Fri Feb 23
4.1-4.3 The vector spaces $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^n$.

<u>Announcements:</u> finish Thursday example. & today's notes.
We are investigating linear combinations, algebraically & geometrically.

<u>Warm-up Exercise</u>: write the <u>vector equation</u>
a.k.a. a linear combination equation
'til 10:47

$$* \quad x_1\begin{bmatrix}1\\-1\\2\end{bmatrix} + x_2\begin{bmatrix}1\\3\\0\end{bmatrix} + x_3\begin{bmatrix}0\\2\\7\end{bmatrix} = \begin{bmatrix}-1\\2\\-3\end{bmatrix}$$

as a matrix equation

$$A\vec{x} = \vec{b}.$$

(i.e. what is the matrix A?)

ans
$$\begin{bmatrix}1 & 1 & 0\\-1 & 3 & 2\\2 & 0 & 7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_2\end{bmatrix} = \begin{bmatrix}-1\\2\\-3\end{bmatrix}$$

long cut. rewrite * in steps

$$\begin{bmatrix}x_1\\-x_1\\2x_1\end{bmatrix} + \begin{bmatrix}x_2\\3x_2\\0\end{bmatrix} + \begin{bmatrix}0\\2x_3\\7x_3\end{bmatrix} = \begin{bmatrix}-1\\2\\-3\end{bmatrix}$$

$$\begin{bmatrix}x_1 + x_2 \\ -x_1 + 3x_2 + 2x_3 \\ 2x_1 + 7x_3\end{bmatrix} = \begin{bmatrix}-1\\2\\-3\end{bmatrix}$$

On Wednes~~dy~~day we interpreted linear combinations geometrically. And, we noticed that to answer natural questions we ended up using matrix theory from Chapter 3 This is because

<u>Exercise 1)</u> By carefully expanding the linear combination below, check than in $\mathbb{R}^m$, the linear combination

$$\cancel{c_1}\begin{bmatrix} c_1a_{11} \\ c_1a_{21} \\ \vdots \\ c_1a_{m1} \end{bmatrix} + \cancel{c_2}\begin{bmatrix} c_2a_{12} \\ c_2a_{22} \\ \vdots \\ c_2a_{m2} \end{bmatrix} + \ldots + \cancel{c_n}\begin{bmatrix} c_na_{1n} \\ c_na_{2n} \\ \vdots \\ c_na_{mn} \end{bmatrix} = \begin{bmatrix} c_1a_{11} + c_2a_{12} + \cdots + c_na_{1n} \\ c_1a_{21} + c_2a_{22} + \cdots + c_na_{2n} \\ \vdots \\ c_1a_{m1} + c_2a_{m2} + \cdots + c_na_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}c_1 + a_{12}c_2 + \cdots + a_{1n}c_n \\ a_{21}c_1 + a_{22}c_2 + \cdots + a_{2n}c_n \\ \vdots \\ a_{m1}c_1 + a_{m2}c_2 + \cdots + a_{mn}c_n \end{bmatrix}$$

is always just the matrix times vector product

$$= \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

vectors in the linear combo are the columns in the matrix. The "weights" $c_1 \ldots c_n$ are the vector $\vec{c}$.

<u>Thus linear combination problems in $\mathbb{R}^m$ can usually be answered using the linear system and matrix techniques we've just been studying in Chapter 3. This will be the main theme of Chapter 4.</u>

Wed example,
finished on Friday

$$\begin{bmatrix}1\\0\\0\end{bmatrix}$$

Exercise 3) Consider the two vectors $\underline{\boldsymbol{v}}_1 = [1, 0, 0]^T$, $\underline{\boldsymbol{v}}_2 = [0, -1, 2]^T \in \mathbb{R}^3$.

3a) Sketch these two vectors as position vectors in $\mathbb{R}^3$, using the axes below.

$= \{c\vec{v}_1,\ c \in \mathbb{R}\} = \left\{c\begin{bmatrix}1\\0\\0\end{bmatrix},\ c \in \mathbb{R}\right\}$

3b) What geometric object is $span\{\underline{\boldsymbol{v}}_1\}$? (Remember, we are identifying position vectors with their endpoints.) Sketch a portion of this object onto your picture below. Remember though, the "span" continues beyond whatever portion you can draw.

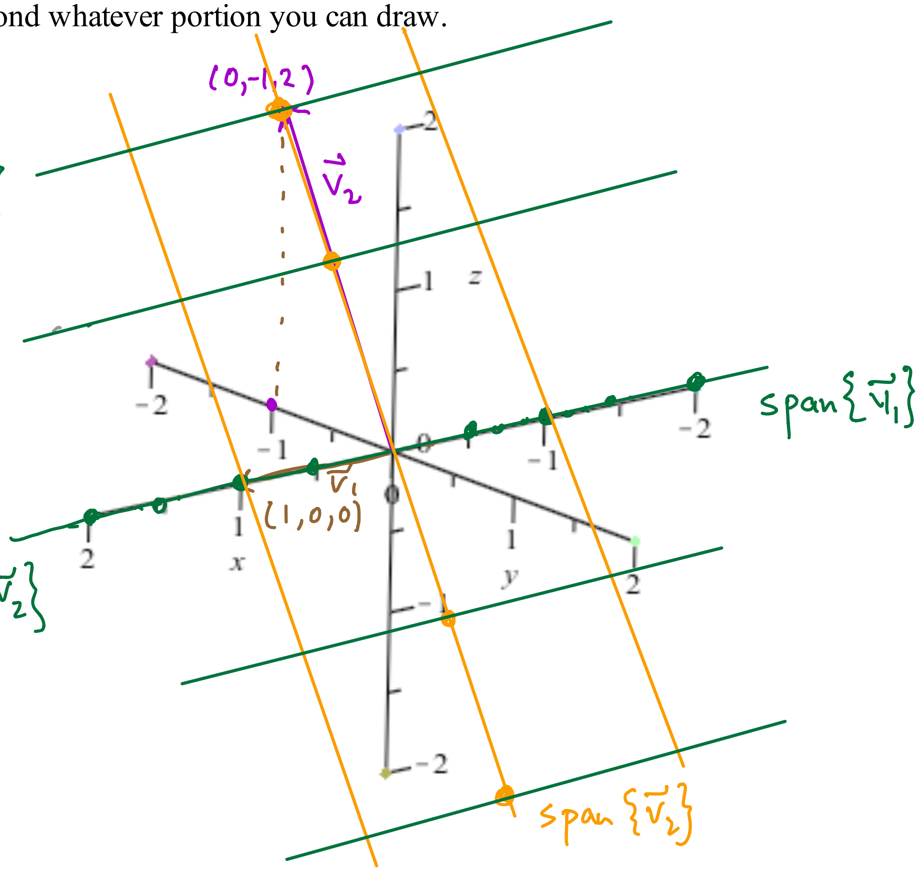
x-axis

3c) What geometric object is $span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2\}$? Sketch a portion of this object onto your picture below. Remember though, the "span" continues beyond whatever portion you can draw.

plane

$\{c_1\vec{v}_1 + c_2\vec{v}_2 : c_1, c_2 \in \mathbb{R}\}$

partial grid
for the plane
that is $span\{\vec{v}_1, \vec{v}_2\}$



$span\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n : c_1, c_2, \ldots c_n \in \mathbb{R}\}$

= collection of all linear combinations

you have a lab problem like this.

3d) What implicit equation must vectors $[b_1, b_2, b_3]^T$ satisfy in order to be in $span\{\underline{v}_1, \underline{v}_2\}$? Hint: For what $[b_1, b_2, b_3]^T$ can you solve the system

$$c_1\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{\vec{v}_1}+c_2\underbrace{\begin{bmatrix}0\\-1\\2\end{bmatrix}}_{\vec{v}_2}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

for c_1, c_2 ? Write this an augmented matrix problem and use row operations to reduce it, to see when you get a consistent system for c_1, c_2.

$$\left[\begin{array}{cc|c}1&0&b_1\\0&-1&b_2\\0&2&b_3\end{array}\right]\begin{array}{l}=x\\=y\\=z\end{array}$$

$$\begin{array}{r}\\-R_2\to R_2\\ \\\end{array}\left[\begin{array}{cc|c}1&0&b_1\\0&1&-b_2\\0&2&b_3\end{array}\right]$$

$$\begin{array}{r}\\ \\-2R_2+R_3\to R_3\end{array}\left[\begin{array}{cc|c}1&0&b_1\\0&1&-b_2\\0&0&2b_2+b_3\end{array}\right]$$

Solutions c_1, c_2 if and only if

$$2b_2+b_3=0$$

i.e. $\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ must lie on the plane

$$2y+z=0$$

compare to picture!

eg the points $(0,0,0)$ in plane

$(1,0,0)$ in plane

$(0,-1,2)$ in plane.

When we are discussing the span of a collection of vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ we would like to know that we are being efficient in describing this collection, and not wasting any free parameters because of redundancies. This has to do with the concept of "linear independence":

Definition: ($\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is linearly independent)

a) The vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly independent if no one of the vectors is a linear combination of (some) of the other vectors. The logically equivalent concise way to say this is that the only way $\underline{\mathbf{0}}$ can be expressed as a linear combination of these vectors, not (1)

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n = \underline{\mathbf{0}},$$

is for all the weights $c_1 = c_2 = \ldots = c_n = 0$. not (2)

b) $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly dependent if at least one of these vectors *is* a linear combination of (some) of the other vectors. (1) The concise way to say this is that there is some way to write $\underline{\mathbf{0}}$ as a linear combination of these vectors

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n = \underline{\mathbf{0}} \quad (2)$$

where not all of the $c_j = 0$. (We call such an equation a linear dependency. Note that if we have any such linear dependency, then any $\underline{\boldsymbol{v}}_j$ with $c_j \neq 0$ is a linear combination of the remaining $\underline{\boldsymbol{v}}_k$ with $k \neq j$. We say that such a $\underline{\boldsymbol{v}}_j$ is linearly dependent on the remaining $\underline{\boldsymbol{v}}_k$.)

start here: 1st way: Some $\vec{v}_j = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n$ (1)

(but no $\vec{v}_j$ term on right side)

$\Rightarrow \vec{0} = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n - \vec{v}_j$ (2)

i.e. some combo of vectors adds up to $\vec{0}$, where not all coeff's $= 0$.

conversely, if

(2) $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$

then I can solve for any $\vec{v}_j$ in terms of the others, if its coeff $c_j \neq 0$ (1)

Note: Two non-zero vectors are linearly independent precisely when they are not multiples of each other. For more than two vectors the situation is more complicated.

two vectors are dependent means at least one is a linear combo of the other,

i.e. $\vec{v}_1 = c\vec{v}_2$ (or $\vec{v}_2 = d\vec{v}_1$)

Example (Refer to Exercise 2 Wednesday):

The vectors $\underline{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \underline{v}_3 = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$ in $\mathbb{R}^2$ are linearly dependent because, as we showed on ~~Tuesday~~ Wednesday and as we can quickly recheck,

$$-3.5\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1.5\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}.$$

We can also write this linear dependency as

$$-3.5\underline{v}_1 + 1.5\underline{v}_2 - \underline{v}_3 = \underline{\mathbf{0}}$$

(or any non-zero multiple of that equation.)

Exercise 2) Are the vectors $\underbrace{\underline{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{\text{ind., not scalar mults}}$ linearly independent? How about $\underline{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\underline{v}_3 = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$?

Same for $\vec{v}_1$ & $\vec{v}_3$.

Exercise 3) For linearly independent vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$, every vector $\underline{v}$ in their span can be written as $\underline{v} = d_1\underline{v}_1 + d_2\underline{v}_2 + \ldots + d_n\underline{v}_n$ uniquely, i.e. for exactly one choice of linear combination coefficients $d_1, d_2, \ldots d_n$. This is not true if vectors are dependent. Explain these facts. (You can illustrate these facts with the vectors in Exercise 2.)

return to this!

Exercise 5) (Recall Exercise 3 in Wednesday's notes):
5a) Are the vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

linearly independent? yes

not scalar multiples... which is all I need to check with just two vectors.

5b) Show that the vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \ \underline{v}_3 = \begin{bmatrix} 3 \\ 4 \\ -8 \end{bmatrix}$$

are linearly dependent (even though no two of them are scalar multiples of each other). What does this mean geometrically about the span of these three vectors?
Hint: You might find this computation useful:

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & 2 & -8 \end{array}\right] \quad \text{reduces to} \quad \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{array}\right]$$

dependent: $\vec{v}_3 = 3\vec{v}_1 - 4\vec{v}_2$

is $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{v}_3$?

$c_1 = 3$
$c_2 = -4$

$$\begin{bmatrix} 3 \\ 4 \\ -8 \end{bmatrix} \overset{?}{=} 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 4\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \ \checkmark$$

OR $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 2 & -8 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

∞'ly many soln's

Exercise 6) Are the vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \ \underline{w}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

linearly independent? What is their span? Hint:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right] \quad \text{reduces to} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0} \implies c_1 = 0$
$c_2 = 0$
$c_3 = 0.$