10.5 Piecewise continuous forcing functions.....e.g. turning the forcing on and off.

• The following Laplace transform material is useful in systems where we turn forcing functions on and off, and when we have right hand side "forcing functions" that are more complicated than what undetermined coefficients can handle. We will continue this discussion on Friday, with a few more table entries including "the delta (impulse) function".

$f(t)$ with $\lvert f(t)\rvert \le Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st}\,dt$ for $s > M$	comments
$u(t-a)$ unit step function	$\frac{e^{-as}}{s}$	for turning components on and off at $t = a$.
$f(t-a)\,u(t-a)$	$e^{-as}F(s)$	more complicated on/off
$\int_0^t f(t-\tau)f(\tau)\,d\tau$	$F(s)G(s)$	"convolution" for inverting products of Laplace transforms

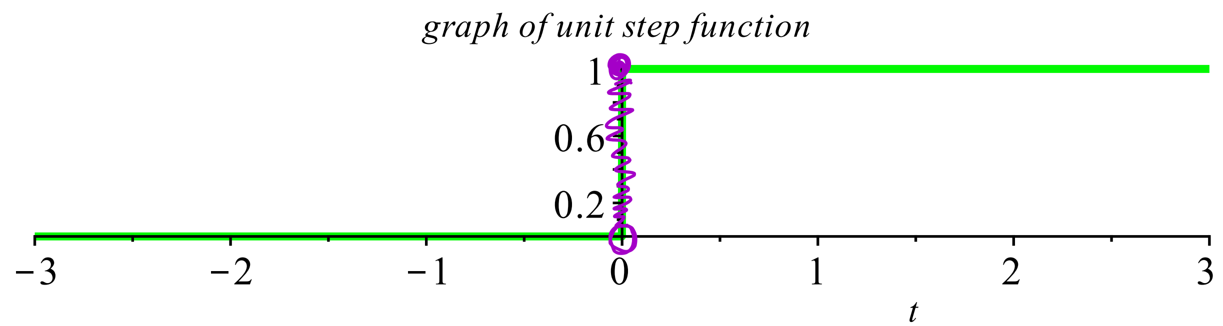
The unit step function with jump at $t = 0$ is defined to be

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \ge 0 \end{cases} .$$

IThis function is also called the "Heaviside" function, e.g. in Maple and Wolfram alpha. In Wolfram alpha it's also called the "theta" function. Oliver Heaviside was a an accomplished physicist in the 1800's. The name is not because the graph is heavy on one side. :-)

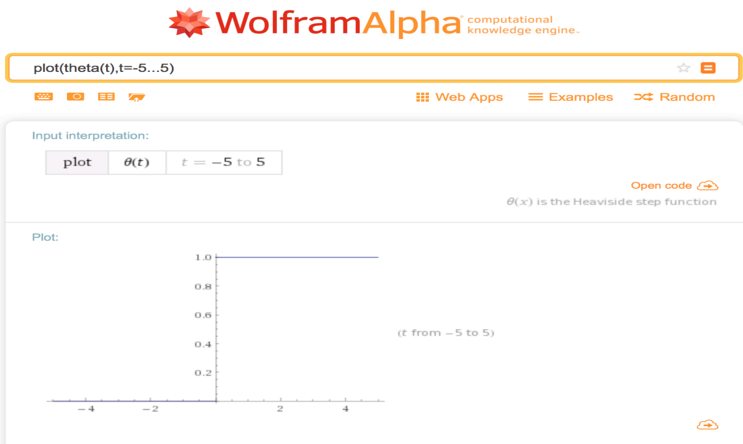
http://en.wikipedia.org/wiki/Oliver_Heaviside

```
> with(plots):
  plot(Heaviside(t), t=-3..3, color=green, title=`graph of unit step function`);
```



Notice that technically the vertical line should not be there - a more precise picture would have a solid point at $(0, 1)$ and a hollow circle at $(0, 0)$, for the graph of $u(t)$. In terms of Laplace transform integral definition it doesn't actually matter what we define $u(0)$ to be.

theta(t)



$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

Then

$$u(t-a) = \begin{cases} 0, & t-a < 0;\ i.e.\ t < a \\ 1, & t-a \geq 0;\ i.e.\ t \geq a \end{cases}$$

and has graph that is a horizontal translation by a to the right, of the original graph, e.g. for $a = 2$:

Exercise 3) Verify the table entries Friday.

$1 \cdot u(t-a)$ unit step function	$\frac{e^{-a s}}{s}$	for turning components on and off at $t = a$.
Friday $f(t-a)\, u(t-a)$	$*$ $e^{-a s}F(s)$	more complicated on/off

$$\mathcal{L}\{1 \cdot u(t-a)\} = e^{-as}\frac{1}{s}.$$

$$\mathcal{L}\{f(t-a)u(t-a)\}(s) = \int_0^\infty f(t-a)u(t-a)e^{-st}\,dt$$

$$= \underbrace{\cancel{\int_0^a 0\,dt}}_{\substack{t<a \\ u(t-a)=0}} + \underbrace{\int_{t=a}^\infty f(t-a)e^{-st}\,dt}_{\substack{t>a \\ u(t-a)=1}} = \int_{\tilde{t}=0}^\infty f(\tilde{t})e^{-s(\tilde{t}+a)}\,d\tilde{t}$$

$\tilde{t} = t-a$
$d\tilde{t} = dt$
$\tilde{t}+a = t$

$$= e^{-as}\boxed{\int_{\tilde{t}=0}^\infty f(\tilde{t})e^{-s\tilde{t}}\,d\tilde{t}}$$

$$'' F(s)$$

Setup: an under-employed mathematician/engineer/scientist
(your choice)

likes to take his/her child to the swings ...

recall pendulum (linearized) eqtn, without forcing, for $\theta = \Theta(t)$

• $L\theta'' + g\theta = 0$

$\dashrightarrow \; x'' + g\frac{x}{L} = 0$

$\dashrightarrow \; mx'' + \frac{mg}{L}x = F_0 \cos\omega t \quad \leftarrow$ parent forcing (?!)

$x(t) = L\sin\theta(t)$
$\approx L\theta$ for small θ

so $x'' \approx L\theta''$

$\dashrightarrow \; x'' + \frac{g}{L}x = \frac{F_0}{m}\cos\omega t$ •

$x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$

parent pushes sinusoidally for exactly 5 cycles, and with $\frac{F_0}{m} = .2$ and then releases:

for resonance $\omega = \omega_0$
construct swing with $L = g \approx 9.8$ m.
so $\omega_0^2 = 1$, $T_0 = 2\pi \approx 6.2$ seconds

Exercise 5a) Explain why the description above leads to the differential equation initial value problem for $x(t)$

$\mathcal{L}$.

$$x''(t) + x(t) = .2\cos(t)\,(1 - u(t - 10\pi)) \quad *$$
$$x(0) = 0$$
$$x'(0) = 0$$

5b) Find $x(t)$. Show that after the parent stops pushing, the child is oscillating with an amplitude of exactly π meters (in our linearized model).

$$F(t) = \begin{cases} \overset{.2}{\cancel{\tfrac{F_0}{m}}} \cos t & 0 \le t \le 10\pi \\ 0 & t \ge 10\pi \end{cases}$$

$$= .2\cos t - u(t-10\pi)\underbrace{(.2\cos t)}_{\text{luck}} \qquad .2\cos(t-10\pi)$$

$$s^2X(s) - s\cdot 0 - 0 + X(s) = .2\frac{s}{s^2+1} - .2u(t-10\pi)\cos(t-10\pi)$$

$$.2e^{-10\pi s}\frac{s}{s^2+1}$$

$$X(s)(s^2+1) = .2\frac{s}{s^2+1} - .2e^{-10\pi s}\frac{s}{s^2+1}$$

$$X(s) = .2\frac{s}{(s^2+1)^2} - .2e^{-10\pi s}\frac{s}{(s^2+1)^2} \qquad f(t) = \frac{t}{2}\sin t$$

$$x(t) = .2\,\frac{t}{2}\sin t - .2u(t-10\pi)\left(\frac{t-10\pi}{2}\right)\sin(t-10\pi)$$

$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$\cos kt$	$\frac{s}{s^2+k^2}$
• $u(t-a)f(t-a)$	$e^{-as}F(s)$
$\frac{t}{2k}\sin kt$	$\frac{s}{(s^2+k^2)^2}$

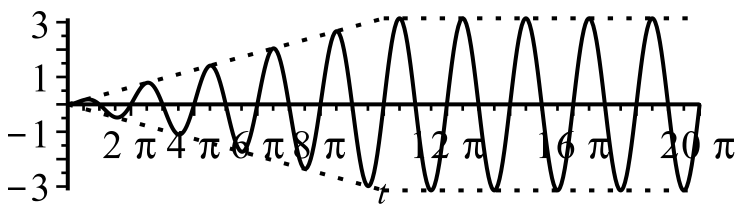
$$x(t) = \begin{cases} .1t\sin t & 0 \le t < 10\pi \\ .1t\cancel{\sin t} - .1(1)(\cancel{t} - 10\pi)\sin t & t \ge 10\pi \end{cases}$$

$= \pi \sin \pi t$

Pictures for the swing:

```
> plot1 := plot(.1·t·sin(t), t = 0..10·Pi, color = black) :
  plot2 := plot(Pi·sin(t), t = 10·Pi..20·Pi, color = black) :
  plot3 := plot(Pi, t = 10·Pi..20·Pi, color = black, linestyle = 2) :
  plot4 := plot(-Pi, t = 10·Pi..20·Pi, color = black, linestyle = 2) :
  plot5 := plot(.1·t, t = 0..10·Pi, color = black, linestyle = 2) :
  plot6 := plot(-.1·t, t = 0..10·Pi, color = black, linestyle = 2) :
  display({plot1, plot2, plot3, plot4, plot5, plot6}, title = `adventures at the swingset`);
```

adventures at the swingset



Alternate approach via Chapter 5:

<u>step 1)</u> solve

$$x''(t) + x(t) = .2\cos(t)$$
$$x(0) = 0$$
$$x'(0) = 0$$

for $0 \le t \le 10\pi$.

<u>step 2)</u> Then solve

$$y''(t) + y(t) = 0$$
$$y(10\pi)\ [redacted]) = x(10\pi)$$
$$y'(10\pi)\ [redacted]) = x'(10\pi)$$

[redacted] .

for $t \geq 10\pi$

$f(t)$, with $\lvert f(t)\rvert \le Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st}\,dt$ for $s > M$	↓ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	☐
1	$\frac{1}{s}$ $(s > 0)$	☐
t	$\frac{1}{s^2}$	☐
t^2	$\frac{2}{s^3}$	☐
t^n, $n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$	☐
$e^{\alpha t}$	$\frac{1}{s-\alpha}$ $(s > \Re(a))$	☐
$\cos(k\,t)$	$\frac{s}{s^2+k^2}$ $(s > 0)$	☐
$\sin(k\,t)$	$\frac{k}{s^2+k^2}$ $(s > 0)$	☐
$\cosh(k\,t)$	$\frac{s}{s^2-k^2}$ $(s > k)$	☐
		☐
$\sinh(k\,t)$	$\frac{k}{s^2-k^2}$ $(s > k)$	☐
$e^{a\,t}\cos(k\,t)$	$\frac{(s-a)}{(s-a)^2+k^2}$ $(s > a)$	☐
$e^{a\,t}\sin(k\,t)$	$\frac{k}{(s-a)^2+k^2}$ $(s > a)$	☐
$e^{a\,t} f(t)$	$F(s-a)$	☐
$u(t-a)$	$\frac{e^{-a\,s}}{s}$	
$f(t-a)\,u(t-a)$	$e^{-a\,s}F(s)$	
$\delta(t-a)$	$e^{-a\,s}$	
$f'(t)$	$s\,F(s) - f(0)$	☐
$f''(t)$	$s^2F(s) - s\,f(0) - f'(0)$	☐
$f^{(n)}(t)$, $n \in \mathbb{N}$	$s^n\,F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$	☐

$\int_0^t f(\tau)\, d\tau$	$\frac{F(s)}{s}$	☐
$t\, f(t)$ $t^2\, f(t)$ $t^n\, f(t),\ n \in \mathbb{Z}$ $\frac{f(t)}{t}$	$-F'(s)$ $F''(s)$ $(-1)^n\, F^{(n)}(s)$ $\int_s^\infty F(\sigma)\, d\sigma$	☐ ☐ ☐
$t\cos(k\,t)$ $\frac{1}{2\,k}\, t\sin(k\,t)$ $\frac{1}{2\,k^3}\,(\sin(k\,t) - k\,t\cos(k\,t))$ $t\, e^{a\,t}$ $t^n\, e^{a\,t},\ n \in \mathbb{Z}$	$\frac{s^2 - k^2}{(s^2 + k^2)^2}$ $\frac{s}{(s^2 + k^2)^2}$ $\frac{1}{(s^2 + k^2)^2}$ $\frac{1}{(s-a)^2}$ $\frac{n!}{(s-a)^{n+1}}$	☐ ☐ ☐ ☐ ☐
$\int_0^t f(\tau) g(t-\tau)\, d\tau$	$F(s)G(s)$	
$f(t)$ with period p	$\frac{1}{1 - e^{-ps}} \int_0^p f(t) e^{-s\,t}\, dt$	

Laplace transform table

Math 2250-4
Fri Apr 6
10.5, EP7.6 piecewise and implulse forcing. • beyond Chapter 5

Monday: "convolution" table entry.
we get a formula for the soltn to
$$mx'' + cx' + kx = F(t)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$
ANY $F(t)$

"Duhamel's Principal"
wikipedia

Announcements:
• Wed notes 1st
• Swing

'til 10:46

Warm-up Exercise: Let
$$f(t) = \begin{cases} 0 & 0 \le t < a \\ g(t) & a \le t < b \\ 0 & t \ge b \end{cases}$$
defined piecewise. Use translated unit step functions to express $f(t)$ by a single formula for $t \ge 0$.

Use $u(t-c) = \begin{cases} 0 & 0 \le t < c \\ 1 & t \ge c \end{cases}$

$$f(t) = g(t)u(t-a) - g(t)u(t-b) = g(t)\left(u(t-a) - u(t-b)\right)$$
$$= \begin{cases} 0 - 0 & t < a \\ g(t)\cdot 1 - 0 & a \le t < b \\ g(t)\cdot 1 - g(t)\cdot 1 = 0 & t \ge b \end{cases}$$
✓

see Lab

Laplace table entries for today.

$f(t)$ with $\lvert f(t)\rvert \le Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st}\,dt$ for $s > M$	comments
$u(t-a)$ unit step function	$\dfrac{e^{-as}}{s}$	for turning components on and off at $t = a$.
$f(t-a)\ u(t-a)$	$e^{-as}F(s)$	more complicated on/off
$\delta(t-a)$	e^{-as}	unit impulse/delta "function"

EP 7.6 impulse functions and the δ operator.

Consider a force $f(t)$ acting on an object for only on a very short time interval $a \le t \le a+\varepsilon$, for example as when a bat hits a ball. This impulse p of the force is defined to be the integral

$$p := \int_a^{a+\varepsilon} f(t)\,dt$$

and it measures the net change in momentum of the object since by Newton's second law

$$m\,v'(t) = f(t)$$

$$\Rightarrow \int_a^{a+\varepsilon} m\,v'(t)\,dt = \int_a^{a+\varepsilon} f(t)\,dt = p$$

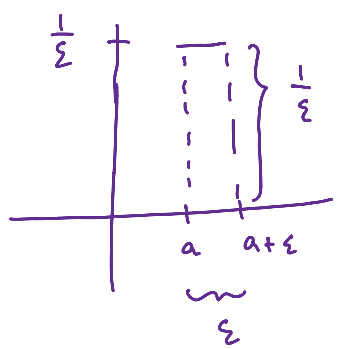
$$\Rightarrow m\,v(t)\Big]_{t=a}^{a+\varepsilon} = p\,.$$

Since the impulse p only depends on the integral of $f(t)$, and since the exact form of f is unlikely to be known in any case, the easiest model is to replace f with a constant force having the same total impulse, i.e. to set

$$f = p\,d_{a,\varepsilon}(t)$$

where $d_{a,\varepsilon}(t)$ is the unit impulse function given by

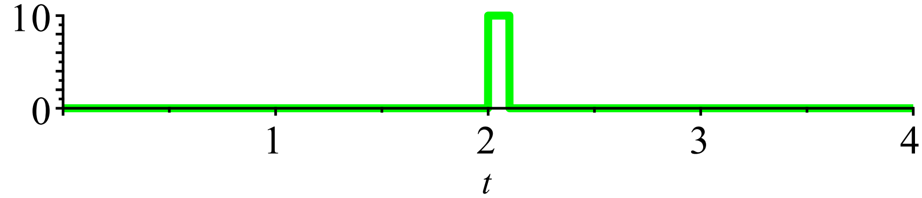
$$d_{a,\varepsilon}(t) = \begin{cases} 0, & t < a \\ \dfrac{1}{\varepsilon}, & a \le t < a+\varepsilon \\ 0, & t \ge a+\varepsilon \end{cases}.$$



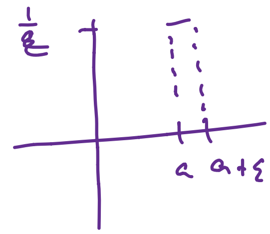
Notice that

$$\int_a^{a+\varepsilon} d_{a,\varepsilon}(t)\,dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\,dt = 1\,.$$

Here's a graph of $d_{2,\,.1}(t)$, for example:


10
0
1
2
3
4
t

Since the unit impulse function is a linear combination of unit step functions, we could solve differential equations with impulse functions so-constructed. As far as Laplace transform goes, it's even easier to take the limit as $\varepsilon \to 0$ for the Laplace transforms $\mathcal{L}\{d_{a,\varepsilon}(t)\}(s)$, and this effectively models impulses on very short time scales.



$$d_{a,\varepsilon}(t) = \frac{1}{\varepsilon}\left[u(t-a) - u(t-(a+\varepsilon))\right]$$

$$\Rightarrow \mathcal{L}\{d_{a,\varepsilon}(t)\}(s) = \frac{1}{\varepsilon}\left(\frac{e^{-a s}}{s} - \frac{e^{-(a+\varepsilon)s}}{s}\right)$$

$$= e^{-a s}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right).$$

$u(t-a)$ | $\frac{e^{-as}}{s}$

In Laplace land we can use L'Hopital's rule (in the variable ε) to take the limit as $\varepsilon \to 0$:

$$\lim_{\varepsilon \to 0} e^{-a s}\underbrace{\left(\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right)} = e^{-a s}\lim_{\varepsilon \to 0}\left(\frac{s\, e^{-\varepsilon s}}{s}\right) = e^{-a s}.$$

The result in time t space is not really a function but we call it the "delta function" $\delta(t-a)$ anyways, and visualize it as a function that is zero everywhere except at $t=a$, and that it is infinite at $t=a$ in such a way that its integral over any open interval containing a equals one. As explained in EP7.6, the delta "function" can be thought of in a rigorous way as a linear transformation, not as a function. It can also be thought of as the derivative of the unit step function $u(t-a)$, and this is consistent with the Laplace table entries for derivatives of functions. In any case, this leads to the very useful Laplace transform table entry

$\delta(t-a)$ unit impulse function	$e^{-a s}$	for impulse forcing

Exercise 1) Revisit the swing from Wednesday's notes and solve the IVP below for $x(t)$. In this case the parent is providing an impulse each time the child passes through equilibrium position after completing a cycle.

$$x''(t) + x(t) = .2\pi[\delta(t) + \delta(t-2\pi) + \delta(t-4\pi) + \delta(t-6\pi) + \delta(t-8\pi)]$$
$$x(0) = 0$$
$$x'(0) = 0.$$

$\mathcal{L}$: $s^2X(s) + X(s) = .2\pi\left[1 + e^{-2\pi s} + e^{-4\pi s} + e^{-6\pi s} + e^{-8\pi s}\right]$

$X(s)(s^2+1)$

$\delta(t-a)$	e^{-as}

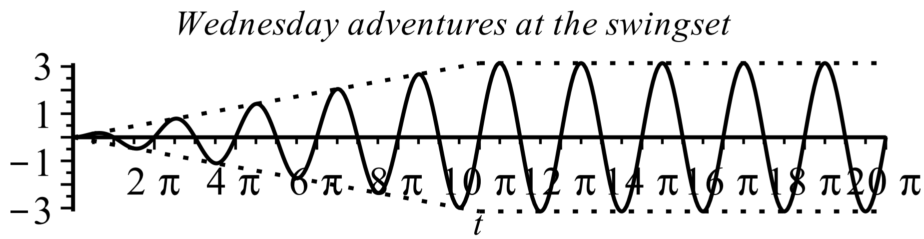
$$X(s) = .2\pi\left[\frac{1}{s^2+1} + e^{-2\pi s}\cdot\frac{1}{s^2+1} + e^{-4\pi s}\cdot\frac{1}{s^2+1} + e^{-6\pi s}\cdot\frac{1}{s^2+1} + e^{-8\pi s}\cdot\frac{1}{s^2+1}\right]$$

$$x(t) = .2\pi\left[\sin t + u(t-2\pi)\sin(t-2\pi) + u(t-4\pi)\sin(t-4\pi) + \dots\right]$$

$\sin kt$	$\frac{k}{s^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$
$f(t)=\sin t$	

$$x(t) = \begin{cases} .2\pi\sin t & 0 \le t < 2\pi \\ .4\pi\sin t & 2\pi \le t < 4\pi \\ .6\pi\sin t & 4\pi \le t < 6\pi \\ .8\pi\sin t & 6\pi \le t < 8\pi \\ \pi\sin t & t \ge 8\pi \end{cases}$$

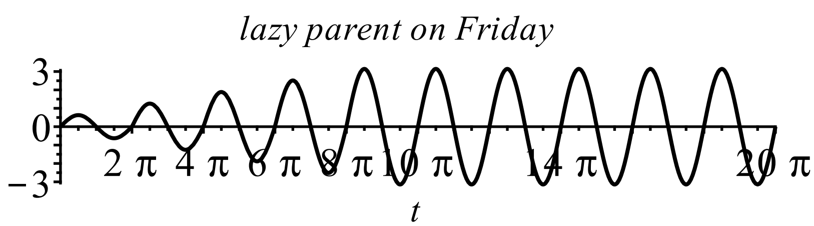
```
> with(plots) :
> plot1 := plot(.1·t·sin(t), t = 0 ..10·Pi, color = black) :
  plot2 := plot(Pi·sin(t), t = 10·Pi ..20·Pi, color = black) :
  plot3 := plot(Pi, t = 10·Pi ..20·Pi, color = black, linestyle = 2) :
  plot4 := plot(−Pi, t = 10·Pi ..20·Pi, color = black, linestyle = 2) :
  plot5 := plot(.1·t, t = 0 ..10·Pi, color = black, linestyle = 2) :
  plot6 := plot(−.1·t, t = 0 ..10·Pi, color = black, linestyle = 2) :
  display({plot1, plot2, plot3, plot4, plot5, plot6}, title = `Wednesday adventures at the swingset`);
```




```
>
```

impulse solution: five equal impulses to get same final amplitude of π meters - Exercise 1:

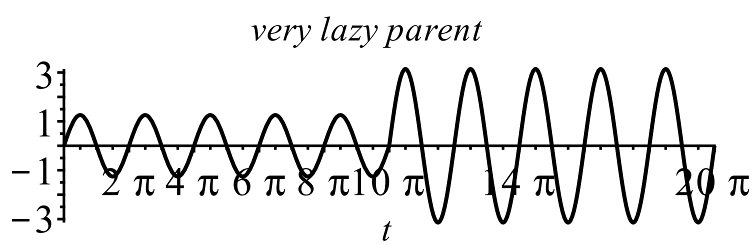
```
> f := t → .2·Pi·sum(Heaviside(t − k·2·Pi)·sin(t − k·2·Pi), k = 0 ..4) :
> plot(f(t), t = 0 ..20·Pi, color = black, title = `lazy parent on Friday`);
```




```
>
```

Or, an impulse at $t = 0$ and another one at $t = 10\pi$.

```
> g := t → .2·Pi·(2·sin(t) + 3·Heaviside(t − 10·Pi)·sin(t − 10·Pi)) :
> plot(g(t), t = 0 ..20·Pi, color = black, title = `very lazy parent`);
```




```
>
```