Math 2250-004
Friday Feb 2
3.1-3.2 systems of linear (algebraic) equations

Announcements: • Chapters 3-4 is about linear algebra.

relates to HW 8 simpson's rule

example.

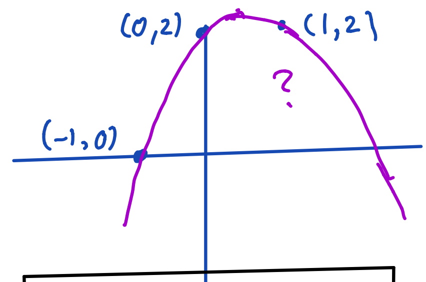
Warm-up Exercise: We're trying to find a parabola,

$$y = ax^2 + bx + c = p(x)$$

passing through the three points

$$(x,y) = (-1,0),\ (0,2),\ (1,2)$$

Find the coefficients a, b, c to make this so!



For example,
@ $x=-1$, $y=0$, so $0 = a(-1)^2 + b(-1) + c$
$0 = a - b + c$

get two more eqtns, then find a,b,c.

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@ $(-1,0)$: $0 = a - b + c$
@ $(0,2)$: $2 = 0 + 0 + c \Rightarrow c = 2$
@ $(1,2)$: $2 = a + b + c$

E_1: $0 = a - b + 2$
E_3: $2 = a + b + 2$

$E_1 + E_3$: $2 = 2a + 4$
$-2 = 2a \Rightarrow a = -1$

$E_1 - E_3$: $-2 = -2b \Rightarrow b = 1$

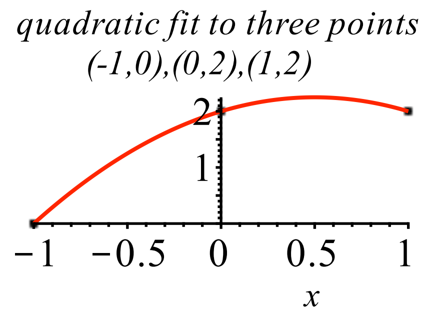
$a = -1$
$b = 1$
$c = 2$

$p(x) = -x^2 + x + 2$
$p(-1) = 0 = -1 - 1 + 2$ ✓
$p(0) = 2 = 0 + 0 + 2$ ✓
$p(1) = 2 = -1 + 1 + 2$ ✓

Exercise 1: (Relates to this week's homework, and introduces systems of linear algebraic equations:

Find the quadratic function $p(x) = a\,x^2 + b\,x + c$ so that the graph $y = p(x)$ interpolates (i.e. passes through) the three points $(-1, 0)$, $(0, 2)$, $(1, 2)$:

we just did this.



In Chapters 3-4 we temporarily leave differential equations in order to study basic concepts in linear algebra. Almost all of you have studied linear systems of equations and matrices before, and that's where Chapter 3 starts. Linear algebra is foundational for many different disciplines, and in this course we'll use the key ideas when we return to higher order linear differential equations and to systems of differential equations. As it turns out, there's an example of solving simultaneous linear equations in this week's homework. It's related to Simpson's rule for numerical integration, which is itself related to the Runge-Kutta algorithm for finding numerical solutions to differential equations.

In 3.1-3.2 our goal is to understand systematic ways to solve systems of (simultaneous) linear equations. Although we used *a*, *b*, *c* for the unknowns in the previous problem, this is not our standard way of labeling.

- We'll often call the unknowns $x_1, x_2, \ldots x_n$, or write them as elements in a vector
$$\underline{\boldsymbol{x}} = [x_1, x_2, \ldots x_n].$$
- Then the general linear system (LS) of *m* equations in the *n* unknowns can be written as

$$\text{(LS)} \qquad \begin{array}{l} a_{11}\,x_1 + a_{12}\,x_2 + \ldots + a_{1\,n}\,x_n = b_1 \\ a_{21}\,x_1 + a_{22}\,x_2 + \ldots + a_{2\,n}\,x_n = b_2 \\ \quad\vdots \qquad\qquad\qquad\qquad\qquad \vdots \quad \vdots \\ a_{m1}\,x_1 + a_{m2}\,x_2 + \ldots + a_{m\,n}\,x_n = b_m \end{array}$$

m equations in n unknowns $x_1, x_2 \ldots x_n$.
a_{ij}, b_i are numbers

where the coefficients $a_{i\,j}$ and the right-side number b_j are known. The goal is to find values for the vector $\underline{\boldsymbol{x}}$ so that all equations are true. (Thus this is often called finding "simultaneous" solutions to the linear system, because all equations will be true at once, for the given vector $\underline{\boldsymbol{x}}$.

<u>Notice</u> that we use two subscripts for the coefficients $a_{i\,j}$ and that the first one indicates which equation it appears in, and the second one indicates which variable its multiplying; in the corresponding *coefficient matrix A*, this numbering corresponds to the row and column of $a_{i\,j}$:

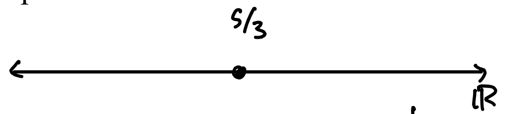
row → i j ← column.

$$A := \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1\,n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2\,n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{m\,n} \end{bmatrix}$$

a_{23} is coeff in 2nd row, 3rd column.
it came from coef. of x_3 in 2nd eqtn.

Let's start small, where geometric reasoning will help us understand what's going on:
<u>Exercise 2</u>: Describe the solution set of each single equation below; describe and sketch its geometric realization in the indicated Euclidean spaces $\mathbb{R}^n$.

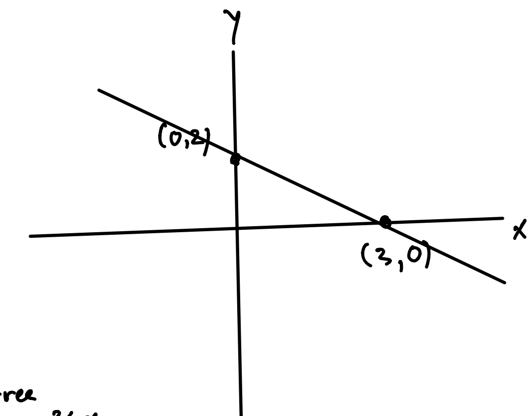
<u>2a)</u> $3x = 5$, for $x \in \mathbb{R}$.



$x = 5/3$

solt'n set was a single point in $\mathbb{R}$

<u>2b)</u> $2x + 3y = 6$, for $[x, y] \in \mathbb{R}^2$.



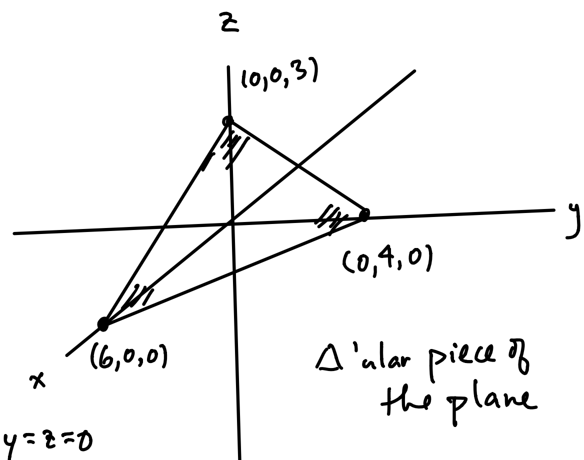
$3y = 6 - 2x$

$y = 2 - \frac{2}{3}x$, x "free", $(x \in \mathbb{R})$

line with y-intercept 2
slope $-2/3$

x free
$y = 2 - \frac{2}{3}x$

<u>2c)</u> $2x + 3y + 4z = 12$, for $[x, y, z] \in \mathbb{R}^3$.



plane in $\mathbb{R}^3$

x free
y free
$z = 3 - \frac{1}{2}x - \frac{3}{4}y$

Δ'ular piece of the plane

$4z = 12 - 2x - 3y$
$z = 3 - \frac{1}{2}x - 3/4\,y$

@ $y = z = 0$
$x = 6$
@ $x = z = 0$
$y = 4$
@ $x = y = 0$
$z = 3$

2 linear equations in 2 unknowns:

$$a_{11}\,x + a_{12}\,y = b_1$$
$$a_{21}\,x + a_{22}\,y = b_2$$

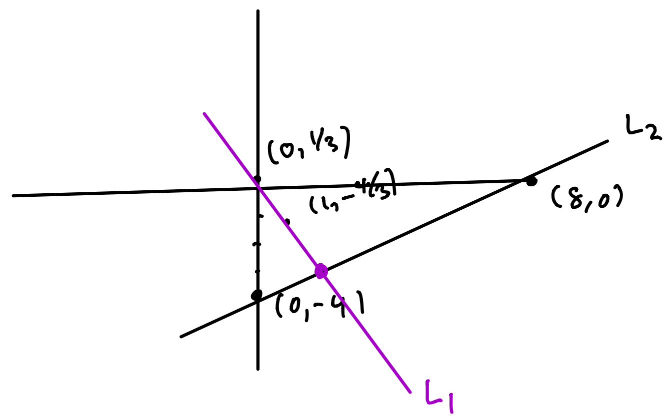
goal: find all $[x, y]$ making both of these equations true. So geometrically you can interpret this problem as looking for the intersection of two lines.

Exercise 3: Consider the system of two equations E_1, E_2:

$$E_1 \qquad 5\,x + 3\,y = 1$$
$$E_2 \qquad x - 2\,y = 8$$

@ $x = 1$ $\quad 5 + 3y = 1$
$3y = -4$
$y = -4/3$

3a) Sketch the solution set in $\mathbb{R}^2$, as the point of intersection between two lines.



step 1

$E_2 \to E_1 \quad 1\cdot x - 2y = 8$
$E_1 \to E_2 \quad 5x + 3y = 1$

3b) Use the following three "elementary equation operations" to systematically reduce the system E_1, E_2 to an equivalent system (i.e. one that has the same solution set), but of the form

$$1\,x + 0\,y = c_1$$
$$0\,x + 1\,y = c_2$$

(so that the solution is $x = c_1, y = c_2$). Make sketches of the intersecting lines, at each stage.

The three types of elementary equation operation are below. Can you explain why the solution set to the modified system is the same as the solution set before you make the modification?

- interchange the order of the equations
- multiply one of the equations by a non-zero constant
- replace an equation with its sum with a multiple of a different equation.

$$x - 2y = 8$$
$$5x + 3y = 1$$

$$-5(x - 2y = 8)$$
$$-5x + 10y = -40$$

$E_1 \quad 5x + 3y = 1$

$E_2 \quad x - 2y = 8$

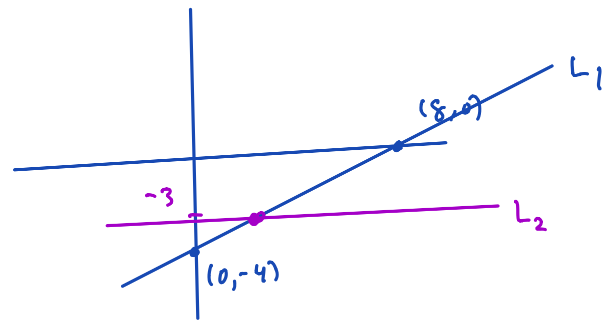
$$x - 2y = 8$$

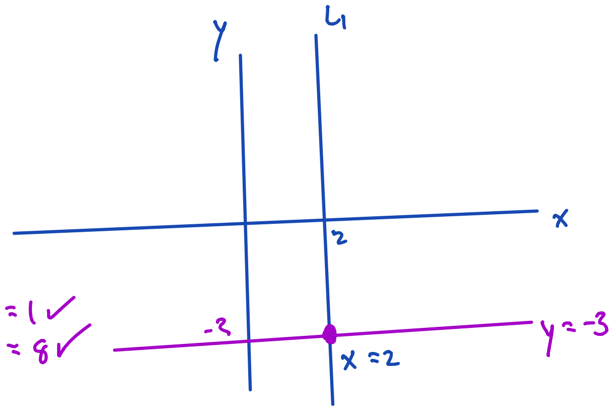
$-5E_1 + E_2 \to E_2$
$$0 + 13y = -39$$

$$x - 2y = 8$$

$E_2/13$
$$y = -3$$

$2E_2 + E_1 \to E_1$
$$x = 2$$
$$y = -3$$





$$(x, y) = (2, -3)$$

$$5x + 3y = 1$$
$$x - 2y = 8$$

$5 \cdot 2 - 9 = 1$ ✓
$2 + 6 = 8$ ✓

synthetically

$$\left[\begin{array}{cc|c} 5 & 3 & 1 \\ 1 & -2 & 8 \end{array}\right]$$

$R_2 \to R_1$, $R_1 \to R_2$
$$\left[\begin{array}{cc|c} 1 & -2 & 8 \\ 5 & 3 & 1 \end{array}\right]$$

$-5R_1 + R_2$
$$\left[\begin{array}{cc|c} 1 & -2 & 8 \\ 0 & 13 & -39 \end{array}\right]$$

$\frac{R_2}{13} \to R_2$
$$\left[\begin{array}{cc|c} 1 & -2 & 8 \\ 0 & 1 & -3 \end{array}\right]$$

$2R_2 + R_1 \to R_1$
$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right] \longrightarrow \begin{array}{l} 1 \cdot x = 2 \\ 1 \cdot y = -3 \end{array} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$2\left[\begin{array}{cc|c} 0 & 1 & -3 \end{array}\right]$$
$$\begin{array}{cc|c} 0 & 2 & -6 \end{array}$$

3c) Look at your work in 3b. Notice that you could have save a lot of writing by doing this computation "synthetically", i.e. by just keeping track of the coefficients and right-side values. Using R_1, R_2 as symbols for the rows, your work might look like the computation below. Notice that when you operate synthetically the "elementary equation operations" correspond to "elementary row operations":

- interchange two rows
- multiply a row by a non-zero number
- replace a row by its sum with a multiple of another row.

$$
\begin{array}{r|rr|r}
 & 5 & 3 & 1 \\
 & 1 & -2 & 8 \\
\hline
R_2 & 1 & -2 & 8 \\
R_1 & 5 & 3 & 1 \\
\hline
 & 1 & -2 & 8 \\
-5R_1+R_2 & 0 & 13 & -39 \\
\hline
 & 1 & -2 & 8 \\
 & 0 & 1 & -3 \\
\hline
2R_2+R_1 & 1 & 0 & 2 \\
 & 0 & 1 & -3
\end{array}
\quad \rightarrow \quad
\begin{array}{l} x = 2 \\ y = -3 \end{array} \; !
$$

3d) What are the possible geometric solution sets to 1, 2, 3, 4 or any number of linear equations in two unknowns? mystery

Solutions to linear equations in 3 unknowns:

What is the geometric question you're answering?

Exercise 4) Consider the system

$$x + 2y + z = 4$$
$$3x + 8y + 7z = 20$$
$$2x + 7y + 9z = 23 .$$

Use elementary equation operations (or if you prefer, elementary row operations in the synthetic version) to find the solution set to this system. There's a systematic way to do this, which we'll talk about. It's called Gaussian elimination.

Hint: The solution set is a single point, $[x, y, z] = [5, -2, 3]$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right]$$

$$\begin{array}{l} \\ -3R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 7 & 15 \end{array}\right]$$

$$\frac{R_2}{2} \to R_2 \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 7 & 15 \end{array}\right]$$

$$\begin{array}{l} \\ \\ -3R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{l} -R_3 + R_1 \to R_1 \\ -2R_3 + R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$-2R_2 + R_1 \to R_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$x = 5$
$y = -2$
$z = 3$

Exercise 5 There are other possibilities. In the two systems below we kept all of the coeffients the same as in Exercise 4, except for a_{33} , and we changed the right side in the third equation, for 4a. Work out what happens in each case.

5a)

$$\begin{aligned} x + 2\,y + z &= 4 \\ 3\,x + 8\,y + 7\,z &= 20 \\ 2\,x + 7\,y + 8\,z &= 20\,. \end{aligned}$$

5b)

$$\begin{aligned} x + 2\,y + z &= 4 \\ 3\,x + 8\,y + 7\,z &= 20 \\ 2\,x + 7\,y + 8\,z &= 23\,. \end{aligned}$$

5c) What are the possible solution sets (and geometric configurations) for 1, 2, 3, 4,... equations in 3 unknowns?