Wed Feb 21
4.1-4.2 The vector spaces $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^n$.

Announcements:
- Homework is due tomorrow in lab
- Quiz is a takehome quiz due in lab.

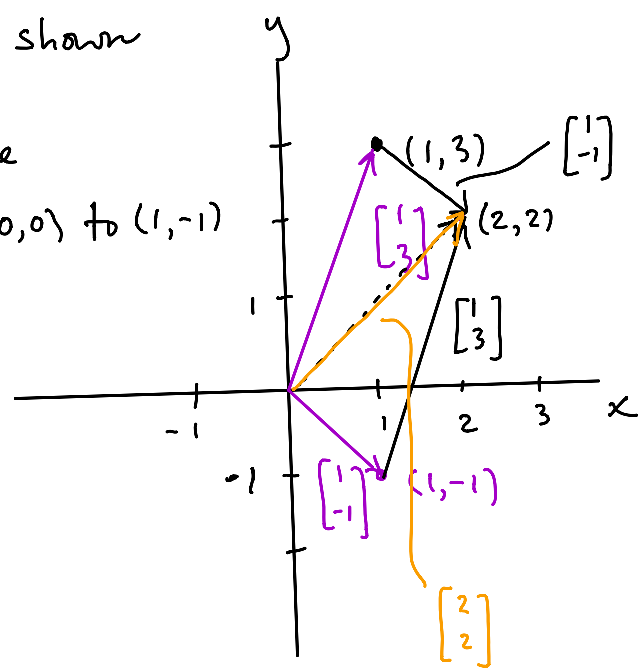
Warm-up Exercise: 'til 10:47

The point $(1,3)$ is plotted below
The position vector displacement
$$2250 \longrightarrow \begin{bmatrix}1\\3\end{bmatrix} = \underbrace{\hat{\imath}}_{2210,\,1320,\,1321} + 3\hat{\jmath}$$ is also shown

a) plot the point $(1,-1)$ and the position vector $\begin{bmatrix}1\\-1\end{bmatrix}$ from $(0,0)$ to $(1,-1)$

b) compute $\begin{bmatrix}1\\3\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$

and plot the point for which it is the position vector
$(2,2)$



4.1-4.2 The vector space $\mathbb{R}^m$ and its subspaces; concepts related to "linear combinations of vectors."

Geometric interpretation of vectors

The space $\mathbb{R}^n$ may be thought of in two equivalent ways. In both cases, $\mathbb{R}^n$ consists of all possible $n - tuples$ of numbers:

(i) We can think of those $n - tuples$ as representing points, as we're used to doing for $n = 1, 2, 3$. In this case we can write

$$\mathbb{R}^n = \{(x_1, x_2, ..., x_n),\ s.t.\ x_1, x_2, ..., x_n \in \mathbb{R}\}.$$

(ii) We can think of those $n - tuples$ as representing vectors that we can add and scalar multiply. In this case we can write

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_n \end{bmatrix},\ s.t.\ x_1, x_2, ..., x_n \in \mathbb{R} \right\}.$$

Since algebraic vectors (as above) can be used to measure geometric displacement, one can identify the two models of $\mathbb{R}^n$ as sets by identifying each point $(x_1, x_2, ...x_n)$ in the first model with the displacement vector $\underline{\boldsymbol{x}} = [x_1, x_2, ...x_m]^T$ from the origin to that point, in the second model, i.e. the "position vector" of the point.

One of the key themes of Chapter 4 is the idea of *linear combinations*. These have an algebraic definition as well as a geometric interpretation as combinations of displacements, as we will review in our first few exercises.

Definition: If we have a collection of n vectors $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n\}$ in $\mathbb{R}^m$, then any vector $\underline{v} \in \mathbb{R}^m$ that can be expressed as a sum of scalar multiples of these vectors is called a linear combination of them. In other words, if we can write

"weighted sum"

$$\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + \ldots + c_n\underline{v}_n \,,$$

then $\underline{v}$ is a linear combination of $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$. The scalars $c_1, c_2, \ldots, c_n$ are called the *weights* or *linear combination coefficients*.

Example You've probably seen linear combinations in previous math/physics classes. For example you might have expressed the position vector $\underline{\mathbf{r}}$ of a point (x, y, z) as a linear combination

$$\underline{\mathbf{r}} = x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\boldsymbol{k}}$$

where $\underline{\mathbf{i}}, \underline{\mathbf{j}}, \underline{\mathbf{k}}$ represent the unit displacements in the Since we can express these displacements using Math 2250 notation as

$$\hat{\underline{\mathbf{i}}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \hat{\underline{\mathbf{j}}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \hat{\underline{\mathbf{k}}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we have

$$x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\boldsymbol{k}} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Remarks: When we had free parameters in our explicit solutions to linear systems of equations $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ back in Chapter 3, we sometimes rewrote the explicit solutions using linear combinations, where the scalars were the free parameters (which we often labeled with letters that were t, t_4, t_3 etc., rather than with "c's"). When we return to differential equations in Chapter 5 -studying higher order differential equations - then the explicit solutions will also be expressed using "linear combinations", just as we did in Chapters 1 -2, where we used the letter "C" for the single free parameter in first order differential equation solutions:

Definition: If we have a collection $\{y_1, y_2, \ldots, y_n\}$ of n functions $y(x)$ defined on a common interval I, then any function that can be expressed as a sum of scalar multiples of these functions is called a linear combination of them. In other words, if we can write

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n ,$$

then y is a linear combination of $y_1, y_2, \ldots, y_n$.

The reason that the same words are used to describe what look like two quite different settings, is that there is a common fabric of mathematics (called vector space theory) that underlies both situations. We shall be exploring these concepts over the next several lectures, using a lot of the matrix algebra theory we've just developed in Chapter 3. This vector space theory will tie in directly to our study of differential equations, in Chapter 5 and subsequent chapters.

Exercise 1) Let $\underline{\boldsymbol{u}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

1a) Plot the points $(1, -1)$ and $(1, 3)$, which have position vectors $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$. Draw these position vectors as arrows beginning at the origin and ending at the corresponding points.

1b) Compute $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}$ and then plot the point for which this is the position vector. Note that the algebraic operation of vector addition corresponds to the geometric process of composing horizontal and vertical displacements.

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

1c) Compute $\underline{\boldsymbol{u}}$ and $2\,\underline{\boldsymbol{v}}$, $\underline{\boldsymbol{u}} + 2\,\underline{\boldsymbol{v}}$ and plot the corresponding points for which these are the position vectors.

1d) Plot the parametric line whose points are the endpoints of the position vectors $\{\underline{\boldsymbol{u}} + t\,\underline{\boldsymbol{v}},\ t \in \mathbb{R}\}$. How else might you have expressed this parametric line in multivariable calculus class? What is the implicit equation of this line?

$\vec{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\quad 2\vec{v} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

$$\vec{u} + 2\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

1d) $\{\vec{u} + t\vec{v}\}_{t \in \mathbb{R}}$

endpoints are a line

$$\vec{u} + t\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 + t \\ -1 + 3t \end{bmatrix}$$

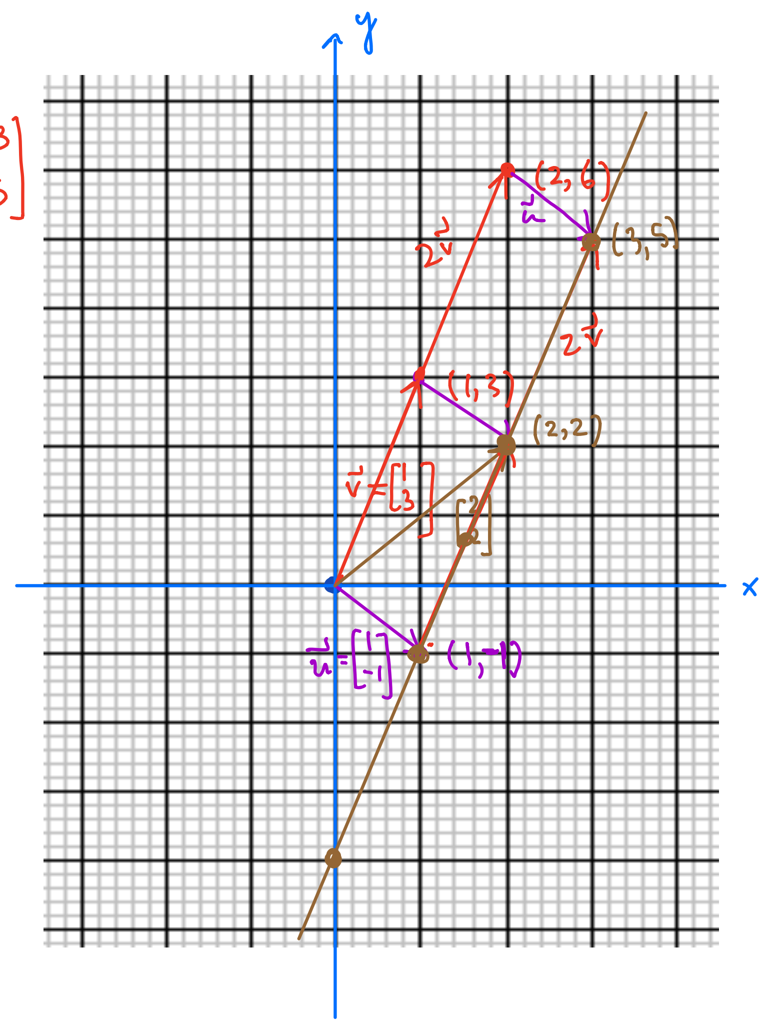
$x = 1 + t \rightarrow t = x - 1$

$y = -1 + 3t \rightarrow$

$$y = -1 + 3(x - 1)$$

$$\boxed{y = 3x - 4}$$

agrees with graph

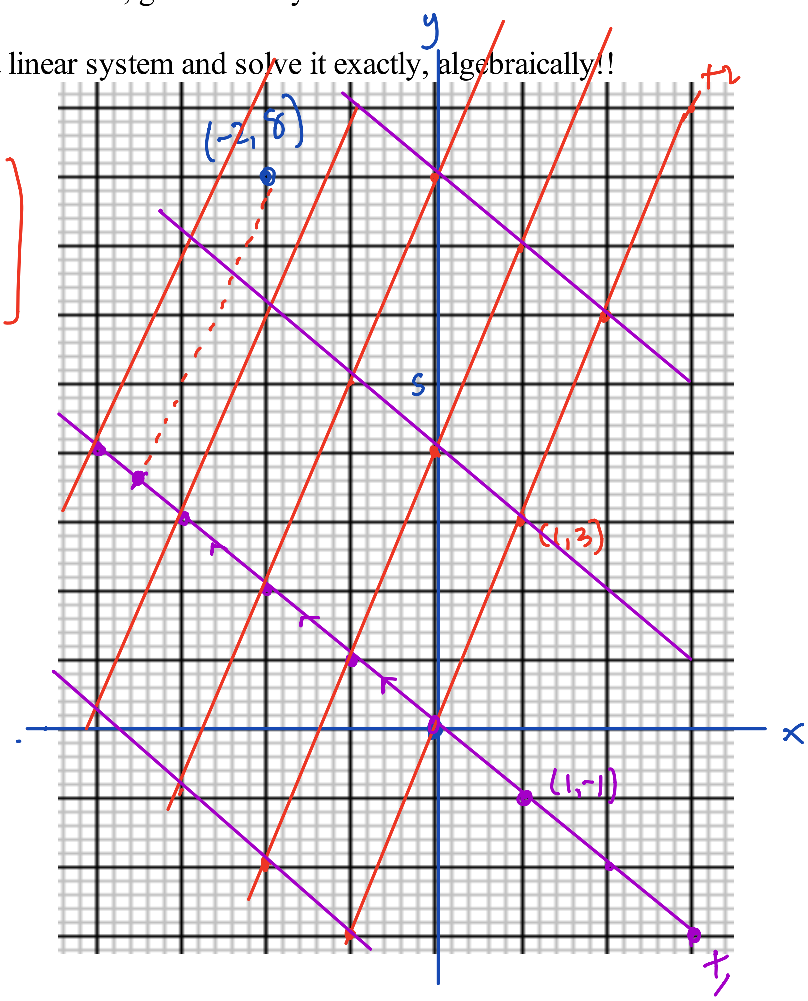


Exercise 2) Can you get to the point $(-2, 8) \in \mathbb{R}^2$, from the origin $(0, 0)$, by moving only in the $(\pm)$ directions of $\underline{\boldsymbol{u}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$? Algebraically, this means we want to solve the linear combination problem

$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}.$$

2a) Superimpose a grid related to the displacement vectors $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$ onto the graph paper below, and, recalling that vector addition yields net displacement, and scalar multiplication yields scaled displacement, try to approximately solve the linear combination problem above, geometrically.

2b) Rewrite the linear combination problem as a linear system and solve it exactly, algebraically!!



2a) $-3.5 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1.5 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$

actually correct!

2b) solve algebraically.

$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 3x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_2 &= -2 \\ -x_1 + 3x_2 &= 8 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 1 & -2 \\ -1 & 3 & 8 \end{array}\right]$$

$R_1 + R_2 \to R_2$
$$\left[\begin{array}{cc|c} 1 & 1 & -2 \\ 0 & 4 & 6 \end{array}\right]$$

$\frac{R_2}{4} \to R_2$
$$\left[\begin{array}{cc|c} 1 & 1 & -2 \\ 0 & 1 & 1.5 \end{array}\right]$$

$-R_2 + R_1 \to R_1$
$$\left[\begin{array}{cc|c} 1 & 0 & -3.5 \\ 0 & 1 & 1.5 \end{array}\right]$$

$x_1 = -3.5$
$x_2 = 1.5$ ✓

2c) Can you get to any point (x, y) in $\mathbb{R}^2$, starting at $(0, 0)$ and moving only in directions parallel to $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$?

Yes. It's clean

Argue geometrically and algebraically. How many ways are there to express $\begin{bmatrix} x \\ y \end{bmatrix}$ as a linear combination of $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$?

only one way

Yes.

$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Solve for x_1, x_2.

$$\left[\begin{array}{cc|c} 1 & 1 & x \\ -1 & 3 & y \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & *_1 \\ 0 & 1 & *_2 \end{array}\right]$$

$x_1 = *_1$
$x_2 = *_2$

reason

$$\text{rref}\left(\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\right) = I$$

Definition The *span* of a collection of vectors, written as *span*$\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$, is the collection of all linear combinations of those vectors.

Examples: We showed in 2c that *span*$\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}\} = \mathbb{R}^2$. ~~Span~~

Remark: The mathematical meaning of the word *span* is related to the English meaning - as in "wing span" or "span of a bridge", but it's also different. The span of a collection of vectors goes on and on and does not "stop" at the vector or associated endpoint:

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Exercise 3) Consider the two vectors $\underline{v}_1 = [1, 0, 0]^T$, $\underline{v}_2 = [0, -1, 2]^T \in \mathbb{R}^3$.

3a) Sketch these two vectors as position vectors in $\mathbb{R}^3$, using the axes below.

3b) What geometric object is $span\{\underline{v}_1\}$? (Remember, we are identifying position vectors with their endpoints.) Sketch a portion of this object onto your picture below. Remember though, the "span" continues beyond whatever portion you can draw.

x-axis

3c) What geometric object is $span\{\underline{v}_1, \underline{v}_2\}$? Sketch a portion of this object onto your picture below. Remember though, the "span" continues beyond whatever portion you can draw.

plane