Math 2250-004 Week 13 April 9-13
EP 7.6 - convolutions; $\underbrace{\text{6.1-6.2 - eigenvalues, eigenvectors and diagonalizability;}}_{T,W}$ $\underbrace{\text{7.1 - systems of}}_{F}$
differential equations.

Mon Apr 9
EP 7.6 Convolutions and Laplace transforms.
(& 10.5)

Announcements:

'til 10:46

Warm-up Exercise: Solve for $x(t)$

$$\begin{cases} x'' + x = 0 \\ x(0) = 0 \\ x'(0) = 1 \end{cases}$$

$f(t)$	$F(s)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$\sin kt$	$\dfrac{k}{s^2+k^2}$
1	$1/s$

Soln

$x(t) = \sin t$
(Also Chptr 5)

$\mathcal{L}$: $s^2X(s) - s\cdot 0 - 1 + X(s) = 0$

$X(s)(s^2+1) = 1$

$X(s) = \dfrac{1}{s^2+1}$

$x(t) = \sin t$

The convolution Laplace transform table entry says it's possible to find the inverse Laplace transform of a product of Laplace transforms. The answer is NOT the product of the inverse Laplace transforms, but a more complicated expression known as a "convolution". If you've had a multivariable Calculus class in which you studied interated integrals and changing the order of integration, you can verify that this table entry is true - I've included the proof as the last page of today's notes.

$f * g(t) := \int_0^t f(\tau) g(t-\tau)\, d\tau$	$F(s)G(s)$	convolution integrals to invert Laplace transform products

Overview: This somewhat amazing table entry allows us to write down solution formulas for *any* constant coefficient nonhomogeneous differential equation, no matter how complicated the right hand side is. Let's focus our discussion on the sort of differential equation that arises in Math 2250, namely the second order case

$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$c_1 f_1 + c_2 f_2$	$c_1 F_1 + c_2 F_2$

$$x'' + a\,x' + b\,x = f(t)$$
$$x(0) = x_0$$
$$x'(0) = v_0\,.$$

Then in Laplace land, this equation is equivalent to

$$s^2 X(s) - s x_0 - v_0 + a\left(s\,X(s) - x_0\right) + b\,X(s) = F(s)$$

$$\Rightarrow X(s)\left(s^2 + a\,s + b\right) = F(s) + x_0\,s + v_0 + a\,x_0.$$

$$\Rightarrow X(s) = F(s)\cdot\frac{1}{s^2 + a\,s + b} + \frac{x_0\,s + v_0 + a\,x_0}{s^2 + a\,s + b}.$$

The inverse Laplace transform of the second fraction contains the initial value information, and its inverse Laplace transform will be a homogeneous solution for the differential equation, and will be zero if $x_0 = v_0 = 0$. (Note that the Chapter 5 characteristic polynomial is exactly $p(r) = r^2 + a\,r + b$, which coincides with the denominator $p(s) = s^2 + a\,s + b$.)

The first fraction is a product of two Laplace transforms

$$F(s)\frac{1}{s^2 + a\,s + b} = F(s)\,W(s)$$

for

$$W(s) := \frac{1}{s^2 + a\,s + b}.$$

and so we can use the convolution table entry to write down an (integral) formula for the inverse Laplace transform. *No matter what* forcing function $f(t)$ appears on the right side of the differential equation, the corresponding solution (to the IVP with $x_0 = v_0 = 0$) is always given by the integral

$$x(t) = f * w(t) = w * f(t) = \int_0^t w(\tau) f(t - \tau)\, d\tau,$$

where

$$w(t) = \mathcal{L}^{-1}\{W(s))\}(t).$$

"weight fcn"

Let's look more closely at that solution formula: The solution to

$W(s) = \dfrac{1}{s^2 + as + b}$

$$x'' + a\,x' + b\,x = f(t)$$
$$x(0) = 0$$
$$x'(0) = 0$$

is given by

$$x(t) = f * w(t) = w * f(t) = \int_0^t w(\tau)\, f(t-\tau)\, d\tau = \int_0^t f(\tau)\, w(t-\tau)\, d\tau\,.$$

<u>Exercise 1)</u> The function $w(t) = \mathcal{L}^{-1}\{W(s)\}(t)$ is called the "weight function" for the differential equation. Verify that it is a solution to the homogeneous DE IVP

$\mathcal{L}:$

$$w'' + a\,w' + b\,w = 0$$
$$w(0) = 0$$
$$w'(0) = 1.$$

w''	$s^2 W(s) - s\,w(0) - w'(0)$
w'	$s\,W(s) - w(0)$

$s^2 W(s) - 1 + a\,s\,W(s) + b\,W(s) = 0$

$W(s)\left(s^2 + as + b\right) = 1$

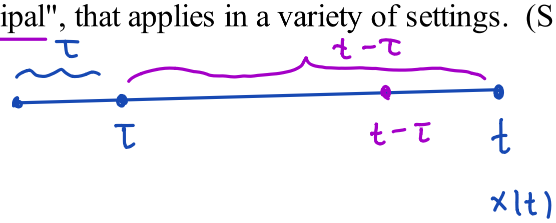
$W(s) = \dfrac{1}{s^2 + as + b}$ ✓

<u>Exercise 2</u> Interpret the convolution formula

$$x(t) = \int_0^t f(\tau)\, w(t-\tau)\, d\tau$$

$\begin{cases} x'' + ax + b = f(t) \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$

in terms of $x(t)$ being a result of the forces $f(\tau)$ for $0 \le \tau \le t$, and how responsive the system through the weight function, and the corresponding times $t \ge t - \tau \ge 0$. This is related to a general principle known as "Duhamal's Principal", that applies in a variety of settings. (See wikipedia.)



$x(t)$ depends on

$f(\tau),\ 0 \le \tau \le t$

and on

$w(t-\tau)$

$t \ge t - \tau \ge 0$

Exercise 3. Let's play the resonance game and practice convolution integrals, first with an old friend, but then with non-sinusoidal forcing functions. We'll stick with our earlier swing, but consider various forcing periodic functions $f(t)$.

$$\mathcal{L}: \quad \begin{aligned} x''(t) + x(t) &= f(t) \\ x(0) &= 0 \\ x'(0) &= 0 \end{aligned}$$

$x''(t)$	$s^2 X(s) - s x(0) - x'(0)$
$f * g(t) = \int_0^t f(\tau) g(t-\tau)\, d\tau$	$F(s)\, G(s)$

a) Find the weight function $w(t)$.

b) Write down the solution formula for $x(t)$ as a convolution integral.

c) Work out the special case of $X(s)$ when $f(t) = \cos(t)$, and verify that the convolution formula reproduces the answer we would've gotten from the table entry

$\frac{t}{2k}\sin(kt)$	$\frac{s}{(s^2+k^2)^2}$

$\int_0^t f(\tau) g(t-\tau)\, d\tau = \int_0^t g(\tau) f(t-\tau)\, d\tau$	$F(s) G(s)$

$$s^2 X(s) + X(s) = F(s)$$

$$X(s)(s^2+1) = F(s)$$

$$X(s) = F(s)\frac{1}{s^2+1} = F(s) W(s)$$

a) $w(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) = \sin t$

b) $x(t) = f * w(t) = \int_0^t f(\tau) w(t-\tau)\, d\tau$
$= \int_0^t w(\tau) f(t-\tau)\, d\tau$

check

$$\int_{\tau=0}^t f(\tau)\, g(\underbrace{t-\tau}_{})\, d\tau \qquad \tilde{\tau} = t - \tau. \quad d\tilde{\tau} = -d\tau$$

$$= \int_{\tilde{\tau}=t}^{\tilde{\tau}=0} f(t-\tilde{\tau})\, g(\tilde{\tau})(-d\tilde{\tau})$$

$$= (-)(-)\int_0^t g(\tilde{\tau}) f(t-\tilde{\tau})\, d\tilde{\tau}$$

c). $\begin{cases} x'' + x = \cos t \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$

$\mathcal{L}$: $X(s)(s^2+1) = \frac{s}{s^2+1}$

$$X(s) = \frac{s}{(s^2+1)^2}$$

table $\boxed{x(t) = \frac{t}{2}\sin t}$

Convolution formula (b)

$x(t) = (\cos * \sin)(t)$

$$= \int_0^t (\cos\tau)\sin(t-\tau)\, d\tau$$

- function to integrate: sin(r)*cos(t-r)
- variable: r
- lower limit: 0
- upper limit: t

Definite integral: Step-by-step solution

$$\int_0^t \sin(r)\cos(t-r)\, dr = \frac{1}{2} t \sin(t)$$

Open code

- function to integrate: cos(r)*sin(t-r)
- variable: r
- lower limit: 0
- upper limit: t

Definite integral: Step-by-step solution

$$\int_0^t \cos(r)\sin(t-r)\, dr = \frac{1}{2} t \sin(t)$$

Open code

(on HW you compute some convolutions)

$$= \int_0^t (\cos\tau)[\cos t(-\sin\tau) + \sin t \cos\tau]\, d\tau \qquad \text{trig!}$$

$$= \cos t \int_0^t -\cos\tau \sin\tau\, d\tau + \sin t \int_0^t \underbrace{\cos^2\tau}_{\frac{1+\cos 2\tau}{2}}\, d\tau$$

$$= \cos t \left[-\frac{(\sin\tau)^2}{2} \right]_{\tau=0}^{t} + \sin t \left[\frac{\tau}{2} + \frac{\sin 2\tau}{4} \right]_0^t$$

$$= \cos t \left[\cancel{-\frac{\sin^2 t}{2} - 0} \right] + \sin t \left[\frac{t}{2} \right] + \cancel{\frac{\sin 2t}{4} - 0}$$

$$= \frac{t}{2} \sin t$$

$$\frac{\sin t \sin 2t}{4} = \frac{(\sin t) 2 \sin t \cos t}{4} = \frac{1}{2} \sin^2 t \cos t$$

<u>Exercise 4</u> The solution $x(t)$ to

$$x'' + a\,x' + b\,x = f(t)$$
$$x(0) = 0$$
$$x'(0) = 0$$

is given by

$$x(t) = f * w(t) = w * f(t) = \int_0^t w(\tau)\, f(t-\tau)\, d\tau = \int_0^t f(\tau)\, w(t-\tau)\, d\tau .$$

We worked out that the solution to our DE IVP will be

$$\begin{cases} x'' + x = f \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

$$x(t) = \int_0^t \sin(\tau)\, f(t-\tau)\, d\tau$$

Since the unforced system has a natural angular frequency $\omega_0 = 1$, we expect resonance when the forcing function has the corresponding period of $T_0 = \frac{2\pi}{w_0} = 2\pi$. We will discover that there is the possibility for resonance if the period of f is a ***multiple*** of T_0. (Also, forcing at the natural period doesn't guarantee resonance...it depends what function you force with.)

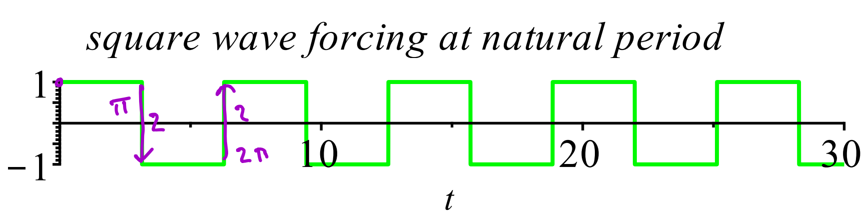
<u>Example 1</u>) A square wave forcing function with amplitude 1 and period 2π. Let's talk about how we came up with the formula (which works until $t = 11\pi$).

> *with(plots)* :

$$x(t) = \int_0^t f(\tau)\, w(t-\tau)\, d\tau$$

> $fl := t \to -1 + 2 \cdot \left(\sum_{n=0}^{10} (-1)^n \cdot \mathrm{Heaviside}(t - n \cdot \mathrm{Pi}) \right)$:
$plot1a := plot(fl(t), t = 0..30, color = green)$:
$display(plot1a, title = \text{`square wave forcing at natural period`});$

$$= -1 + 2u(0) - 2u(t-\pi) + 2u(t-2\pi) -$$

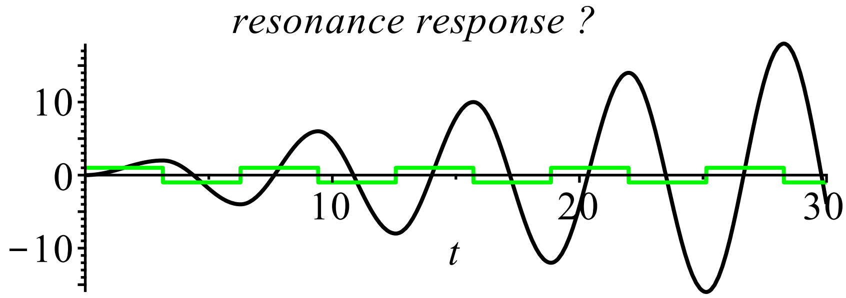


<u>1)</u> What's your vote? Is this square wave going to induce resonance, i.e. a response with linearly growing amplitude?

> $x1 := t \rightarrow \int_0^t \sin(\tau) \cdot f1(t - \tau)\, d\tau$:

$plot1b := plot(x1(t), t = 0\,..30, color = black)$:

$display(\{plot1a, plot1b\}, title = \text{`resonance response ?`})$;

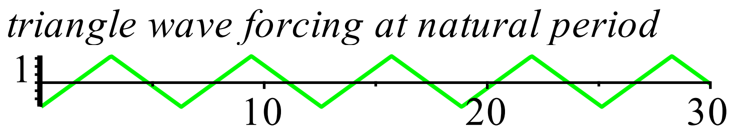


"$\pi/2$"

Example 2) A triangle wave forcing function, same period

> $f2 := t \to \int_0^t f1(s)\, ds - 1.5$: # *this antiderivative of square wave should be triangle wave*

$plot2a := plot(f2(t), t = 0..30, color = green)$:
$display(plot2a, title = \text{`triangle wave forcing at natural period`})$;

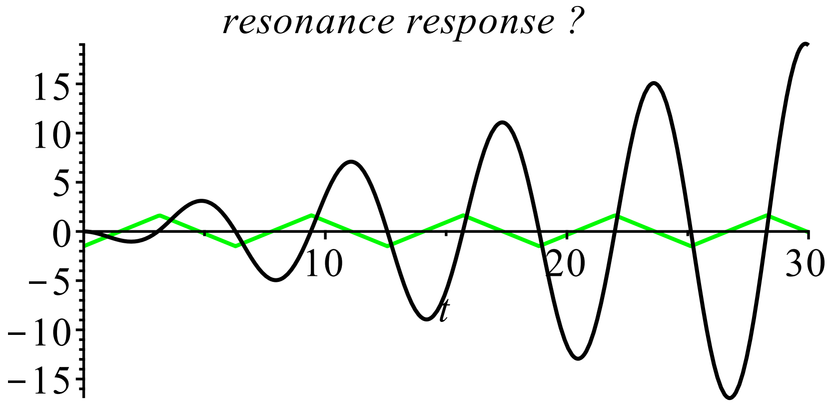


2) Resonance?

> $x2 := t \rightarrow \int_0^t \sin(\tau) \cdot f2(t-\tau)\, \mathrm{d}\tau :$

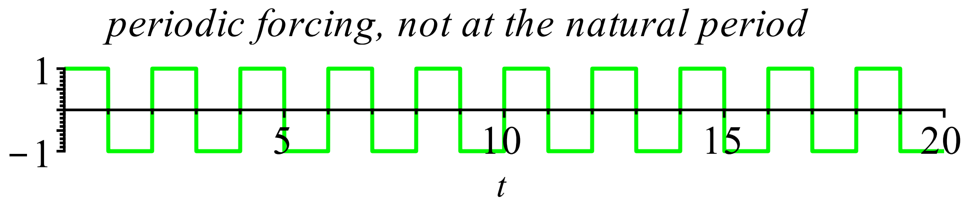
$plot2b := plot(x2(t), t = 0..30, color = black) :$

$display(\{plot2a, plot2b\}, title = \text{`resonance response ?`});$



Example 3) Forcing not at the natural period, e.g. with a square wave having period $T = 2$.

> $f3 := t \rightarrow -1 + 2 \cdot \sum_{n=0}^{20} (-1)^n \cdot \text{Heaviside}(t - n)$:
$plot3a := plot(f3(t), t = 0..20, color = green)$:
$display(plot3a, title = \text{`periodic forcing, not at the natural period`})$;

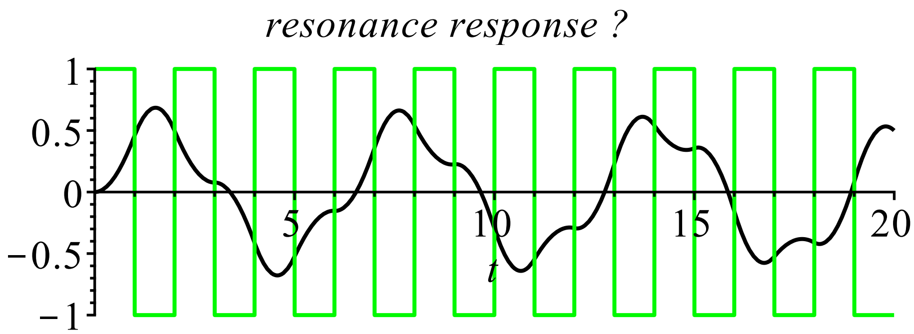


3) Resonance?

> $x3 := t \rightarrow \int_0^t \sin(\tau) \cdot f3(t - \tau)\, d\tau :$

$plot3b := plot(x3(t), t = 0..20, color = black) :$

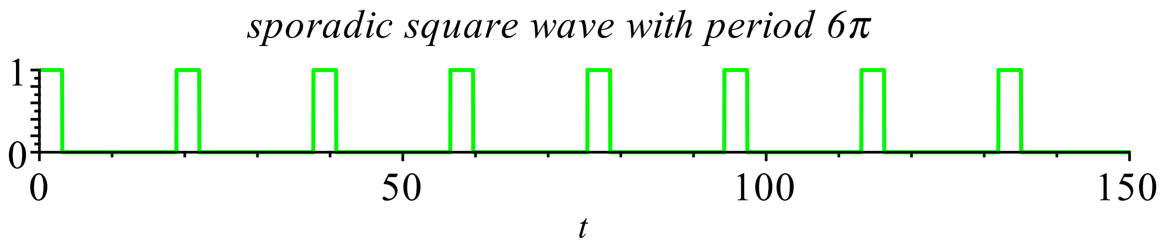
$display(\{plot3a, plot3b\}, title = \text{`resonance response ?`});$



Example 4) Forcing not at the natural period, e.g. with a particular wave having period $T = 6\pi$.

$$> f4 := t \rightarrow \sum_{n=0}^{10} \left(\text{Heaviside}(t - 6 \cdot n \cdot \pi) - \text{Heaviside}(t - (6 \cdot n + 1) \cdot \pi) \right) :$$

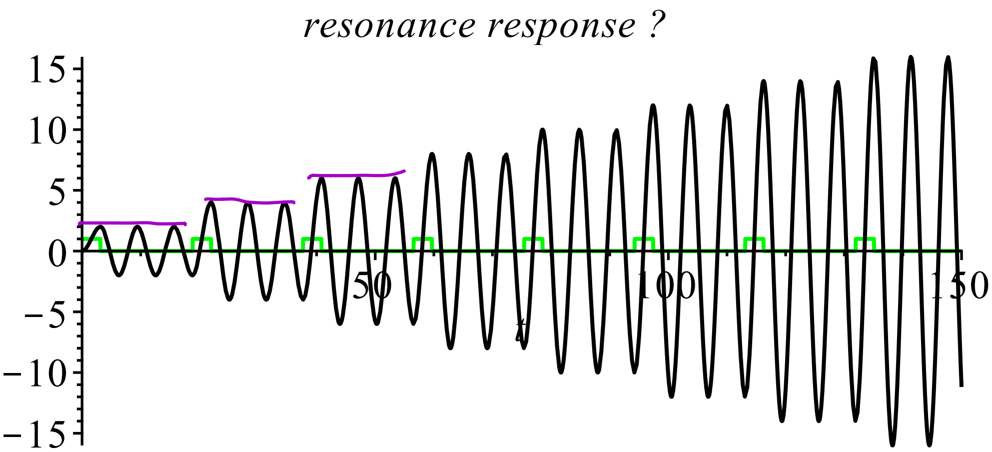
$plot4a := plot(f4(t), t = 0..150, color = green) :$
$display(plot4a, title = sporadic\ square\ wave\ with\ period\ 6\pi);$



4) Resonance?

> $x4 := t \rightarrow \int_0^t \sin(\tau) \cdot f4(t-\tau) \, \mathrm{d}\tau :$

$plot4b := plot(x4(t), t = 0 ..150, color = black) :$
$display(\{plot4a, plot4b\}, title = \text{`resonance response ?`});$



>

Hey, what happened???? How do we need to modify our thinking if we force a system with something which is not sinusoidal, in terms of worrying about resonance? In the case that this was modeling a swing (pendulum), how is it getting pushed?

<u>**Precise Answer:**</u> It turns out that any periodic function with period P is a (possibly infinite) superposition of a constant function with *cosine* and *sine* functions of periods $\left\{P, \frac{P}{2}, \frac{P}{3}, \frac{P}{4}, \ldots\right\}$. Equivalently, these functions in the superposition are

$\{1, \cos(\omega\, t), \sin(\omega\, t), \cos(2\,\omega\, t), \sin(2\,\omega\, t), \cos(3\,\omega\, t), \sin(3\,\omega\, t), \ldots\}$ with $\omega = \frac{2\cdot\pi}{P}$. This is the theory of Fourier series, which you will study in other courses, e.g. Math 3150, Partial Differential Equations. If the given periodic forcing function $f(t)$ has non-zero terms in this superposition for which $n\cdot\omega = \omega_0$ (the natural angular frequency) (equivalently $\frac{P}{n} = \frac{2\cdot\pi}{\omega_0} = T_0$), there will be resonance; otherwise, no resonance. We could already have understood some of this in Chapter 5, for example

<u>Exercise 5)</u> The natural period of the following DE is (still) $T_0 = 2\,\pi$. Notice that the period of the first forcing function below is $T = 6\,\pi$ and that the period of the second one is $T = T_0 = 2\,\pi$. Yet, it is the first DE whose solutions will exhibit resonance, not the second one. Explain, using Chapter 5 superposition ideas:
<u>a)</u>

$$x''(t) + x(t) = \cos(t) + \sin\left(\frac{t}{3}\right).$$

<u>b)</u>

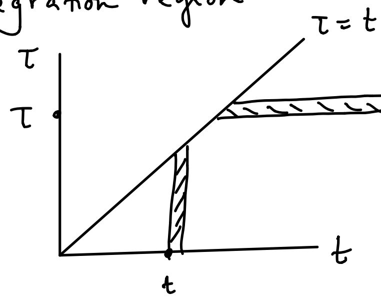
$$x''(t) + x(t) = \cos(2\,t) - 3\,\sin(3\,t).$$

proof of the convolution theorem:

$$\mathcal{L}\{f*g\}(s) = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)g(t-\tau)\,d\tau\right)dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(\tau) g(t-\tau)\,d\tau\,dt \quad (1)$$

integration region:



$0 \le \tau < \infty$ (2)
$\tau \le t < \infty$

interchange limits:

$0 \le t < \infty$
$0 \le \tau \le t$
(1)

$$= \int_0^\infty \int_\tau^\infty e^{-st} f(\tau) g(t-\tau)\,dt\,d\tau$$

$$= \int_0^\infty \int_\tau^\infty e^{-s\tau} f(\tau)\, e^{-s(t-\tau)} g(t-\tau)\,dt\,d\tau$$ pattern recognition

$$= \int_0^\infty e^{-s\tau} f(\tau)\left(\int_\tau^\infty e^{-s(t-\tau)} g(t-\tau)\,dt\right)d\tau$$

$\tilde{t} = t - \tau$
$d\tilde{t} = dt$

$$\left(\underbrace{\int_0^\infty e^{-s\tilde{t}} g(\tilde{t})\,d\tilde{t}}_{G(s)}\right)$$

$$= G(s)\int_0^\infty e^{-s\tau} f(\tau)\,d\tau$$

$$= G(s)F(s) \;!!$$