3.4 Matrix algebra

Matrix vector algebra that we've already touched on, but that we want to record carefully:

Vector addition and scalar multiplication:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ \vdots \\ x_n + y_n \end{bmatrix} ; \qquad c \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} c\,x_1 \\ c\,x_2 \\ c\,x_3 \\ \vdots \\ c\,x_n \end{bmatrix}$$

Vector dot product , which yields a scalar (i.e. number) output (regardless of whether vectors are column vectors or row vectors):

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} := x_1\, y_1 + x_2\, y_2 + \ldots + x_n\, y_n .$$

Matrix times vector: If A is an $m \times n$ matrix and $\underline{\boldsymbol{x}}$ is an n column vector, then

$$A\underline{\boldsymbol{x}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1\,n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2\,n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{m\,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1\,n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2\,n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{m\,n}x_n \end{bmatrix} = \begin{bmatrix} Row_1(A) \cdot \underline{\boldsymbol{x}} \\ Row_2(A) \cdot \underline{\boldsymbol{x}} \\ \vdots \\ Row_m(A) \cdot \underline{\boldsymbol{x}} \end{bmatrix}$$

Compact way to write our usual linear system:

$$\underbrace{A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}}\, .$$

Exercise 1a) Compute   done!

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 2 & 1 & -2 \\ -5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}.$$

\# of columns in A ("n"), = \#rows in B ("n")

Matrix times matrix: Let $\boxed{A_{m \times n}, B_{n \times p}}$ be two matrices such that the number of columns of A equals the number of rows of B. ~~Then the product AB~~ is an $m \times p$ matrix, with

- $$col_j(AB) = A\, col_j(B).$$

In other words, you just compute matrix times vector, for each column of B, to get the corresponding column of the product AB. So, the resulting matrix will have as many columns as B and as many rows as A.

Exercise 1b) Compute   did in warm-up!

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 2 & 1 & -2 \\ -5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -4 & 1 \\ 0 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ -8 & -3 \\ -7 & 2 \end{bmatrix}$$

* In warmup Friday, we just computed

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 2 & 1 & -2 \\ -5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -7 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 2 & 1 & -2 \\ -5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 2 \end{bmatrix}$$

Summary of different ways to think of the matrix product $A\,B$:

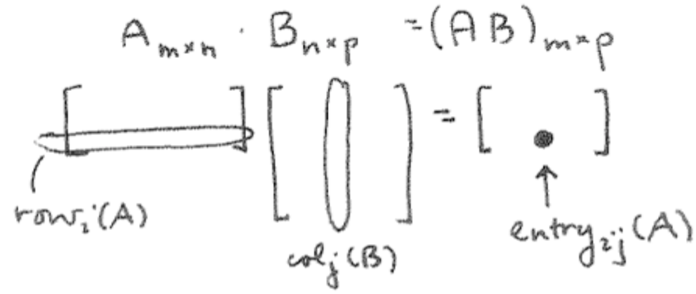
- The j^{th} column of AB is given by A times the j^{th} column of B

$$col_j(AB) = A\,col_j(B)$$

- The i^{th} entry in the j^{th} column of $A\,B$, i.e. $entry_{i\,j}(AB)$ is the dot product of the i^{th} row of A with the j^{th} column of B:

$$entry_{i\,j}(AB) := row_i(A) \bullet col_j(B) = \sum_{k=1}^{n} a_{i\,k} b_{k\,j}\,.$$

This stencil might help:



More matrix operations:

- addition and scalar multiplication: Let $A_{m \times n}$, $B_{m \times n}$ be two matrices of the same dimensions (m rows and n columns). Let $entry_{ij}(A) = a_{ij}$, $entry_{ij}(B) = b_{ij}$. (In this case we write $A = [a_{ij}]$, $B = [b_{ij}]$.) Let c be a scalar. Then

$$entry_{ij}(A + B) := a_{ij} + b_{ij}.$$
$$entry_{ij}(c\,A) := c\,a_{ij}.$$

In other words, addition and scalar multiplication are defined analogously as for vectors. In fact, for these two operations you can just think of matrices as vectors written in a rectangular rather than row or column format.

Exercise 3) Let $A := \begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 0 & 3 \end{bmatrix}$ and $B := \begin{bmatrix} 0 & 27 \\ 5 & -1 \\ -1 & 1 \end{bmatrix}$. Compute $4\,A - B$.   3×2 matrices.

$$4A = \begin{bmatrix} 4 & -8 \\ 12 & -4 \\ 0 & 12 \end{bmatrix} \qquad -B = \begin{bmatrix} 0 & -27 \\ -5 & 1 \\ 1 & -1 \end{bmatrix}$$

$$4A - B = 4A + (-B) = \begin{bmatrix} 4 & -35 \\ 7 & -3 \\ 1 & 11 \end{bmatrix}$$

try to get so that you can write down 4A−B entry by entry, in one step.

Properties for the algebra of matrix addition and multiplication :

- Multiplication is not commutative in general (AB usually does not equal BA, even if you're multiplying square matrices so that at least the product matrices are the same size).

size: $A_{m\times n} B_{n\times m} = (AB)_{m\times m}$

$B_{n\times m} A_{m\times n} = (BA)_{n\times n}$

not even same size, unless $m=n$ (i.e. square matrices), even for square matrices, almost never true that $AB=BA$

$$\begin{bmatrix} 2 & 6 \\ 1 & 7 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 0 & 0 \end{bmatrix}$$

But other properties you're used to do hold:

- $+$ is commutative $A + B = B + A$

ij entry: $a_{ij} + b_{ij} = b_{ij} + a_{ij}$

- $+$ is associative $(A + B) + C = A + (B + C)$

ij entry on both sides $a_{ij} + b_{ij} + c_{ij}$

- scalar multiplication distributes over $+$ $c(A + B) = cA + cB$.

ij entry on both sides: $c(a_{ij} + b_{ij}) = ca_{ij} + cb_{ij}$

$$c\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = c\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} c4 \\ c6 \end{bmatrix}$$

$$c\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} c \\ 2c \end{bmatrix} + \begin{bmatrix} 3c \\ 4c \end{bmatrix} = \begin{bmatrix} 4c \\ 6c \end{bmatrix}$$

- multiplication is associative $(AB)C = A(BC)$.

* magic.

you can do this by brute force also, see HW for example.

- matrix multiplication distributes over $+$ $A(B + C) = AB + AC$; $(A + B)C = AC + BC$

e.g. $$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\left(\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_1 + c_1 \\ b_2 + c_2 \end{bmatrix} = \begin{bmatrix} a_{11}(b_1+c_1) + a_{12}(b_2+c_2) \\ a_{21}(b_1+c_1) + a_{22}(b_2+c_2) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} a_{11}b_1 + a_{12}b_2 \\ a_{21}b_1 + a_{22}b_2 \end{bmatrix}}_{b\text{ terms}} + \underbrace{\begin{bmatrix} a_{11}c_1 + a_{12}c_2 \\ a_{21}c_1 + a_{22}c_2 \end{bmatrix}}_{c\text{ terms}}$$

$= A\vec{b} + A\vec{c}$

- If A is an $m \times n$ matrix, and we use the letter I for identity matrices, then $I_{m\times m}A_{m\times n} = A$ and $A_{m\times n}I_{n\times n} = A$.

$I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ...

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 2 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 2 & 0 \\ 1 & -1 \end{bmatrix};$$

$$\begin{bmatrix} 3 & 3 \\ 2 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 2 & 0 \\ 1 & -1 \end{bmatrix}$$

Math 2250-004

Fri Feb 9
- 3.4 Matrix algebra
- 3.5 Matrix inverses

Announcements:
- finish Wed notes on matrix algebra
- start 3.5 on matrix inverses
  ... to be continued on Monday

exam thru §3.5

* what's it all good for?

Warm-up Exercise: Compute

'til 10:47

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 2 & 1 & -2 \\ -5 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -7 \end{bmatrix}$$

$3\cdot 2 + 2\cdot(-4) + 1\cdot 0 - 2\cdot 3$
$= 6 - 8 - 6 = -8$

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 2 & 1 & -2 \\ -5 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 2 \end{bmatrix}$$

$-5\cdot 0 + 0\cdot 1 + 0\cdot(-1) + 1\cdot 2$
$= 2$

so!

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 2 & 1 & -2 \\ -5 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -4 & 1 \\ 0 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ -8 & -3 \\ -7 & 2 \end{bmatrix}$$

$$col_j(AB) = A\, col_j(B)$$

$$A_{3\times 4} B_{4\times 2} = (AB)_{3\times 2}$$

this is how we define matrix multiplication, column by column.

We've been talking about matrix algebra: addition, scalar multiplication, multiplication, and how these operations combine. If necessary, finish those notes.

But I haven't told you what all that algebra is good for. Today we'll start to find out. By way of comparison, think of a scalar linear equation with known numbers a, b, c, d and an single unknown number x,

$$a\,x + b = c\,x + d$$

We know how to solve it by collecting terms and doing scalar algebra:

$$a\,x - c\,x = d - b$$

$$(a - c)\,x = d - b \qquad *$$

$$x = \frac{d - b}{a - c}\,.$$

How would you solve such an equation if A, B, C, D were square matrices, and X was a vector (or matrix) ? Well, you could use the matrix algebra properties we've been discussing to get to the $*$ step. And then if X was a vector you could solve the system * with Gaussian elimination. In fact, if X was a matrix, you could solve for each column of X (and do it all at once) with Gaussian elimination.

But you couldn't proceed as with scalars and do the final step after the * because it is not possible to divide by a matrix. Today we'll talk about a potential shortcut for that last step that is an analog of of dividing, in order to solve for X . It involves the concept of *inverse matrices.*

$$AX + B = CX + D$$

$$-CX \;\; = -CX$$
$$-B \qquad -B$$

$$AX - CX = D - B$$

$$(A - C)\,X = D - B.$$

can't divide by a matrix

$$\cancel{X = \frac{D-B}{A-C}}$$

Matrix inverses: A square matrix $A_{n \times n}$ is invertible if there is a matrix $B_{n \times n}$ so that

$$AB = BA = I,$$

where I is the $n \times n$ identity matrix. In this case we call B the inverse of A, and write $B = A^{-1}$.

Remark: A matrix A can have at most one inverse, because if we have two candidates B, C with

$$AB = BA = I \quad \text{and also} \quad AC = CA = I$$

then

$$(BA)C = IC = C$$
$$B(AC) = BI = B$$

so since the associative property $(BA)C = B(AC)$ is true, it must be that

$$B = C.$$

Exercise 1a) Verify that for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ the inverse matrix is $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$BA = I$ too!

Inverse matrices are very useful in solving algebra problems. For example

Theorem: If A^{-1} exists then the only solution to $\boxed{A\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}}$ is $\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$.

$\Rightarrow$
$$A\vec{x} = \vec{b}$$
$$A^{-1}(A\vec{x}) = A^{-1}\vec{b}$$
$$(A^{-1}A)\vec{x} = A^{-1}\vec{b}$$
$$I\vec{x} = A^{-1}\vec{b}$$
$$\boxed{\vec{x} = A^{-1}\vec{b}}$$

Exercise 1b) Use the theorem and A^{-1} in 1a, to write down the solution to the system

$$x + 2y = 5$$
$$3x + 4y = 6$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 9/2 \end{bmatrix}.$$

check $x + 2y = -4 + 9 = 5$ ✓
$3x + 4y = -12 + 18 = 6$ ✓