$f(t)$	$F(s)$
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
1	$1/s$

$k=2$

Exercise 5) Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\}(t)$

a) using the result of 4d.

b) using partial fractions.

return here on Tuesday

$F(s) = \frac{1}{s^2+4}$

$\frac{F(s)}{s} = \frac{1}{s(s^2+4)}$

$f(t) = \mathcal{L}^{-1}\{F(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{2}{s^2+4}\right\}(t) = \frac{1}{2}\sin 2t$

$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\}(t) = \int_0^t \frac{1}{2}\sin 2\tau\, d\tau$

$= -\frac{1}{4}\cos 2\tau\Big]_0^t = \boxed{-\frac{1}{4}\cos 2t + \frac{1}{4}}$

b) $\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$

with resp. to common denom $s(s^2+4)$, equate numerators

$1 = A(s^2+4) + s(Bs+C)$

$1 + 0s + 0s^2$

$= 4A + Cs + (A+B)s^2$

so,

$$\left.\begin{aligned} 1 &= 4A \\ 0 &= C \\ 0 &= A+B \end{aligned}\right\} \begin{aligned} A &= \tfrac{1}{4} \\ B &= -\tfrac{1}{4} \\ C &= 0 \end{aligned} \Rightarrow \frac{1}{s(s^2+4)} = \frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4}$$

$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\}(t)$

$\boxed{= \frac{1}{4} - \frac{1}{4}\cos 2t}$

Aside. ~~partial frac.~~

$\frac{1}{s^2(s^2+4)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4} + \frac{Es+F}{(s^2+4)^2}$

fact: leads to linear syst for A, B, C, D, E, F with unique soltn.

Exercise 6) Show $\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}$, $n \in \mathbb{N}$, using the results of 4, namely

$\mathcal{L}\{\underbrace{f^{(n)}(t)}_{n!}\}(s) = s^n F(s) - \cancel{s^{n-1} f(0)} - \cancel{s^{n-2} f'(0)} - \dots - \cancel{f^{(n-1)}}(0)$, $n \in \mathbb{N}$.

$\mathcal{L}\{n!\}(s) = \frac{n!}{s} = s^n F(s)$

$\frac{n!}{s^{n+1}} = F(s)$

$\mathcal{L}\{1\}(s) = \frac{1}{s} = \frac{0!}{s}$

$\mathcal{L}\{t\}(s)$

$f(t) = t$

$f'(t) = 1$

$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\} - f(0)$

$\mathcal{L}\{1\}(s) = s\mathcal{L}\{t\}(s) - 0$

$\frac{1}{s} = s\mathcal{L}\{t\}(s)$

$\frac{1}{s^2} = \mathcal{L}\{t\}(s)$

- $f(t) = t^2$

$f''(t) = 2. \Rightarrow \mathcal{L}\{2\}(s) = s^2\mathcal{L}\{t^2\}(s) - t\cdot 0 - 0$

$\frac{2}{s^3} = \mathcal{L}\{t^2\}(s)$

$f(t)$	$F(s)$
1	$1/s$
• $f'(t)$	$sF(s) - f(0)$
• $f''(t)$	$s^2F(s) - tf(0) - f'(0)$
t	$\frac{1}{s^2}$
t^2	$\frac{2}{s^3}$
t^n	$\frac{n!}{s^{n+1}}$

Math 2250-004
Tues Apr 3

10.2-10.3 Laplace transform, and application to DE IVPs, including Chapter 5.
Today we'll continue to fill in the Laplace transform table, and to use the table entries to solve linear differential equations. One focus today will be to review partial fractions, since the table entries are set up precisely to show the inverse Laplace transforms of the components of partial fraction decompositions.

Announcements:
- post exam soltns
- check the addition on your score
- HW due in lab
- quiz tomorrow will be solving an IVP with Laplace.
- save last 10-15 minutes for pendulum

'til 10:46

Warm-up Exercise: Use Laplace transforms to solve one of our favorite sort of IVPs from Chapters 1-2, for $x(t)$:

$$\begin{cases} x' + 3x = 6 \\ x(0) = 4 \end{cases}$$

	$f(t)$	$F(s)$
*	e^{at}	$\frac{1}{s-a}$
	1	$1/s$
*	$f'(t)$	$sF(s) - f(0)$
	$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$

$\mathcal{L}$: $sX(s) - 4 + 3X(s) = \frac{6}{s}$

$X(s)[s+3] = \frac{6}{s} + 4$

$X(s) = \frac{6}{s(s+3)} + \frac{4}{s+3}$

do not combine!
you want to "decombine" !!

$X(s) = 6\,\frac{1}{s(s+3)} + \frac{4}{s+3}$

$= 6\left[\frac{1}{3}\left(\frac{1}{s} - \frac{1}{s+3}\right)\right] + \frac{4}{s+3}$

$\frac{1}{s} - \frac{1}{s+3} = \frac{\cancel{s}+3-\cancel{s}}{s(s+3)}$

$X(s) = \frac{2}{s} - \frac{2}{s+3} + \frac{4}{s+3}$

$X(s) = \frac{2}{s} + \frac{2}{s+3}$

$\mathcal{L}^{-1}$: $x(t) = \underbrace{2}_{x_p} + \underbrace{2e^{-3t}}_{x_H}$

Chptr 5: for $x(0) = 4$

$$F(s) = \mathcal{L}\{f(t)\}(s) := \int_0^\infty f(t)e^{-st}\,dt.$$

Exercise 1) Check why this table entry is true - notice that it generalizes how the Laplace transforms of $\cos(kt)$, $\sin(kt)$ are related to those of $e^{at}\cos(kt)$, $e^{at}\sin(kt)$:

$e^{at}f(t)$	$F(s-a)$

$$\mathcal{L}\{e^{at}f(t)\}(s) = \int_0^\infty e^{at}f(t)\,e^{-st}\,dt$$

$$= \int_0^\infty f(t)\,e^{(a-s)t}\,dt$$

$$= \int_0^\infty f(t)\,e^{-(s-a)t}\,dt$$

$$\mathcal{L}\{f(t)\}(s-a)$$

Exercise 2) Use the table entry

$t^n,\ n \in \mathbb{Z}$	$\dfrac{n!}{s^{n+1}}$

and Exercise 1 to get the table entry

$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$\cos kt$	$\dfrac{s}{s^2+k^2}$
$\sin kt$	$\dfrac{k}{s^2+k^2}$
$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$ ✓
$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$ ✓

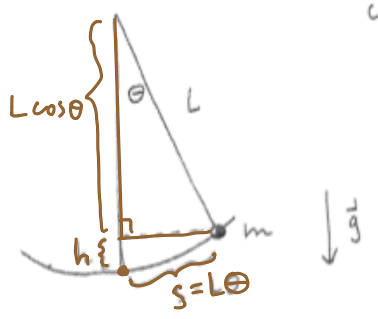
$f(t)$, with $\lvert f(t)\rvert \le Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st}\,dt$ for $s > M$	↓ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	☐
1 t t^2 $t^n,\ n \in \mathbb{N}$	$\frac{1}{s} \quad (s > 0)$ $\frac{1}{s^2}$ $\frac{2}{s^3}$ $\frac{n!}{s^{n+1}}$	☐
$e^{\alpha t}$	$\frac{1}{s-\alpha} \quad (s > \Re(a))$	☐
$\cos(kt)$ $\sin(kt)$ $\cosh(kt)$ $\sinh(kt)$ $e^{at}\cos(kt)$ $e^{at}\sin(kt)$ $e^{at} f(t)$	$\frac{s}{s^2+k^2} \quad (s > 0)$ $\frac{k}{s^2+k^2} \quad (s > 0)$ $\frac{s}{s^2-k^2} \quad (s > k)$ $\frac{k}{s^2-k^2} \quad (s > k)$ $\frac{(s-a)}{(s-a)^2+k^2} \quad (s > a)$ $\frac{k}{(s-a)^2+k^2} \quad (s > a)$ $F(s-a)$	☐ ☐ ☐ ☐ ☐ ☐
$f'(t)$ $f''(t)$ $f^{(n)}(t),\ n \in \mathbb{N}$ $\int_0^t f(\tau)\,d\tau$	$s\,F(s) - f(0)$ $s^2F(s) - s\,f(0) - f'(0)$ $s^n\,F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$ $\frac{F(s)}{s}$	☐ ☐ ☐ ☐
$t\,f(t)$ $t^2\,f(t)$ $t^n\,f(t),\ n \in \mathbb{Z}$ $\frac{f(t)}{t}$	$-F'(s)$ $F''(s)$ $(-1)^n\,F^{(n)}(s)$ $\int_s^\infty F(\sigma)\,d\sigma$	
$t\cos(kt)$	$\frac{s^2-k^2}{(s^2+k^2)^2}$	

$\frac{1}{2\,k}\,t\,\sin(k\,t)$	$\frac{s}{(s^2+k^2)^2}$	
$\frac{1}{2\,k^3}\,(\sin(k\,t)-k\,t\,\cos(k\,t))$	$\frac{1}{(s^2+k^2)^2}$	
$t\,e^{a\,t}$	$\frac{1}{(s-a)^2}$	
$t^n\,e^{a\,t},\ n\in\mathbb{Z}$	$\frac{n!}{(s-a)^{n+1}}$	

Laplace transform table

The pendulum application that we didn't cover carefully in Chapter 5....we'll use this for a sequence of examples using Laplace transform over the next several lectures.

1) pendulum



$\theta = \theta(t)$

conservative system $KE + PE = \text{const.}$ (once you set sys into motion)

$\frac{1}{2} m v^2 + mgh = \text{const}$

$s = L\theta$

$v = \frac{ds}{dt} = L\theta'(t)$ $\quad h = L - L\cos\theta = L(1-\cos\theta)$

so, $\frac{1}{2} m \underbrace{L^2(\theta'(t))^2}_{v^2} + mg\underbrace{L(1-\cos(\theta(t)))}_{h} \equiv \text{const}$

D_t: $mL^2\theta'\theta'' + mgL(\sin\theta)\theta' \equiv 0$

$mL\theta'(L\theta'' + g\sin\theta) \equiv 0$

$\neq 0$ except at isolated times

$\leadsto$ deduce eqtn of motion is

$$\boxed{\theta'' + \frac{g}{L}\sin\theta = 0}$$

linearize

$$\boxed{\theta'' + \frac{g}{L}\theta = 0}$$

$\omega_0 = \sqrt{\frac{g}{L}}$

$\theta(t) = C\cos(\omega_0 t - \alpha)$

$T = \frac{2\pi}{\omega_0}$

non-linear DE

but $\sin\theta = \theta - \frac{\theta^3}{3!} + \ldots$

$\sin\theta \approx \theta$ θ small is excellent approx (alternating series test)

$|\theta| < .1$ $\quad |\sin\theta - \theta| < \frac{.001}{6}$

$< .0002$

measured:

$L = 1.52$ m

$g = 9.806$

$\omega_0 = \sqrt{\frac{g}{L}} = 2.54$

$T = \frac{2\pi}{\omega_0} = 2.474$ sec.

experiment: 2.48 sec

(we measured time for 10 cycles and took the average)