Math 2250-004
Tuesday, April 24
Course review

• you may hand in labs & hw now or later

Final exam: Friday April 27, 10:30 a.m. -12:30 p.m. This is the official University time and location – in our MWF lecture room WEB L105. (I will let you start working at 10:15 and stay until 12:45 if you wish and if the room is available.) As usual the exam is closed book and closed note, and the only sort of calculator that is allowed is a simple scientific one. Your cell phones need to be put away for the entire exam. You will be provided the Laplace Transform table from the front cover of our text. The algebra and math on the exam should all be doable by hand.

12-4 today, 2-4 Thursday JWB 332

Wed 2-4 LCB loft, 4th floor

Review of previous final exam practice exams: Dihan Dai and Jose Yanez will announce their office hours and possible review sessions. Two old final exams and solutions are posted in CANVAS. I will hold my usual Tuesday office hours in JWB 240, 4:30-6:00 p.m. I will go over an old final exam on Wednesday April 25, 10:30-12:30, in JWB 335.

The final exam will be comprehensive, but weighted to more recent material. Rough percentage ranges per chapter are below – these percentage ranges add up to more than 100% because many topics and problems recur in several chapters. Also consult the "course learning objectives" from our syllabus (next page).

Chapters

- 1-2: 10-20% first order DEs
- 3-4: 20-40% matrix algebra and vector spaces
- 5: 15-30% linear differential equations and applications
- 6.1-6.2: 15-30% eigenvalues and eigenvectors, including complex case
- 7.1-7.4: 20-40% linear systems of differential equations and applications
- 10.1-10.5, EP 7.6: 20-40% Laplace transform

On the 3rd page, repeated on page 4, is a detailed list of the topics we've investigated this semester. They are more inter-related than you may have realized at the time, so we'll discuss the connections in class. Then we'll look at an extended problem that highlights these connections and connects perhaps 70% of the key ideas in this course. We won't have time to work out everything in class, but filling in the details might help you consolidate these ideas.

Learning Objectives for 2250

The goal of Math 2250 is to master the basic tools and problem solving techniques important in differential equations and linear algebra. These basic tools and problem solving skills are described below.

The essential topics

Be able to model dynamical systems that arise in science and engineering, by using general principles to derive the governing differential equations or systems of differential equations. These principles include linearization, compartmental analysis, Newton's laws, conservation of energy and ~~Kirchoff's law~~.

Learn solution techniques for first order separable and linear differential equations. Solve initial value problems in these cases, with applications to problems in science and engineering. Understand how to approximate solutions even when exact formulas do not exist. Visualize solution graphs and numerical approximations to initial value problems via slope fields. Understand phase diagram analysis for autonomous first order differential equations.

Become fluent in matrix algebra techniques, in order to be able to compute the solution space to linear systems and understand its structure; by hand for small problems and with technology for large problems.

Be able to use the basic concepts of linear algebra such as linear combinations, span, independence, basis and dimension, to understand the solution space to linear equations, linear differential equations, and linear systems of differential equations.

Understand the natural initial value problems for first order systems of differential equations, and how they encompass the natural initial value problems for higher order differential equations and general systems of differential equations.

Learn how to solve constant coefficient linear differential equations via superposition, particular solutions, and homogeneous solutions found via characteristic equation analysis. Apply these techniques to understand the solutions to the basic unforced and forced mechanical and ~~electrical oscillation~~ problems.

Learn how to use Laplace transform techniques to solve linear differential equations, with an emphasis on the initial value problems of mechanical systems, electrical circuits, and related problems.

Be able to find eigenvalues and eigenvectors for square matrices. Apply these matrix algebra concepts to find the general solution space to first and second order constant coefficient homogeneous linear systems of differential equations, especially those arising from compartmental analysis and mechanical systems.

not really

Understand and be able to use linearization as a technique to understand the behavior of nonlinear dynamical systems near equilibrium solutions. Apply these techniques to non-linear mechanical oscillation problems. (Additional material, subject to time availability: Apply linearization to autonomous systems of two first order differential equations, including interacting populations. Relate the phase portraits of non-linear systems near equilibria to the linearized data, in particular to understand stability.)

Develop your ability to communicate modeling and mathematical explanations and solutions, using technology and software such as Maple, Matlab or internet-based tools as appropriate.

Problem solving fluency

Students will be able to read and understand problem descriptions, then be able to formulate equations modeling the problem usually by applying geometric or physical principles. Solving a problem often requires specific solution methods listed above. Students will be able to select the appropriate operations, execute them accurately, and interpret the results using numerical and graphical computational aids.

Students will also gain experience with problem solving in groups. Students should be able to effectively transform problem objectives into appropriate problem solving methods through collaborative discussion. Students will also learn how to articulate questions effectively with both the instructor and TA, and be able to effectively convey how problem solutions meet the problem objectives.

Green = existence, uniqueness
Blue = vector space concepts & applications to DE's.

1-2: first order DEs
- slope fields, Euler approximation
- phase diagrams for autonomous DEs
  - equilibrium solutions
  - stability
- existence-uniqueness thm for IVPs
- methods:
  - separable
  - linear
- applications
  - populations
  - velocity-acceleration models
  - input-output models

$\begin{cases} x'(t) = f(t,x) \\ x(0) = x_0 \end{cases}$ scalar IVP

3-4 matrix algebra and vector spaces
- linear systems and matrices
- reduced row echelon form
- matrix and vector algebra
  - manipulating and solving matrix-vector equations for unknown vectors or matrices.
- matrix inverses
- determinants
- vector space concepts
  - vector spaces and subspaces
  - linear combinations
  - linear dependence/independence
  - span
  - basis and dimension
  - linear transformations
    - aka superposition
    - fundamental theorem for solution space to L(y)=f when L is linear

vector version
$\begin{cases} \vec{x}'(t) = \vec{f}(t,\vec{x}) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$

nth order DE IVP's are equivalent to ones for 1st order systems

5 Linear differential equations
- IVP existence and uniqueness
- Linear DEs
  - Homogeneous solution space, its dimension, and why
  - superposition, $\underline{x}(t) = \underline{x}_P + \underline{x}_H$
- Constant coefficient linear DEs
  - $\underline{x}_H$ via characteristic polynomial
    - Euler's formula, complex roots
  - $\underline{x}_P$ via undetermined coefficients
  - solving IVPs

5: applications:
- mechanical configurations
  - unforced: undamped and damped
    - cos and sin addition angle formulas and amplitude-phase form
  - forced undamped: beating, resonance
  - forced damped: $\underline{\mathbf{x}}_{sp} + \underline{\mathbf{x}}_{tr}$, practical resonance
  - RLC circuits
  - Using conservation of total energy (=KE+PE) to derive equations of motion, especially for mass-spring and pendulum

6.1-6.1 eigenvalues, eigenvectors (eigenspaces), diagonalizable matrices; real+complex eigendata.

7.1-7.4 linear systems of DEs
- first order systems of DEs and tangent vector fields.
- existence-uniqueness thm for first order IVPs
- superposition, $\underline{\mathbf{x}} = \underline{\mathbf{x}}_P + \underline{\mathbf{x}}_H$
- dimension of solution space for $\underline{\mathbf{x}}_H$.
- equivalence of DE IVPs or systems to first order system IVPs.
- Constant coefficient systems and methods for $\underline{\mathbf{x}} = \underline{\mathbf{x}}_P + \underline{\mathbf{x}}_H$:
  - $\underline{\mathbf{x}}'(t) = A\underline{\mathbf{x}}$
  - $\underline{\mathbf{x}}'(t) = A\underline{\mathbf{x}} + \underline{\mathbf{f}}(t)$
  - $\underline{\mathbf{x}}''(t) = A\underline{\mathbf{x}}$ (from conservative systems)
  - $\underline{\mathbf{x}}''(t) = A\underline{\mathbf{x}} + \underline{\mathbf{f}}(t)$
- applications: phase portrait interpretation of unforced oscillation problems; input-output modeling; forced and unforced mass-spring systems and phenomena.

Linear.

10.1-10.5, EP7.6: Laplace transform
- definition, for direct computation
- using table for Laplace and inverse Laplace transforms ... including for topics before/after the second midterm, i.e. on/off and impulse, forcing, convolution solutions
- Solving linear DE (or system of DE) IVPs with Laplace transform.

black = computation basics.
red = applications.
brown = methods for solving linear DE's & systems.

1-2: first order DEs
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- phase diagrams for autonomous DEs
  - equilibrium solutions
  - stability
- existence-uniqueness thm for IVPs
- methods:
  - separable
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- applications
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  - input-output models

applications build upon each other

for sure on exam

3-4 matrix algebra and vector spaces
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Matrix Algebra.

5 Linear differential equations
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- Linear DEs
  - Homogeneous solution space, its dimension, and why
  - superposition, $\underline{x}(t)=\underline{x}_P+\underline{x}_H$
- Constant coefficient linear DEs
  - $\underline{x}_H$ via characteristic polynomial
  - Euler's formula, complex roots
  - $\underline{x}_P$ via undetermined coefficients
  - solving IVPs

$$e^{i\theta}=\cos\theta+i\sin\theta$$
$$e^{(a+ib)t}=\underbrace{\boxed{e^{at}\cos bt}}_{x_1(t)}+i\underbrace{\boxed{e^{at}\sin bt}}_{x_2(t)}$$

5: applications:
- mechanical configurations
  - unforced: undamped and damped
    - cos and sin addition angle formulas and amplitude-phase form
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no complex eigenvectors

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e.g. Chptr 5.

We can illustrate many ideas in this course, and how they are tied together by studying the following two differential equations in as many ways as we can think of.

$$x''(t) + 5x'(t) + 4x(t) = 0, \quad \left.\begin{matrix} x(0) = x_0 \\ x'(0) = v_0 \end{matrix}\right]$$

Chapter 5 $\quad p(r) = r^2 + 5r + 4 = (r+4)(r+1)$

$$x_H(t) = c_1 e^{-4t} + c_2 e^{-t}$$

Solve IVP with choice of c_1 & c_2.

Chapter 10 ($\mathcal{L}$)

$$s^2 X(s) - s x_0 - v_0 + 5(sX(s) - x_0) + 4X(s) = 0$$

$$X(s)\left[s^2 + 5s + 4\right] = s x_0 + v_0 + 5x_0$$

$$X(s) = \frac{s x_0 + 5x_0 + v_0}{s^2 + 5s + 4}$$

... partial fractions
convolution.

Chapter 7 (systems of DE's)

$$\begin{bmatrix} x \\ x' \end{bmatrix}' = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} x' \\ -4x - 5x' \end{bmatrix}$$

$$\begin{bmatrix} x \\ x' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}\begin{bmatrix} x \\ x' \end{bmatrix}$$

$$\begin{cases} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \begin{bmatrix} x_1(0) \\ x_2(x_1) \end{bmatrix} = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}. \end{cases}$$

basis $e^{\lambda t}\vec{v}$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -4 & -5-\lambda \end{vmatrix} = \lambda^2 + 5\lambda + 4 = (\lambda+4)(\lambda+1)$$

$$E_{\lambda=-4} = \mathrm{span}\{\vec{v}\}$$

$$E_{\lambda=-1} = \mathrm{span}\{\vec{w}\}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{-4t}\vec{v} + c_2 e^{-t}\vec{w}.$$

$$x''(t) + 5x'(t) + 4x(t) = 3\cos(2t)$$

Chptr 5 $\quad x = x_p + x_H$

undet. coef's $\quad x_p(t) = A\cos 2t + B\sin 2t$

find A & B.

$$x = \underbrace{A\cos 2t + B\sin 2t}_{C\cos(2t-\alpha)} + c_1 e^{-4t} + c_2 e^{-t}$$

" $x_{sp}(t)$ " steady periodic.

Chapter 10

$$X(s) = \frac{s x_0 + 5x_0 + v_0}{s^2 + 5s + 4} + \frac{3s}{(s^2+4)(s^2+5s+4)}$$

yipes

Chapter 7

$$\begin{bmatrix} x \\ x' \end{bmatrix}' = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} x' \\ -4x - 5x' + 3\cos 2t \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \cos 2t \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\vec{x} = \vec{x}_p + \vec{x}_H$$

$$\vec{x}_p = \cos 2t\, \vec{c}_1 + \sin 2t\, \vec{c}_2$$