Tues Feb 13

- 3.6 determinants

Announcements:

- no quiz tomorrow (exam on Friday)
- I posted 2 practice exams (& solns) on CANVAS (one has a det. question, we won't).
- In HW this week, I included some determinant questions that are part of next week's HW
- a couple odds & ends from Monday, then proceed.

Warm-up Exercise: recall that for a 2×2 matrix A that has an inverse,

'til 10:47

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}; \quad \det(A) := a_{11}a_{22} - a_{12}a_{21}$$

- $$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

use the magic formula for A^{-1} to solve the system

$$3x + 7y = 5$$
$$5x + 4y = 8$$

ans $= \begin{bmatrix} x \\ y \end{bmatrix} =$

$$A = \begin{bmatrix} 3 & 7 \\ 5 & 4 \end{bmatrix}.$$

$$\begin{bmatrix} 3 & 7 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

$$A\vec{x} = \vec{b}$$
$$\underbrace{A^{-1}A}_{I}\vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}\vec{b}.$$

$$\det(A) \text{ "=" } |A| = 3\cdot4 - 5\cdot7 = 12 - 35 = -23.$$

so $$A^{-1} = \frac{1}{-23} \begin{bmatrix} 4 & -7 \\ -5 & 3 \end{bmatrix} = -\frac{1}{23} \begin{bmatrix} 4 & -7 \\ -5 & 3 \end{bmatrix}$$

soln is

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{23} \begin{bmatrix} 4 & -7 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

$$= -\frac{1}{23}\begin{bmatrix} -36 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{36}{23} \\ \frac{1}{23} \end{bmatrix}$$

- Determinants are scalars defined for square matrices $A_{n \times n}$ They always determine whether or not the inverse matrix A^{-1} exists, (i.e. whether the reduced row echelon form of A is the identity matrix): In fact, the determinant of A is non-zero if and only if A^{-1} exists. The determinant of a 1×1 matrix $[a_{11}]$ is defined to be the number a_{11}; determinants of 2×2 matrices are defined as in yesterday's notes; and in general determinants for $n \times n$ matrices are defined recursively, in terms of determinants of $(n-1) \times (n-1)$ submatrices:

Definition: Let $A_{n \times n} = [a_{ij}]$. Then the determinant of A, written $det(A)$ or $|A|$, is defined by

cofactor

$$det(A) := \sum_{j=1}^{n} a_{1j}(-1)^{1+j}M_{1j} = \sum_{j=1}^{n} a_{1j}C_{1j}.$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & & \end{bmatrix}$$

Here M_{1j} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and the j^{th} column, and C_{1j} is simply $(-1)^{1+j}M_{1j}$.

More generally, the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A is called the $i\,j$ Minor M_{ij} of A, and $C_{ij} := (-1)^{i+j}M_{ij}$ is called the $i\,j$ Cofactor of A.

Exercise 1 Check that the messy looking definition above gives the same answer we talked about yesterday in the 2×2 case, namely

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$(-1)^{i+j}$ $\pm$: $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$.

$$|A| = a_{11}(-1)^{1+1}M_{11} + a_{12}(-1)^{1+2}M_{12}$$
$$= a_{11}\cdot 1 \cdot a_{22} + a_{12}\cdot(-1)\cdot a_{21}$$
$$= a_{11}a_{22} - a_{12}a_{21}$$

There's a nice formula for the inverses of 2 × 2 matrices, and it turns out this formula will lead to the next text section 3.6 on determinants:

Theorem: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ exists if and only if the determinant D $= ad - bc$ of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-zero. And in this case,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(Notice that the diagonal entries have been swapped, and minus signs have been placed in front of the off-diagonal terms. This formula should be memorized.)

Exercise 6a) Check that this formula for the inverse works, for $D \neq 0$. (We could have derived it with elementary row operations, but it's easy to check since we've been handed the formula.)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \overset{?}{=} \begin{bmatrix} -2 & 1 \\ +3/2 & -1/2 \end{bmatrix}$$

A

$\det(A) = 1\cdot 4 - 3\cdot 2 = -2.$

formula:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \checkmark$$

6b) Even with systems of two equations in two unknowns, unless they come from very special problems the algebra is likely to be messier than you might expect (without the formula above). Use the magic formula to solve the system

$$3x + 7y = 5$$
$$5x + 4y = 8.$$

this was Tuesday warmup.

6a).
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & \overset{0}{\cancel{-ab+ba}} \\ \underset{0}{\cancel{cd-dc}} & -bc+ad \end{bmatrix} = \det(A)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

so if $|A| \neq 0$, divide both sides by it, to get magic formula.

from the last page, for our convenience:

<u>Definition:</u> Let $A_{n \times n} = [a_{ij}]$. Then the determinant of A, written $det(A)$ or $|A|$, is defined by

$$det(A) := \sum_{j=1}^{n} a_{1j}(-1)^{1+j}M_{1j} = \sum_{j=1}^{n} a_{1j}C_{1j}.$$

Here M_{1j} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and the j^{th} column, and C_{1j} is simply $(-1)^{1+j}M_{1j}$.

<u>Exercise 2</u> Work out the expanded formula for the determinant of a 3×3 matrix. It's not worth memorizing (as opposed to the recursive formula above), but it's good practice to write out at least once, and we might point to it later.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$[(-1)^{i+j}] = \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

sum of 6 terms.
each is $\pm$ product : each column row is used exactly once in each term.

see wikipedia formula for more info.

Theorem: (proof is in text appendix) $det(A)$ can be computed by expanding across any row, say row i:

$$det(A) := \sum_{j=1}^{n} a_{ij}(-1)^{i+j}M_{ij} = \sum_{j=1}^{n} a_{ij}C_{ij}$$ $\leftarrow$ across $\text{row}_i(A)$

or by expanding down any column, say column j:

$$det(A) := \sum_{i=1}^{n} a_{ij}(-1)^{i+j}M_{ij} = \sum_{i=1}^{n} a_{ij}C_{ij}.$$ $\leftarrow$ down $\text{col}_j(A)$

Exercise 3a) Let $A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$. Compute $det(A)$ using the definition. (On the next page we'll use other rows and columns to do the computation.)

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$|A| = 1\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 3 \\ 2 & -2 \end{vmatrix}$$

$$= 1\cdot 5 - 2(-2) - 1(-6)$$
$$= 5 + 4 + 6 = 15$$

From previous page,

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}.$$

3b) Verify that the matrix of all the cofactors of A is given by $[C_{ij}] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$. Then expand $det(A)$ down various columns and rows using the a_{ij} factors and C_{ij} cofactors. Verify that you always get the same value for $det(A)$, as the Theorem on the previous page guarantees. Notice that in each case you are taking the dot product of a row (or column) of A with the corresponding row (or column) of the cofactor matrix.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$[C_{ij}] = \begin{bmatrix} +\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 3 \\ 2 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} \\ +\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad [C_{ij}] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

$row_1(A) \cdot row_1(cof(A))$

$|A| = 1\cdot 5 + 2\cdot 2 + (-1)(-6) = 5 + 4 + 6 = 15$

$col_2(A) \cdot col_2(cof(A))$

$|A| = 4 + 9 + 2 = 15.$

middle rows : $|A| = 0 + 9 + 6 = 15.$

3c) What happens if you take dot products between a row of A and a *different* row of $[C_{ij}]$? A column of A and a *different* column of $[C_{ij}]$? The answer may seem magic. We'll come back to this example when we talk about the magic formula for the inverses of 3×3 (or $n \times n$) invertible matrices.

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad [C_{ij}] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

$\text{row}_1(A) \cdot \text{row}_3(\text{cof}(A)) = 5 - 2 - 3 = 0$

$\text{col}_3(A) \cdot \text{col}_2(\text{cof}(A)) = -2 + 3 - 1 = 0$

we will return.

Exercise 4) Compute the following determinants by being clever about which rows or columns to use:

4a) $\begin{vmatrix} 1 & 38 & 106 & 3 \\ 0 & 2 & 92 & -72 \\ 0 & 0 & 3 & 45 \\ 0 & 0 & 0 & -2 \end{vmatrix}$; 4b) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ \pi^2 & 2 & 0 & 0 \\ 0.476 & 88 & 3 & 0 \\ 1 & 22 & 33 & -2 \end{vmatrix}$.

$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$

4a). $|A| = 1 \begin{vmatrix} 2 & 92 & -72 \\ 0 & 3 & 45 \\ 0 & 0 & -2 \end{vmatrix} - 0 + 0 - 0$

$= 1\left(2 \begin{vmatrix} 3 & 45 \\ 0 & -2 \end{vmatrix} - 0 + 0\right)$

$= 1 \cdot 2 \cdot \left(3(-2) - 0\right)$

$= 1 \cdot 2 \cdot 3 \cdot (-2)$. = product of diagonal entries (upper triangular)

Exercise 5) Explain why it is always true that for an upper triangular matrix (as in 2a), or for a lower triangular matrix (as in 2b), the determinant is always just the product of the diagonal entries.