the eigenspace is more than one-dimensional), for the following matrices. You can check your work with technology, but you don't have to hand in the technology check.

a) $A := \begin{bmatrix} 2 & 9 & 3 \\ -2 & -5 & 0 \\ 2 & 6 & 1 \end{bmatrix}$

b) $E := \begin{bmatrix} 1 & 6 & 6 \\ 0 & -1 & -2 \\ 0 & 4 & 5 \end{bmatrix}$

c) $F := \begin{bmatrix} 5 & 3 & -9 \\ -4 & -5 & 4 \\ 4 & 2 & -7 \end{bmatrix}$. In this problem you may use technology to compute and factor the characteristic polynomial, and to find the eigenspace bases. Be careful with what the technology is telling you.

w13.2)
a) Which of the matrices in 13.1 are diagonalizable and which are not?
b) What happens to the diagonal matrix D of eigenvalues for a diagonalizable matrix, if you change the order of the eigenvector columns in *P*, in the identity $A\,P = P\,D$. Explain.

7.1: modeling coupled mass-spring systems or multi-component input-output systems with systems of differential equations;converting single differential equations or systems of differential equations into equivalent first order systems of differential equations by introducing functions for the intermediate derivatives; comparing solutions to these equivalent systems.

7.1: 1, 3, **2**, 5, **8**, 11,12, 21, **24**, **25**, **26**.

w13.3) This is related to problems11, 21 ideas above.

a) Consider the IVP for the first order system of differential equations for $x(t), v(t)$:

$$x'(t) = v$$
$$v'(t) = -4x$$
$$x(0) = x_0.$$
$$v(0) = v_0 .$$

Find the equivalent second order differential equation initial value problem for the function $x(t)$.

b) Use Chapter 5 techniques to solve the second order IVP for $x(t)$ in part **a**.

c) Use your result from **b** to deduce the solutions $[x(t), v(t)]^T$ for the IVP in **a.**

d) Use your solution formulas for $x(t), v(t)$ from **c** along with algebra and trig identities to verify that the parametric solution curves $[x(t), v(t)]^T$ to the IVP in **a** lie on ellipses in the $x - v$ plane, satisfying implicit equations for ellipses given by

$$x^2 + \frac{v^2}{4} = C, \text{ where } C = x_0^2 + \frac{v_0^2}{4}.$$

e) Reproduce the result of **d** without using the solution formulas, but instead by showing that whenever $x'(t) = v$ and $v'(t) = -4x$ then it must be true also that

$$\frac{d}{dt}\left(x(t)^2 + \frac{v(t)^2}{4}\right) \equiv 0 ,$$

so that $x(t)^2 + \frac{v(t)^2}{4}$ must be constant for any solution trajectory. Hint: use the chain rule to compute the time derivative above, then use the DE's in **a** to show that the terms cancel out.

f) Your result in **e** is connected conservation of energy for the undamped harmonic oscillator with mass $m = 1$ and spring constant $k = 4$, because the total energy for a moving undamped mass-spring configuration with mass m and spring constant k is

$$TE = KE + PE = \frac{1}{2}m v^2 + \frac{1}{2}k x^2$$

(Recall, the work done to stretch the spring from equilibrium to position x is $W = \int_0^x F(s)\, ds = \int_0^x ks\, ds = \frac{1}{2}k x^2$, which is where the PE term comes from.) Verify that the expression in **e** which has zero time derivative is just a multiple of the total energy.

g) Use pplane or other technology to create a picture of the tangent field for the first order system of differential equations in this problem, as well as selected solution trajectories. This "phase plane" picture should be consistent with your work above. Print out a screen shot to hand in. Each solution trajectory follows a path on which the total energy is constant.

7.2) recognizing homogeneous and non-homogeneous linear systems of first order differential equations; writing these systems in vector-matrix form; statement of existence and uniqueness for IVP's in first order systems of DE's and its consequences for the dimension of the solution space to the first order system, and for the general solution to the non-homogeneous system. Using the Wronskian to check for bases.
7.2: **1**, 9, **12**, 13, **14**, **23**.

w13.4) This is a continuation of **14**, **23**
a) Use the eigenvalue-eigenvector method of section 7.3 (see text as well as class notes Friday April 13, Tuesday April 17, and the April 12 lab) to generate the basis for the general solution that the text told you in **7.2.14**.
b) Use pplane or other technology to draw the phase portrait for this first order system along with the parametric curve of the solution $[x(t), y(t)]^T$ to the initial value problem in **23**. Print out a screen shot of your work to hand in.

These problems are postponed until the next (and last) homework assignment:

7.3) the eigenvalue-eigenvector method for finding the solution space to homogeneous constant coefficient first order systems of differential equations: real and complex eigenvalues.
7.3: 3, 13, 29, 31, **34**, 36. *In* **34** *you may use technology to find the eigendata to save time, or if you want practice working by hand, just use technology to check your answer.*

w13.5) Use the eigenvalue-eigenvector method (with complex eigenvalues) to solve the first order system initial value problem which is equivalent to the second order differential equation IVP on the Monday April 16 notes, This is the reverse procedure from Monday, where we use the solutions from the equivalent second order DE IVP to deduce the solution to the first order system IVP. Of course, your answer here should agree with our work there!

$$\begin{bmatrix} x'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$
$$\begin{bmatrix} x(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$