Math 2250-004 Week 12 April 2-6
continue 10.1-10.3; also cover parts of 10.4-10.5, EP 7.6

Mon Apr 2:
10.1-10.3 Laplace transform and initial value problems like we studied in Chapter 5

Announcements:
- Exams will be returned tomorrow (a couple people still need to take it)
- HW on 10.1-10.3 due in Lab on Thursday.
- on exam, you'll be given a Laplace transform table, but...

'til 10:46

Warm-up Exercise: Laplace transforms!!

$|f(t)| \le Ce^{Mt}$ $s > M$

	$f(t)$	$F(s) = \int_0^\infty f(t) e^{-st}\, dt$
a •	1	$1/s$
a •	e^{at}	$\frac{1}{s-a}$
b)	$\sin kt$	$\frac{k}{s^2+k^2}$
	$\cos kt$	$\frac{s}{s^2+k^2}$
a •	$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$

a) Find $\mathcal{L}\{3 - 4e^{2t}\}(s)$

b) Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}(t)$

a) $\frac{3}{s} - 4\frac{1}{s-2}$

b) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{2}{s^2+4}\right\}(t) = \frac{1}{2}\sin 2t$

$k=2$ in $\sin kt$ entry.

Recall,

- The Laplace transform is a linear transformation "$\mathcal{L}$" that converts piecewise continuous functions $f(t)$, defined for $t \geq 0$ and with at most exponential growth ($|f(t)| \leq Ce^{Mt}$ for some values of C and M), into functions $F(s)$ defined by the transformation formula

$$F(s) = \mathcal{L}\{f(t)\}(s) := \int_0^\infty f(t)e^{-st}\, dt\,.$$

- Notice that the integral formula for $F(s)$ is only defined for sufficiently large s, and certainly for $s > M$, because as soon as $s > M$ the integrand is decaying exponentially, so the improper integral from $t = 0$ to ∞ converges.

- The convention is to use lower case letters for the input functions and (the same) capital letters for their Laplace transforms, as we did for $f(t)$ and $F(s)$ above. Thus if we called the input function $x(t)$ then we would denote the Laplace transform by $X(s)$.

Exercise 1) (to review) Use the table entries we computed last Wednesday, to compute

1a) $\mathcal{L}\{4 - 5\cos(3t) + 2e^{-4t}\sin(12t)\}(s)$ $= \frac{4}{s} - 5\frac{s}{s^2+9} + 2\cdot\frac{12}{(s+4)^2+144}$

1b) $\mathcal{L}^{-1}\left\{\frac{2}{s-2} + \underbrace{\frac{1}{s^2+2s+5}}_{s^2+2s+5=(s+1)^2+4}\right\}(t)$. $= 2e^{2t} + \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{\cancel{1}2}{(s+1)^2+4}\right\}(t)$

$= 2e^{2t} + \frac{1}{2}e^{-t}\sin 2t$

$f(t)$ $\lvert f(t)\rvert \le Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st}\,dt$ for $s > M$
$c_1f_1(t) + c_2f_2(t)$	$c_1F_1(s) + c_2F_2(s)$
1	$\frac{1}{s}$ $(s > 0)$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$ $(s > \Re(a))$
$\cos(kt)$	$\frac{s}{s^2+k^2}$ $(s > 0)$
$\sin(kt)$	$\frac{k}{s^2+k^2}$ $(s > 0)$
$e^{at}\cos(kt)$	$\frac{(s-a)}{(s-a)^2+k^2}$ $(s > a)$
$e^{at}\sin(kt)$	$\frac{k}{(s-a)^2+k^2}$ $(s > a)$
$f'(t)$	$s\,F(s) - f(0)$
$f''(t)$	$s^2F(s) - s\,f(0) - f'(0)$

$\to \mathcal{L}\{0\}(s) = 0$

$a = 2$

$k = 3$

$a = -4$, $k = 12$ $\qquad a = -1$, $k = 2$

Laplace transform table

Exercise 2) (to review) Use Laplace transforms to solve the IVP for an underdamped, unforced oscillator DE. Compare to Chapter 5 method.

$\mathcal{L}$:

$$x''(t) + 6\,x'(t) + 34\,x(t) = 0$$
$$x(0) = 3$$
$$x'(0) = 1$$

$x(t)$	$X(s)$
$x''(t)$	$s^2 X(s) - s\,x(0) - x'(0)$
$x'(t)$	$s\,X(s) - x(0)$
$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$

soltn $x(t)$ makes DE true,
so $\mathcal{L}$ of each side is equal

$$s^2 X(s) - 3s - 1 + 6\left(sX(s) - 3\right) + 34\,X(s) = 0$$

$$X(s)\left[s^2 + 6s + 34\right] = 3s + 1 + 18 = 3s + 19$$

$$X(s) = \frac{3s+19}{s^2+6s+34}$$

$$= \frac{3s+19}{(s+3)^2+25}$$

"complete the linear"

$$= \frac{3(s+3) + 10}{(s+3)^2+25}$$

$\boxed{\begin{aligned} k &= 5 \\ a &= -3 \end{aligned}}$

$$X(s) = 3\,\frac{s+3}{(s+3)^2+25} + \frac{10}{(s+3)^2+25} \rightarrow 2\,\frac{5}{(s+3)^2+25}$$

$$\Rightarrow \boxed{x(t) = 3\,e^{-3t}\cos 5t + 2\,e^{-3t}\sin 5t}$$

$x(0) = 3$ ✓
$x'(0) = -9 + 10$
$= 1$ ✓

We'll fill in more table entries today. (Compare to front cover of your text, which contains this information but maybe more compactly.)

$f(t)$, with $\lvert f(t)\rvert \le C e^{M t}$	$F(s) := \int_0^\infty f(t) e^{-s t}\, dt$ for $s > M$	↓ verified	
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	✓	
1	$\frac{1}{s}$ $(s > 0)$	✓	
t	$\frac{1}{s^2}$		
t^2	$\frac{2}{s^3}$		
t^n, $n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$		
$e^{\alpha t}$	$\frac{1}{s - \alpha}$ $(s > \Re(\alpha))$	✓	
$\cos(k t)$	$\frac{s}{s^2 + k^2}$ $(s > 0)$	✓	
$\sin(k t)$	$\frac{k}{s^2 + k^2}$ $(s > 0)$	✓	
$\cosh(k t)$	$\frac{s}{s^2 - k^2}$ $(s > k)$		shortcut to avoid partial fractions
$\sinh(k t)$	$\frac{k}{s^2 - k^2}$ $(s > k)$		
$e^{a t}\cos(k t)$	$\frac{(s - a)}{(s - a)^2 + k^2}$ $(s > a)$	✓	
$e^{a t}\sin(k t)$	$\frac{k}{(s - a)^2 + k^2}$ $(s > a)$	✓	
$f'(t)$	$s F(s) - f(0)$	✓	
$f''(t)$	$s^2 F(s) - s f(0) - f'(0)$	✓	
$f^{(n)}(t)$, $n \in \mathbb{N}$	$s^n F(s) - s^{n-1} f(0) - \ldots - f^{(n-1)}(0)$		
$\int_0^t f(\tau)\, d\tau$	$\frac{F(s)}{s}$		
$t f(t)$	$-F'(s)$		
$t^2 f(t)$	$F''(s)$		
$t^n f(t)$, $n \in \mathbb{Z}$	$(-1)^n F^{(n)}(s)$		
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\, d\sigma$		
$t \cos(k t)$	$\frac{s^2 - k^2}{(s^2 + k^2)^2}$		
$\frac{1}{2k} t \sin(k t)$	$\frac{s}{(s^2 + k^2)^2}$		

$\frac{1}{2\,k^3}\left(\sin(k\,t) - k\,t\,\cos(k\,t)\right)$	$\frac{1}{(s^2+k^2)^2}$	
$e^{a\,t} f(t)$ $t\,e^{a\,t}$ $t^n\,e^{a\,t},\ n \in \mathbb{Z}$	$F(s-a)$ $\frac{1}{(s-a)^2}$ $\frac{n!}{(s-a)^{n+1}}$	

Laplace transform table

work down the table ...

3a) $\mathcal{L}\{cosh(k\,t)\}(s) = \frac{s}{s^2-k^2}$

3b) $\mathcal{L}\{\sinh(k\,t)\}(s) = \frac{k}{s^2-k^2}$.

"recall" $\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$ $\quad \left(= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$

$\cosh kt = \frac{1}{2}\left(e^{kt} + e^{-kt}\right)$ $\quad \cosh x = \cos(ix)$

$\mathcal{L}\{\cosh kt\}(s) = \frac{1}{2}\left(\frac{1}{s-k} + \frac{1}{s+k}\right)$

$= \frac{1}{2}\,\frac{2s}{(s-k)(s+k)}$

$= \frac{s}{s^2-k^2}$

$f(t)$	$F(s)$
e^{at}	$\frac{1}{s-a}$

$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$

$\sinh kt = \frac{1}{2}\left(e^{kt} - e^{-kt}\right)$ ...

$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t) e^{-st}\, dt$ $\qquad$ $\mathcal{L}\{g'(t)\}(s) = \int_0^\infty \underbrace{g'(t)}_{} \underbrace{e^{-st}\, dt}_{du}$

$|f(t)| \le C e^{Mt}$, $s > M$ $\qquad$ $u = g(t)$

Exercise 4) Recall we used integration by parts on Wednesday to derive

$$* \quad \mathcal{L}\{g'(t)\}(s) = s\mathcal{L}\{g(t)\}(s) - g(0) .$$

Wed $\qquad$ $dv = -se^{-st}dt$

Use that identity to show

a) $\mathcal{L}\{f''(t)\}(s) = s^2F(s) - s f(0) - f'(0)$,

b) $\mathcal{L}\{f'''(t)\}(s) = s^3F(s) - s^2 f(0) - s f'(0) - f''(0)$,

c) $\mathcal{L}\{f^{(n)}(t)\}(s) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0),\ n \in \mathbb{N}$.

d) $\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\}(s) = \dfrac{F(s)}{s}$.

These are the identities that make Laplace transform work so well for initial value problems such as we studied in Chapter 5.... with Laplace transforms the "free parameters" when you write down the solution $x = x_P + x_H$ are exactly the initial values for the differential equation, rather than the linear combinations coefficients in the general homogeneous solution, so you definitely save a step there, in solving IVPs.

a) $*$ for $g(t) = f'(t)$
$g'(t) = f''(t)$

$$\mathcal{L}\{f''(t)\}(s) = s\mathcal{L}\{f'(t)\}(s) - f'(0)$$
$$\overset{*}{=} s\left(s\mathcal{L}\{f(t)\}(s) - f(0)\right) - f'(0)$$
$$\overset{*\text{ for } g = f}{=} s^2F(s) - sf(0) - f'(0)$$

b) $*$ for $g(t) = f''(t)$
$g'(t) = f'''(t)$

$$\mathcal{L}\{f'''(t)\}(s) = s\mathcal{L}\{f''(t)\}(s) - f''(0)$$
$$\underset{a}{\overset{*}{=}} s\left(s^2F(s) - sf(0) - f'(0)\right) - f''(0)$$
$$= s^3F(s) - s^2 f(0) - sf'(0) - f''(0)$$

c) inductively.

d) $g(t) = \int_0^t f(\tau)\, d\tau$
$g'(t) = f(t)$ $\quad$ FTC.

$$* \quad \mathcal{L}\{g'(t)\}(s) = s\mathcal{L}\{g(t)\} - g(0)$$
$$F(s) = s\mathcal{L}\{g(t)\}(s) - 0.$$
$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}(s) = \frac{F(s)}{s}$$

Exercise 5) Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s^2+4)}\right\}(t)$

a) using the result of 4d.

b) using partial fractions.

$f(t)$	$F(s)$
$\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$
$\sin kt$	$\dfrac{k}{s^2+k^2}$

$k=2$

$$F(s) = \frac{1}{s^2+4}$$

$$\frac{F(s)}{s} = \frac{1}{s(s^2+4)}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{\cancel{1}\,2}{s^2+4}\right\}(t) = \frac{1}{2}\sin 2t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\}(t) = \int_0^t \frac{1}{2}\sin 2\tau\,d\tau$$

$$= -\frac{1}{4}\cos 2\tau\Big]_0^t = \boxed{-\frac{1}{4}\cos 2t + \frac{1}{4}}$$

Exercise 6) Show $\mathcal{L}\{t^n\}(s) = \dfrac{n!}{s^{n+1}}$, $n \in \mathbb{N}$, using the results of 4, namely

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0),\ n \in \mathbb{N}.$$