Wed Feb 28
4.3 - 4.4. linear independence, bases and dimension for vector spaces.

Announcements:

- quiz today
- go to a piece of Friday's notes, where we treat functions as vectors (& relate to DE's!!)
- then continue in notes from Tuesday.

Warm-up Exercise:
'til 10:47

Recall, a sub (vector) space of a vector space is closed under addition and scalar multiplication.

True / False (explain).

T ① If $\begin{bmatrix}1\\2\end{bmatrix}$ is in a subspace W of $\mathbb{R}^2$, then so is $\begin{bmatrix}-5\\-10\end{bmatrix}$.
because $\begin{bmatrix}-5\\-10\end{bmatrix} = -5\begin{bmatrix}1\\2\end{bmatrix}$

T ② If $\begin{bmatrix}1\\2\end{bmatrix}$ is in the subspace W then so is $\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$
$= \left\{c_1\begin{bmatrix}1\\2\end{bmatrix}, c_1 \in \mathbb{R}\right\}$

T ③ If $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\0\end{bmatrix}$ are in the subspace W, then W must be all of $\mathbb{R}^2$. !

closed under + & scalar multiplication

$\Longrightarrow$ closed under all linear combinations.

(because $c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}1\\0\end{bmatrix}$ is built out of taking scalar multiples and then adding the results)

In fact the only subspaces of $\mathbb{R}^2$ are

(0) $\{\vec{0}\}$ is closed under + & scalar.

(1) $\text{span}\{\vec{u}\}$ $\vec{u} \neq \vec{0}$ i.e. line thru $\vec{0}$

(2) $\text{span}\{\vec{u}, \vec{v}\}$ $\vec{u}, \vec{v}$ independent : all of $\mathbb{R}^2$.

(from Friday's notes)

Exercise 0) In Chapter 5 we focus on the vector space

- $V = C(\mathbb{R}) := \{f: \mathbb{R} \rightarrow \mathbb{R} \ s.t. \ f \text{ is a continuous function}\}$

and its subspaces. Verify that the vector space axioms for linear combinations are satisfied for this space of functions. Recall that the function $f + g$ is defined by $(f + g)(x) := f(x) + g(x)$ and the scalar multiple $c f(x)$ is defined by $(c f)(x) := c f(x)$. What is the zero vector for functions?

$(f+g)' = f' + g'$

Definition: A vector space is a collection of objects together with and "addition" operation "+", and a scalar multiplication operation, so that the rules below all hold.

(α) Whenever $f, g \in V$ then $f + g \in V$. (closure with respect to addition)

(β) Whenever $f \in V$ and $c \in \mathbb{R}$, then $c \cdot f \in V$. (closure with respect to scalar multiplication)

As well as:

(a) $f + g = g + f$ (commutative property)

(b) $f + (g + h) = (f + g) + h$ (associative property)

(c) $\exists \ 0 \in V$ so that $f + 0 = f$ is always true. the zero fcn "0": $0(x) = 0$ for all x

(d) $\forall \ f \in V \ \exists -f \in V$ so that $f + (-f) = 0$ (additive inverses)

(e) $c \cdot (f + g) = c \cdot f + c \cdot g$ (scalar multiplication distributes over vector addition)

(f) $(c_1 + c_2) \cdot f = c_1 \cdot f + c_2 \cdot f$ (scalar addition distributes over scalar multiplication)

(g) $c_1 \cdot (c_2 \cdot f) = (c_1 c_2) \cdot f$ (associative property)

(h) $1 \cdot f = f$, $(-1) \cdot f = -f$, $0 \cdot f = 0$ (these last two actually follow from the others).

(From Friday's notes)

Because the vector space axioms are exactly the arithmetic rules we used to work with linear combination equations, all of the concepts and vector space theorems we talked about for $\mathbb{R}^m$ and its subspaces make sense for the function vector space V and its subspaces. In particular we can talk about

- the span of a finite collection of functions $f_1, f_2, \ldots f_n$, $span\{f_1, f_2, \ldots f_n\}$. $= \{c_1 f_1 + c_2 f_2 + \ldots + c_n f_n$ such that $c_1, c_2, \ldots c_n \in \mathbb{R}\}$

- linear independence/dependence for a collection of functions $\{f_1, f_2, \ldots f_n\}$.

  independence: if $c_1 f_1 + c_2 f_2 + \ldots + c_n f_n \equiv 0$ then $c_1 = c_2 = \ldots c_n = 0$

  is identically equal

  0 fcn, i.e. 0 for all x

- subspaces of V

  satisfy α)

  β)

- bases and dimension for finite dimensional subspaces. (The function space V in Exercise 0 itself is infinite dimensional, meaning that no finite collection of functions spans it.)

(From Friday's notes, covered Wednesday)

Exercise 1 Consider the three functions with domain $\mathbb{R}$, given by

$$f_1(x) = 1, \; f_2(x) = x, \; f_3(x) = x^2.$$

1a) Describe $span\{f_1, f_2, f_3\}$. $= \{ c_1 f_1 + c_2 f_2 + c_3 f_3 : c_1, c_2, c_3 \in \mathbb{R} \}$

so at x, we get $c_1 \cdot 1 + c_2 \cdot x + c_3 \cdot x^2$

so span is set of polys of degree ≤ 2.

1b) Is the set $\{f_1, f_2, f_3\}$ linearly dependent or linearly independent?

dependency eqtn $c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$

at each x: If $c_1 \cdot 1 + c_2 \cdot x + c_3 x^2 = 0$ for all x

does $c_1 = c_2 = c_3 = 0$?

@ $x = 0$: $c_1 + c_2 \cdot 0 + c_3 \cdot 0 = 0 \Rightarrow c_1 = 0$

@ $x = 1$: $\cancel{c_1} + c_2 + c_3 = 0$

@ $x = -1$: $\cancel{c_1} - c_2 + c_3 = 0$

$E_2 + E_3 \Rightarrow c_3 = 0$

$E_2 - E_3 \Rightarrow c_2 = 0$

OR

$c_1 + c_2 x + c_3 x^2 \equiv 0 \qquad \Rightarrow c_1 = 0$

D_x: $c_2 + 2x c_3 \equiv D_x 0 \equiv 0 \qquad \Rightarrow c_2 = 0$

D_x: $2 c_3 \equiv 0 \qquad \Rightarrow c_3 = 0$

so f_1, f_2, f_3 are linearly independent.

Consider the 3rd order ("linear") differential eqtn for $y(x)$

$$y''' = 0$$

find all soltns: antidiff: $\Rightarrow y'' = a$ a const

$\Rightarrow y' = ax + b$ b const

$\Rightarrow y(x) = \frac{1}{2} a x^2 + bx + c$

$= c_1 \cdot 1 + c_2 \cdot x + c_3 \cdot x^2$

solution space to $y''' = 0$

is span$\{1, x, x^2\}$ = span$\{f_1, f_2, f_3\}$

In fact $\{f_1, f_2, f_3\}$ is a basis for space of solutions to $y''' = 0$

Reviewed Wednesday, from Tuesday's notes:

We've been talking about vector spaces and subspaces, with examples in $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^n$.

<u>Key facts about how subspaces (sub vector spaces) DO arise:</u>

There are two main ways that subspaces arise: (These ideas will be important when we return to differential equations, in Chapter 5, although it's probably difficult to envision what they have to do with differential equations right now.)

1) $W = span\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ is always a subspace. ← Explicit way to describe a vector space

Expressing a subspace this way is an <u>explicit</u> way to describe the subspace W, because you are "listing" all of the vectors in it.

Why $W = span\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ is a subspace: Let $\underline{v}, \underline{w} \in W$. In other words, we can express

$$\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_n\underline{v}_n$$
$$\underline{w} = d_1\underline{v}_1 + d_2\underline{v}_2 + \dots + d_n\underline{v}_n .$$

So,

$$\underline{v} + \underline{w} = \left(c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_n\underline{v}_n\right) + \left(d_1\underline{v}_1 + d_2\underline{v}_2 + \dots + d_n\underline{v}_n\right).$$

After using the vector space axioms (addition is commuative and associative, and scalar addition distributes over scalar multiplication), we can rewrite

$$\underline{v} + \underline{w} = \left(c_1 + d_1\right)\underline{v}_1 + \left(c_2 + d_2\right)\underline{v}_2 + \dots + \left(c_n + d_n\right)\underline{v}_n \in W .$$

This verifies (α). closed under +.

Now let $c \in \mathbb{R}$. Then

$$c\,\underline{v} = c\left(c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_n\underline{v}_n\right) = cc_1\underline{v}_1 + cc_2\underline{v}_2 + \dots + cc_n\underline{v}_n \in W$$

which verifies (β). closed under scalar multiplication

(And, notice that $0\,\underline{v} = \underline{\mathbf{0}} \in W$.)

<u>Example:</u> From yesterday's discussion, an example subspace in $\mathbb{R}^3$:

$$W = span\left\{\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}\right\}$$

is a 2-dimensional subspace, i.e. plane through the origin.

Implicit way to describe a subspace

The other way subspaces arise (in $\mathbb{R}^n$) :

$\boxed{\text{2) Let } A \text{ be an } m \times n \text{ matrix. Let } V = \{\underline{\mathbf{x}} \in \mathbb{R}^n \textit{ such that } A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}\}. \text{ Then } V \text{ is a subspace. (We call this collection of vectors the "homogeneous solution space" or "null space" of } A.}$

Note that this is an <u>implicit way</u> to describe the subspace V because we're only specifying a homogeneous matrix equation that the vectors in V must satisfy, but you're not saying what the vectors are.

Why V is a subspace: Let $\underline{\mathbf{v}}, \underline{\mathbf{w}} \in V \Rightarrow$

- $A\underline{\mathbf{v}} = \underline{\mathbf{0}},\ A\underline{\mathbf{w}} = \underline{0} \quad \Rightarrow \quad A(\underline{\mathbf{v}} + \underline{\mathbf{w}}) = A\underline{\mathbf{v}} + A\underline{\mathbf{w}} = \underline{\mathbf{0}} + \underline{\mathbf{0}} = \underline{\mathbf{0}}, \quad \Rightarrow \quad \underline{\mathbf{v}} + \underline{\mathbf{w}} \in V$ (verifies α)

and let $c \in \mathbb{R} \Rightarrow$

- $A\underline{\mathbf{v}} = \underline{0} \quad \Rightarrow \quad A(c\,\underline{\mathbf{v}}) = c\,A\,\underline{\mathbf{v}} = c\,\underline{\mathbf{0}} = \underline{\mathbf{0}}, \quad \Rightarrow \quad c\,\underline{\mathbf{v}} \in V$ (verfies β).

(and $\underline{\mathbf{0}} \in V$, since $A\,\underline{\mathbf{0}} = \underline{\mathbf{0}}$.)

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<u>Example:</u> Continuing the example from the previous page, the plane

$$W = span\left\{ \underline{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \underline{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}$$

could have been described implicitly as the collection of position vectors for points (x, y, z) satisfying the very small homogeneous matrix equation

$$0\,x + 2\,y + z = 0.$$

$$\begin{bmatrix} 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ x \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}.$$