Wed Feb 14

- 3.6 determinants
- midterm Friday!

Announcements:

1) test Friday, 10:40 – 11:40
2) review for at least last 20 minutes today (labs tomorrow are review).
3) HW due next week!, due in lab, though.

Warm-up Exercise:

Compute the determinant!

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 4 & 8 \\ -3 & 1 & 6 & -9 \\ 4 & 0 & 2 & 10 \end{vmatrix}$$

'til 10:47

can expand across any row or down any column

using first row $\det = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$

$$= 1 \cdot \begin{vmatrix} 0 & 4 & 8 \\ 1 & 6 & -9 \\ 0 & 2 & 10 \end{vmatrix} + 0 + 0 + 0$$

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

$$= 1 \cdot \left(\cancel{0 \cdot C_{11}} + 1 \cdot C_{21} + \cancel{0 \cdot C_{31}}\right)$$

$$= 1 \cdot \left(- \begin{vmatrix} 4 & 8 \\ 2 & 10 \end{vmatrix} \right)$$

$$= -(40 - 16) = \underline{-24}$$

The effective way to compute determinants for larger-sized matrices without lots of zeroes is to not use the definition, but rather to use the following facts, which track how elementary row operations affect determinants:

- (1a) Swapping any two rows changes the sign of the determinant.

  proof: This is clear for 2×2 matrices, since

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \qquad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad\,.$$

For 3×3 determinants, expand across the row *not* being swapped, and use the 2×2 swap property to deduce the result. Prove the general result by induction: once it's true for $n \times n$ matrices you can prove it for any $(n+1) \times (n+1)$ matrix, by expanding across a row that wasn't swapped, and applying the $n \times n$ result.

swap rows 1 & 3:

expand unswapped matrix across 2nd row

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{21}\cdot\left(-\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}\right) + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{33} \end{vmatrix}$$

what happens if we swap 1st & 3rd row of original matrix, on the right side?

ans rows swap in 2×2 matrices

so by fact for 2×2 dets, the sign on the right side (& left side) changes

(1b) Thus, if two rows in a matrix are the same, the determinant of the matrix must be zero: on the one hand, swapping those two rows leaves the matrix and its determinant unchanged; on the other hand, by (1a) the determinant changes its sign. The only way this is possible is if the determinant is zero.

• (2a) If you factor a constant out of a row, then you factor the same constant out of the determinant.
Precisely, using $\mathcal{R}_i$ for i^{th} row of A, and writing $\mathcal{R}_i = c\,\mathcal{R}_i^*$

$$\begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_i \\ \mathcal{R}_n \end{vmatrix} = \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ c\,\mathcal{R}_i^* \\ \mathcal{R}_n \end{vmatrix} = c \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_i^* \\ \mathcal{R}_n \end{vmatrix} .$$

proof: expand across the i^{th} row, noting that the corresponding cofactors don't change, since they're computed by deleting the i^{th} row to get the corresponding minors:

$$det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} c\, a_{ij}^* C_{ij} = c \sum_{j=1}^{n} a_{ij}^* C_{ij} = c\, det\left(A^*\right) .$$

row$_i$

$$\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} \stackrel{?}{=} 2 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$6-4=2 \stackrel{?}{=} 2(3-2)=2$$

$$\underbrace{\begin{vmatrix} 3 & 6 \\ 4 & 12 \end{vmatrix}}_{36-24 = 12 \checkmark} = 3 \cdot 4 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 12$$

(2b) Combining (2a) with (1b), we see that if one row in A is a scalar multiple of another, then $det(A) = 0$. !

• (3) If you replace row i of A, $\mathcal{R}_i$ by its sum with a multiple of another row, say $\mathcal{R}_k$ then the determinant is unchanged! Expand across the i^{th} row:

$$\begin{array}{l} \\ \\ \\ i^{th} \text{ row location} \rightarrow \\ \\ \end{array} \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_k \\ \mathcal{R}_i + c\,\mathcal{R}_k \\ \mathcal{R}_n \end{vmatrix} = \sum_{j=1}^{n} \underbrace{(a_{ij} + c\,a_{kj})}C_{ij} = \sum_{j=1}^{n} a_{ij}C_{ij} + c\sum_{j=1}^{n} a_{kj}C_{ij} = det(A) + c \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_k \\ \mathcal{R}_k \\ \mathcal{R}_n \end{vmatrix} = det(A) + 0\,.$$

Remark: The analogous properties hold for corresponding "elementary column operations". In fact, the proofs are almost identical, except you use column expansions.

Exercise 1) Recompute $\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix}$ from yesterday (using row and column expansions we always got an answer of 15 then.) This time use elementary row operations (and/or elementary column operations).

$$\underset{-2R_1+R_3}{=} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & -6 & 3 \end{vmatrix} \underset{2R_2+R_3 \to R_3}{=} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{vmatrix} = 1 \cdot 3 \cdot 5 = 15$$

Exercise 2) Compute $\begin{vmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ -1 & 0 & -2 & 1 \end{vmatrix}$.

$$= -0 + 1 \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ -1 & -2 & 1 \end{vmatrix} - 0 + 0$$

$$= 1 \cdot \begin{vmatrix} 1 & -1 & 2 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{vmatrix} \quad \begin{matrix} -2R_1+R_2 \to R_2 \\ R_1+R_3 \to R_3 \end{matrix}$$

$$= 0$$ two rows are multiples!

Theorem: Let $A_{n \times n}$. Then A^{-1} exists if and only if $det(A) \neq 0$.

proof: We already know that A^{-1} exists if and only if the reduced row echelon form of A is the identity matrix. Now, consider reducing A to its reduced row echelon form, and keep track of how the determinants of the corresponding matrices change: As we do elementary row operations,

- if we swap rows, the sign of the determinant switches.
- if we factor non-zero factors out of rows, we factor the same factors out of the determinants.
- if we replace a row by its sum with a multiple of another row, the determinant is unchanged.

Thus,

$$|A| = c_1|A_1| = c_1c_2|A_2| = \ldots = c_1c_2 \ldots c_N|rref(A)|$$

where the nonzero c_k's arise from the three types of elementary row operations. If $rref(A) = I$ its determinant is 1, and $|A| = c_1c_2 \ldots c_N \neq 0$. If $rref(A) \neq I$ then its bottom row is all zeroes and its determinant is zero, so $|A| = c_1c_2 \ldots c_N(0) = 0$. Thus $|A| \neq 0$ if and only if $rref(A) = I$ if and only if A^{-1} exists !

Remark: Using the same ideas as above, you can show that $det(AB) = det(A)det(B)$. This is an important identity that gets used, for example, in multivariable change of variables formulas for integration, using the Jacobian matrix. (It is not true that $det(A + B) = det(A) + det(B)$.) Here's how to show $det(AB) = det(A)det(B)$: The key point is that if you do an elementary row operation to AB, that's the same as doing the elementary row operation to A, and then multiplying by B. With that in mind, if you do exactly the same elementary row operations as you did for A in the theorem above, you get

$$|AB| = c_1|A_1B| = c_1c_2|A_2B| = \ldots = c_1c_2 \ldots c_N|rref(A)B|.$$

If $rref(A) = I$, then from the theorem above, $|A| = c_1c_2 \ldots c_N$, and we deduce $|AB| = |A||B|$. If $rref(A) \neq I$, then its bottom row is zeroes, and so is the bottom row of $rref(A)B$. Thus $|AB| = 0$ and also $|A||B| = 0$.

There is a "magic" formula for the inverse of square matrices A (called the "adjoint formula") that uses the determinant of A along with the cofactor matrix of A. We'll talk about the magic formula on Monday next week, after the midterm.

Exam notes:

The exam is this Friday February 16, from 10:40-11:40 a.m. Note that it will start 5 minutes before the official start time for this class, and end 5 minutes afterwards, so you should have one hour to work on the exam. Get to class early, and bring your University I.D. card, which we might ask you to show if we don't recognize you from sections or lecture.

2.1-2.3 + 2.4 not 2.5, 2.6

This exam will cover textbook material from 1.1-1.5, 2.1-2.4, 3.1-3.5. The exam is closed book and closed note. You may use a scientific (but not a graphing) calculator, although symbolic answers are accepted for all problems, so no calculator is really needed. (Using a graphing calculator which can do matrix computations for example, is grounds for receiving grade of 0 on your exam. So please ask before the exam if you're unsure about your calculator. And of course, your cell phones must be put away.)

I recommend trying to study by organizing the conceptual and computational framework of the course so far. Only then, test yourself by making sure you can explain the concepts and do typical problems which illustrate them. The class notes and text should have explanations for the concepts, along with worked examples. Old homework assignments and quizzes are also a good source of problems. It could be helpful to look at quizzes/exams from my previous Math 2250 classes, which go back several years from the link

http://www.math.utah.edu/~korevaar/oldclasses.html.

Your lab meetings tomorrow will be exam review sessions.

Chapter 3:

3a) Can you recognize an algebraic linear system of equations?

- $a_{11}x_1 + a_{12}x_2 + \cdots = b_1$
  $\vdots$
  $a_{m1}x_1 + \cdots \quad + a_{mn}x_n = b_m$

$A\vec{x} = \vec{b}$

3b) Can you interpret the solution set geometrically when there are 2 or three unknowns?

common intersection set for lines

intersection set of planes.

3c) Can you use Gaussian elimination to compute reduced row echelon form for matrices? Can you apply this algorithm to augmented matrices to solve linear systems ?

you better be able to!

*3d) What does the shape of the reduced row echelon form of a matrix A tell you about the possible solution sets to A**x** = **b** (perhaps depending, and perhaps not depending on **b**)? Focus especially on whether each row of the reduced matrix has a pivot or not; and on whether each column of the reduced matrix has a pivot or not.*

if each row of rref(A) has a pivot, you can always solve $A\vec{x} = \vec{b}$

if each col. of rref(A) has a pivot then solutions that exist are unique (because no free parameters).

3e) What properties do (and do not) hold for the matrix algebra of addition, scalar multiplication, and matrix multiplication?

division does not hold.

$AB \neq BA$ usually.

the other algebra rules work, as long as matrices have the right #'s of rows and columns to allow the indicated operations

3f) What is the matrix inverse, A^{-1} for a square matrix A? Does every square matrix have an inverse? How can you tell whether or not a matrix has an inverse, using reduced row echelon form? What's the row operations way of finding A^{-1}, when it exists? Can you use matrix algebra to solve matrix equations for unknown vectors $\underline{\mathbf{x}}$ *or matrices* $\mathbf{X}$, *possibly using matrix inverses and other algebra manipulations?*

- A has an inverse means there is a matrix B so that $AB = I$, $BA = I$. We write A^{-1} for B
- $A_{n\times n}$ has an inverse if and only if $\mathrm{rref}(A) = I$
- In this case we find the columns of A^{-1} & so all of A^{-1} by augmenting A with the identity matrix and reducing: $A \mid I \longrightarrow I \mid A^{-1}$
- When solving matrix eqtns such as

$$AX = B$$

or

$$XA = C$$

multiply both sides of the eqtn by A^{-1}, but since multiplication doesn't commute put the A^{-1} where it can cancel the A

$$AX = B \Rightarrow A^{-1}AX = A^{-1}B$$
$$\Rightarrow X = A^{-1}B$$
$$XA = C \Rightarrow XAA^{-1} = CA^{-1}$$
$$\Rightarrow X = CA^{-1}$$

Chapters 1-2:

highest order deriv. in the eqtn

1a) What is a differential equation? What is its order? What is an initial value problem, for a first or second order DE?

→ an eqtn involving a function & its derivatives, e.g. for $y = y(x)$, $f(x, y, y', \ldots y^{(n)}) = 0$.

1st order IVP: $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$

1b) How do you check whether a function solves a differential equation? An initial value problem?

substitute the function into the DE & see whether a true identity results. For IVP see if the function satisfies the initial condition(s)

1c) What is the connection between a first order differential equation and a slope field for that differential equation? The connection between an IVP and the slope field?

the graphs of solutions $y(x)$ to the DE $y' = f(x,y)$ will have slopes at (x,y) equal to $f(x,y)$, so the graphs are tangent to the slope fields. Graphs of solutions to IVP's go through the initial points.

1d) Do you expect solutions to IVP's to exist, at least for values of the input variable close to its initial value? Why? Do you expect uniqueness? What can cause solutions to not exist beyond a certain input variable value?

Expect soltns, expect uniqueness, although this can fail if the conditions of the existence-uniqueness theorem aren't satisfied

1e) What is Euler's numerical method for approximating solutions to first order IVP's, and how does it relate to slope fields?

for $y' = f(x,y)$: $\quad x_{i+1} = x_i + \Delta x$

$y_{i+1} = y_i + f(x_i, y_i)\Delta x$

You're using the value of the slope function at (x_i, y_i) as a constant slope to estimate y_{i+1}.

1f) What's an autonomous differential equation? What's an equilibrium solution to an autonomous differential equation? What is a phase diagram for an autonomous first order DE, and how do you construct one? How does a phase diagram help you understand stability questions for equilibria? What does the phase diagram for an autonomous first order DE have to do with the slope field?

autonomous: for $y(x)$: $y' = f(y)$ (so $x(t)$, $x' = f(x)$, etc.)

phase diagram contains equilibrium points (constant solutions), and arrows on the intervals between the equilibrium points, to indicate whether solutions are increasing or decreasing.

You can decide whether constant solutions are stable or unstable (or asymptotically stable or semi-stable) using phase diagram. You can think of the phase diagram as a projection of the slope field onto the vertical axis

1g) *Can you recognize the first order differential equations for which we've studied solution algorithms, even if the DE is not automatically given to you pre-set up for that algorithm? Do you know the algorithms for solving these particular first order DE's?*

separable for $y(x)$, $y' = f(x)g(y)$ or equivalent
($\text{for } x(t)$, $x' = f(t)g(x)$, etc.).

linear: for $y(x)$, $y' + P(x)y = Q(x)$
(for $x(t)$, $x' + P(t)x = Q(t)$, etc.)

know how to recognize & solve!

2) *Can you convert a description of a dynamical system in terms of rates of change, or a geometric configuration in terms of slopes, into a differential equation? What are the models we've studied carefully in Chapters 1-2? What sorts of DE's and IVP's arise? Can you solve these basic application DE's, once you've set up the model as a differential equation and/or IVP?*

- recognize whether DE is separable, linear, (or something you can't solve) & be able to solve

applications!
- population models ... $P'(t) = aP^2 + bP + c$
  improved velocity models (mainly linear drag).
  input-output modeling "tanks"
  Newton's law of cooling, exponential growth & decay.

be able to set up and/or interpret the model, and then be able to solve the resulting DE's & IVP's, and interpret the results