Math 2250-004 Week 8 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes include material from 4.2-4.4 and an introduction to Chapter 5.

Mon Feb 26
4.3, 4.2 linear combinations and independence of vectors, vector subspaces of $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^n$.

span of sets of vectors

Announcements: continue 4.1 - 4.3

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Warm-up Exercise: Recall that the set of vectors $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_p\}$ is linearly dependent if

$$(*) \quad c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p = \vec{0}$$

holds where not all of $c_1, c_2, \dots c_p$ are equal to zero.
Why do you know that

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -18 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 42 \\ 813 \\ \pi \end{bmatrix} \right\}$$ pi

must be dependent? Hint: what's the augmented matrix for the dependency eqtn $(*)$ in this case?

$$* \quad c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -18 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 42 \\ 813 \\ \pi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -18 & 3 & 42 \\ 2 & 0 & 2 & 813 \\ 3 & 1 & 1 & \pi \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{cccc|c} 1 & -18 & 3 & 42 & 0 \\ 2 & 0 & 2 & 813 & 0 \\ 3 & 1 & 1 & \pi & 0 \end{array} \xrightarrow{\text{reduce}} \text{3 rows} \left\{ \underbrace{\qquad\qquad}_{\text{4 cols}} \right. \begin{array}{|c} 0 \\ 0 \\ 0 \end{array}$$

at most 3 pivots
at least 1 free parameter, so, lots of solns $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$

We've been discussing ideas related to linear combinations of vectors. After the weekend, we should review, to recall these concepts:

A *linear combination* of the vectors in the set $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ is

any vector $\underline{\boldsymbol{v}}$ that is a sum of scalar multiples of those vectors,

i.e. any $\underline{\boldsymbol{v}}$ expressible as

$$\underline{\boldsymbol{v}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n.$$

The *span* of the vectors in the set $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ is

the collection of all possible linear combinations:

- *span* $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\} := \{\underline{\boldsymbol{v}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n$ such that each $c_i \in \mathbb{R}, i = 1, 2, \ldots n\}$

The vectors in the set $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ are *linearly dependent* means

it is possible to satisfy $c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n = \underline{\boldsymbol{0}}$ *with not all of the weights* $c_1, c_2, \ldots c_n = 0$.

(Equivalently, at least one $\underline{\boldsymbol{v}}_j$ can be expressed as a linear combination of some of the other vectors in the collection.)

The vectors in the set $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ are *linearly independent* means

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n = 0 \quad \Rightarrow \quad c_1 = c_2 = \ldots = c_n = 0.$$

(Equivalently, no $\underline{\boldsymbol{v}}_j$ can be expressed as a linear combination of some of the other vectors in the collection.)

NEW:

<u>Definition</u>: Let $V = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ \dots \ \underline{\boldsymbol{v}}_n\}$. If the vectors in the set $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ \dots \ \underline{\boldsymbol{v}}_n\}$ are linearly independent, then we say that they are a *basis* for V. And, we say that the *dimension* of V is n.

"standard basis"

$$= \{\hat{i}, \hat{j}, \hat{k}\} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$$

<u>Example</u> The set of vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ spans $\mathbb{R}^3$ and is linearly independent, so is a *basis* for $\mathbb{R}^3$. The *dimension* of $\mathbb{R}^3$ is 3!

Let's review what we were doing at the end of class on Friday, in light of these new definitions...

span?
$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
so the three vectors span $\mathbb{R}^3$

independent?
$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \begin{matrix} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{matrix}$$
yes, indep.

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span, basis, dimension

$\begin{bmatrix}1\\0\\0\end{bmatrix}$ $\begin{bmatrix}0\\-1\\2\end{bmatrix}$

Wed example, finished on Friday

Exercise 3) Consider the two vectors $\underline{v}_1 = [1, 0, 0]^T$, $\underline{v}_2 = [0, -1, 2]^T \in \mathbb{R}^3$.

3a) Sketch these two vectors as position vectors in $\mathbb{R}^3$, using the axes below.

$= \{c\vec{v}_1,\ c \in \mathbb{R}\} = \left\{c\begin{bmatrix}1\\0\\0\end{bmatrix},\ c \in \mathbb{R}\right\}$

3b) What geometric object is $span\{\underline{v}_1\}$? (Remember, we are identifying position vectors with their endpoints.) Sketch a portion of this object onto your picture below. Remember though, the "span" continues beyond whatever portion you can draw.

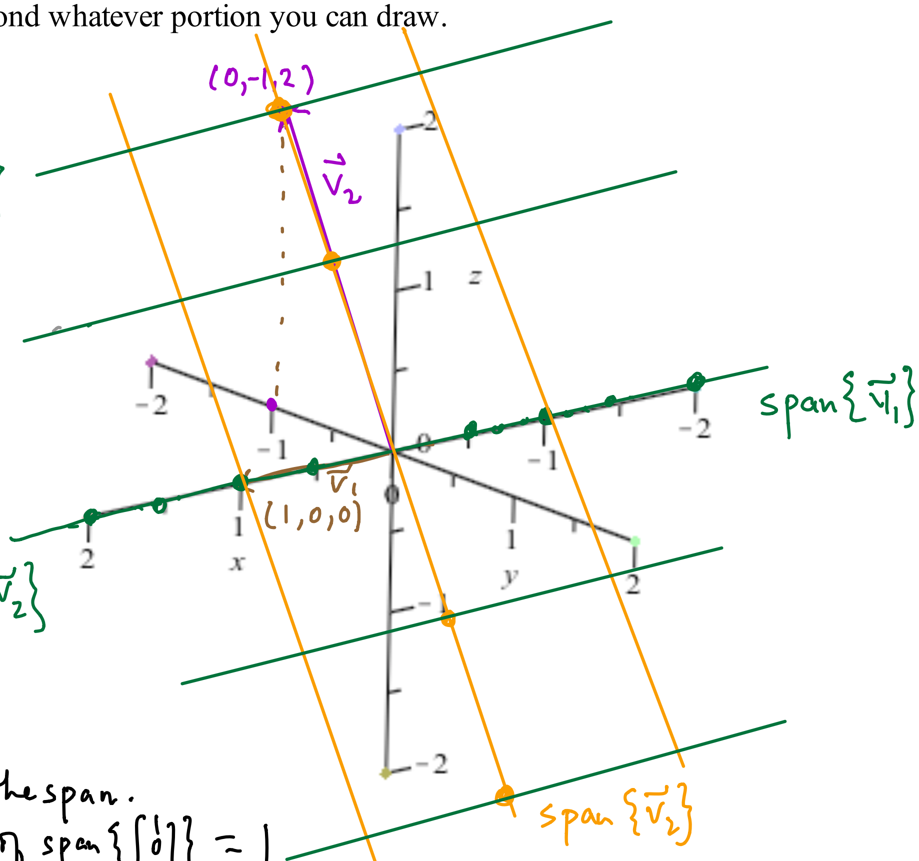
x-axis

3c) What geometric object is $span\{\underline{v}_1, \underline{v}_2\}$? Sketch a portion of this object onto your picture below. Remember though, the "span" continues beyond whatever portion you can draw.

plane

$\{c_1\vec{v}_1 + c_2\vec{v}_2 : c_1, c_2 \in \mathbb{R}\}$

partial grid for the plane that is $span\{\vec{v}_1, \vec{v}_2\}$



3b) $span\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$ = x-axis.

$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$ is a basis for the span.

so dimension of $span\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\} = 1$

notice by adding or scalar multiplying vectors in $span\{\vec{v}_1\}$, I can never leave this set $= span\{\vec{v}_1\}$

3c) $span\{\vec{v}_1, \vec{v}_2\}$ is plane thru $\vec{0}$.

$\{\vec{v}_1, \vec{v}_2\} = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\2\end{bmatrix}\right\}$. Since $\vec{v}_1$ & $\vec{v}_2$ are independent they're a basis for their span, and the dimension of this plane is 2!!

$span\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n : c_1, c_2, \ldots c_n \in \mathbb{R}\}$

= collection of all linear combinations

notice, $span\{\vec{v}_1, \vec{v}_2\}$ is also "closed" under vector addition & scalar multiplication

e.g. $(2\vec{v}_1 + 3\vec{v}_2) + (-\vec{v}_1 + 7\vec{v}_2) = \vec{v}_1 + 10\vec{v}_2$

$3(2\vec{v}_1 + 3\vec{v}_2) = 6\vec{v}_1 + 9\vec{v}_2$

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span, basis, dimension

you have a lab problem like this.

3d) What implicit equation must vectors $[b_1, b_2, b_3]^T$ satisfy in order to be in $span\{\underline{v}_1, \underline{v}_2\}$? Hint: For what $[b_1, b_2, b_3]^T$ can you solve the system

$$c_1\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{\vec{v}_1}+c_2\underbrace{\begin{bmatrix}0\\-1\\2\end{bmatrix}}_{\vec{v}_2}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

for c_1, c_2? Write this an augmented matrix problem and use row operations to reduce it, to see when you get a consistent system for c_1, c_2.

$$\left[\begin{array}{cc|c}1 & 0 & b_1\\0 & -1 & b_2\\0 & 2 & b_3\end{array}\right]\begin{array}{l}=x\\=y\\=z\end{array}$$

$$\begin{array}{r}\\-R_2\to R_2\\ \\ \end{array}\left[\begin{array}{cc|c}1 & 0 & b_1\\0 & 1 & -b_2\\0 & 2 & b_3\end{array}\right]$$

$$\begin{array}{r}\\ \\ -2R_2+R_3\to R_3\end{array}\left[\begin{array}{cc|c}1 & 0 & b_1\\0 & 1 & -b_2\\0 & 0 & 2b_2+b_3\end{array}\right]$$

Solutions c_1, c_2 if and only if

$$2b_2+b_3=0$$

i.e. $\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ must lie on the plane

$$2y+z=0$$

compare to picture!

eg the points $(0,0,0)$ in plane
$(1,0,0)$ in plane
$(0,-1,2)$ in plane.

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span, basis, dimension

5b) Show that the vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \underline{v}_3 = \begin{bmatrix} 3 \\ 4 \\ -8 \end{bmatrix}$$

are linearly dependent (even though no two of them are scalar multiples of each other). What does this mean geometrically about the span of these three vectors?
Hint: You might find this computation useful:

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & 2 & -8 \end{array}\right] \text{ reduces to } \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{array}\right]$$

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dependent: $\vec{v}_3 = 3\vec{v}_1 - 4\vec{v}_2$

$$\begin{bmatrix} 3 \\ 4 \\ -8 \end{bmatrix} \overset{?}{=} 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 4\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \checkmark$$

is $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{v}_3$?

$c_1 = 3$
$c_2 = -4$

$$c_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 3 \\ 4 \\ -8 \end{bmatrix} = \vec{0}$$

OR $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$

~~augmented~~ matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 2 & -8 & 0 \end{array}\right] \xrightarrow{\text{reduced}} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

∞'ly many soltn's

backsolve
$c_1 = -3t$
$c_2 = 4t$
$c_3 = t$

dependencies:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = -3t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4t\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 3 \\ 4 \\ -8 \end{bmatrix}$$

e.g. $t=1$
$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \\ -8 \end{bmatrix}$$

(same as orange eqtn)

$\{\vec{v}_1, \vec{v}_2, \vec{w}_3\}$ not a basis for the plane, because not independent

Exercise 6) Are the vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \underline{w}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

linearly independent? What is their span? Hint:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right] \text{ reduces to } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

independent: $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{w}_3 = \vec{0} \implies c_1 = 0, c_2 = 0, c_3 = 0$

span$\{\vec{v}_1, \vec{v}_2, \vec{w}_3\} = \mathbb{R}^3$:

$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{w}_3 = \vec{b} \in \mathbb{R}^3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & b_1 \\ 0 & -1 & 1 & b_2 \\ 2 & 2 & 1 & b_3 \end{array}\right] \xrightarrow{\text{reduce}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & d_1 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 1 & d_3 \end{array}\right]$$

$c_1 = d_1$
$c_2 = d_2$
$c_3 = d_3$

So $\{\vec{v}_1, \vec{v}_2, \vec{w}_3\}$ is alternate basis for $\mathbb{R}^3$

Exercise 1) Use properties of reduced row echelon form matrices to answer the following questions:

$= \{\vec{v}_1, \vec{v}_2, .. \vec{v}_p\}$ $p > 3$

1a) Why must more than 3 vectors in $\mathbb{R}^3$ always be linearly dependent?

$$c_1\vec{v}_1 + c_2\vec{v}_2 + .. + c_p\vec{v}_p = \vec{0}$$

has augmented matrix $3\left\{\underbrace{\left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \vec{v}_p\right]}_{p} \begin{matrix}0\\0\\0\end{matrix}\right.$ $\xrightarrow{\text{reduce}}$

we will get at least $p-3$ free parameters, (because at most 3 pivots) so lots of dependencies.

1b) Why can fewer than 3 vectors never span $\mathbb{R}^3$?
(So every basis of $\mathbb{R}^3$ must have exactly three vectors.)

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{b}$$

$$3\left[\begin{array}{cc|c} \underbrace{\quad}_{2} & & \begin{matrix}b_1\\b_2\\b_3\end{matrix}\end{array}\right] \xrightarrow{\text{reduce}} \left[\begin{array}{cc|c} & & d_1 \\ & & d_2 \\ 0 & 0 & d_3 \end{array}\right]$$

at most 2 pivots, so lots inconsistent systems if want arbitrary $\vec{b}$ to be in span $\{\vec{v}_1, \vec{v}_2\}$.
in the 1st 2 cols

1c) If you are given a set of 3 vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \underline{\boldsymbol{v}}_3\}$ in $\mathbb{R}^3$, what is the condition on the reduced row echelon form of the 3×3 matrix that has these vectors as columns, $[\underline{\boldsymbol{v}}_1\ \underline{\boldsymbol{v}}_2\ \underline{\boldsymbol{v}}_3]$ that guarantees they're linearly independent? That guarantees they span $\mathbb{R}^3$? That guarantees they're a basis of $\mathbb{R}^3$?

$$\text{rref}\left[\vec{v}_1 \middle| \vec{v}_2 \middle| \vec{v}_3\right] = I$$

guarantees independence & span.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \begin{bmatrix}0\\0\\0\end{bmatrix} \text{ or } \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

$$\left[\begin{array}{c|c|c|cc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}b_1\\b_2\\b_3\end{matrix}\end{array}\right] \xrightarrow{\text{reduce}} \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & d_1 \\ 0 & 1 & 0 & 0 & d_2 \\ 0 & 0 & 1 & 0 & d_3 \end{array}\right]$$

1d) What is the dimension of $\mathbb{R}^3$?

3

1e) How does this discussion generalize to $\mathbb{R}^n$?