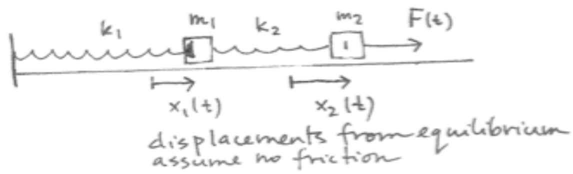
A larger example of converting higher order DE's and systems of DE's into first-order ones....

lilo §7.1

Example:
Consider this configuration of two coupled masses and springs:



Exercise 4) Use Newton's second law to derive a system of two second order differential equations for $x_1(t), x_2(t)$, the displacements of the respective masses from the equilibrium configuration. What initial value problem do you expect yields unique solutions in this case? (See homework in section 7.1)

$m_1 x_1'' =$ net forces

$m_2 x_2'' =$ net forces

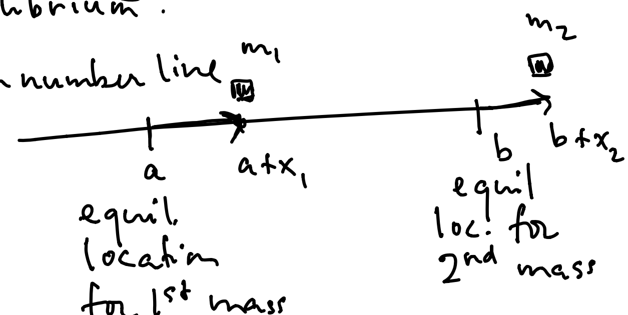
$$m_1 x_1'' = \underbrace{-k_1 x_1}_{\text{from 1st spring}} + k_2 \underbrace{(x_2 - x_1)}$$

net stretch from equil of 2nd spring is $x_2 - x_1$
(easy to recall since if $x_2 = x_1$ both masses have equal displacement from equil. position, so spring is same length as when at equilibrium.

$$m_2 x_2'' = -k_2 \underbrace{(x_2 - x_1)}_{\text{stretch amt}} + F(t)$$

spring is attached on left side of mass

Details on number line



We expect to specify
$x_1(0)$, $x_1'(0)$ } int. pos & velocity for m_1
$x_2(0)$
$x_2'(0)$.

equil length of spring 2 $= b - a$
length when masses displaced is $(b + x_2) - (a + x_1)$

$$= \underbrace{(b-a)}_{\text{equil length}} + \underbrace{(x_2 - x_1)}_{\text{stretch amt.}}$$

Exercise 5) Consider the IVP from Exercise 4, with the special values $m_1 = 2, m_2 = 1; k_1 = 4, k_2 = 2; F(t) = 40\sin(3t)$:

$$1 \quad \begin{cases} x_1'' = -3x_1 + x_2 \\ x_2'' = 2x_1 - 2x_2 + 40\sin(3t) \\ x_1(0) = b_1, x_1'(0) = b_2 \\ x_2(0) = c_1, x_2'(0) = c_2 . \end{cases}$$

5a) Show that if $x_1(t), x_2(t)$ solve the IVP above, and if we define

$$v_1(t) := x_1'(t)$$
$$v_2(t) := x_2'(t)$$

then $x_1(t), x_2(t), v_1(t), v_2(t)$ solve the first order system IVP

$$2 \quad \begin{cases} x_1' = v_1 \\ x_2' = v_2 \\ v_1' = -3x_1 + x_2 \\ v_2' = 2x_1 - 2x_2 + 40\sin(3t) \\ x_1(0) = b_1 \\ x_1'(0) = v_1(0) = b_2 \\ x_2(0) = c_1 \\ x_2'(0) = v_2(0) = c_2 . \end{cases}$$

$1 \Rightarrow 2$

$x_1' = v_1$ by definition

$x_2' = v_2$ " "

$v_1' = x_1'' = -3x_1 + x_2$ from 1

$v_2' = x_2'' = 2x_1 - 2x_2 + 40\sin 3t$ from 1

$x_1(0) = b_1$

$v_1(0) = x_1'(0) = b_2$

$x_2(0) = c_1$

$v_2(0) = x_2'(0) = c_2$

5b) Conversely, show that if $x_1(t), x_2(t), v_1(t), v_2(t)$ solve the IVP of four first order DE's, then $x_1(t), x_2(t)$ solve the original IVP for two second order DE's.

$2 \Rightarrow 1$

$x_1' = v_1$

$\Rightarrow x_1'' = v_1' = -3x_1 + x_2$ from 2

$x_2' = v_2$

$\Rightarrow x_2'' = v_2' = 2x_1 - 2x_2 + 40\sin 3t$ from 2

$x_1(0) = b_1$

$x_1'(0) = v_1(0) = b_2$

$x_2(0) = c_2$

$x_2'(0) = v_2(0) = c_2$

Tues Apr 17

7.2-7.3 Linear systems of differential equations (7.2); Solving homogeneous linear systems $\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}}(t)$ by finding basis solutions of the form $e^{\lambda t}\underline{\mathbf{v}}$. (7.3).

Announcements:
- please pick up graded HW & quizzes
- There will be one more HW & lab (after tomorrow's & Thursday's) They'll be due next Tuesday @ 6:00 p.m.
- We'll do today's notes first & then go back to the multi mass-spring model in Monday's notes

Warm-up Exercise: Find the eigenvalues and eigenvectors (eigenspace bases) for the matrix $A = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}$.

$\lambda = 0, -6$

$E_{\lambda=0} = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

$E_{\lambda=-6} = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

The focus of sections 7.2-7.3 is linear systems of first order differential equations, and their associated initial value problems:

Chptr 1

$$\underline{\mathbf{x}}'(t) = A(t)\underline{\mathbf{x}}(t) + \underline{\boldsymbol{f}}(t)$$
$$\underline{\mathbf{x}}(t_0) = \underline{\mathbf{x}}_0$$

$\vec{x}(t) \in \mathbb{R}^n$
$A_{n\times n}$

If the matrix $A(t)$ and the vector function $\underline{\boldsymbol{f}}(t)$ are continuous on an open interval I containing t_0 then a solution $\underline{\mathbf{x}}(t)$ exists and is unique, on the entire interval. We had a similar fact for the scalar version of this system, in Chapter 1. There, we had an integrating factor technique to find the solutions. Not so, here. When we want to emphasize the *linear* nature of this sort of system we may re-write it as

$$\underline{\mathbf{x}}'(t) - A(t)\underline{\mathbf{x}}(t) = \underline{f}(t)$$

$$\underline{\mathbf{x}}(t_0) = \underline{\mathbf{x}}_0$$

$x'(t) + p(t)x = q(t)$
$e^{\int p(t)dt}$ I.F.
...

We checked on Friday that the operator on the left, namely

$$L(\underline{\mathbf{x}}(t)) := \underline{x}'(t) - A(t)\underline{\mathbf{x}}(t)$$

is linear:

$$L(\underline{\mathbf{x}}(t) + \underline{\mathbf{z}}(t)) = L(\underline{\mathbf{x}}(t)) + L(\underline{\mathbf{z}}(t))$$

$$L(c\,\underline{\mathbf{x}}(t)) = c\,L(\underline{\mathbf{x}}(t)).$$

This is how the check went (we suppress the t):

$$L(\underline{\mathbf{x}}(t) + \underline{\mathbf{z}}(t)) = L(\underline{\mathbf{x}} + \underline{\mathbf{z}}) := (\underline{\mathbf{x}} + \underline{\mathbf{z}})' - A(\underline{\mathbf{x}} + \underline{\mathbf{z}})$$

$$= \underline{\mathbf{x}}' + \underline{\mathbf{z}}' - A\,\underline{\mathbf{x}} - A\,\underline{\mathbf{z}} = (\underline{\mathbf{x}}' - A\,\underline{\mathbf{x}}) + (\underline{\mathbf{z}}' - A\,\underline{\mathbf{z}}) = L(\underline{\mathbf{x}}) + L(\underline{\mathbf{z}}).$$

$$L(c\,\underline{\mathbf{x}}(t)) = L(c\,\underline{\mathbf{x}}) = (c\,\underline{\mathbf{x}})' - A\,(c\,\underline{\mathbf{x}}) = c\,\underline{\mathbf{x}}' - c\,A\,\underline{\mathbf{x}} = c(\underline{\mathbf{x}}' - A\,\underline{\mathbf{x}}) = c\,L(\underline{\mathbf{x}}).$$

Thus, from vector space theory,

1a) The solution space to the homogeneous linear system of DE's

$$\underline{\boldsymbol{x}}'(t) - A(t)\underline{\boldsymbol{x}}(t) = \underline{\mathbf{0}}.$$

is a subspace.

1b) If $\underline{\boldsymbol{x}}(t)$ is an n-vector and A is an $n \times n$ matrix, then the initial value problems

$$\begin{cases} \underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t) \\ \underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0 \end{cases}$$

have n free parameters (i.e. the n entries of the intial vector $\underline{\boldsymbol{x}}_0$), so it will take a set of n linearly independent solutions $\{\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), \ldots \underline{\boldsymbol{X}}_n(t)\}$ to uniquely solve each IVP with a linear combination

$$\underline{\boldsymbol{x}}(t) = c_1 \underline{\boldsymbol{X}}_1(t) + c_2 \underline{\boldsymbol{X}}_2(t) + \ldots c_n \underline{\boldsymbol{X}}_n(t).$$

In other words, the solution space has dimension n.

By the way, the matrix that has n solutions in its columns is called the *Wronskian matrix* of the solutions, and its determinant is called the *Wronskian determinant.*

2) The general solution to the inhomogeneous system

$$\underline{\boldsymbol{x}}'(t) - A(t)\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{f}}(t)$$

which we often write in the form

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{f}}(t)$$

is

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{x}}_p(t) + \underline{\boldsymbol{x}}_H(t)$$

where $\underline{\boldsymbol{x}}_p(t)$ is any single particular solution and $\underline{\boldsymbol{x}}_H(t)$ is the general solution to the homogeneous problem

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t).$$

Just like in Chapter 5!

Section 7.3 How to find a basis for the solution space to homogeneous first order systems of differential equations

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$

when the matrix A is constant and diagonalizable.

Here's how!! If $\underline{\boldsymbol{v}} \neq \underline{\boldsymbol{0}}$ is an eigenvector of A, i.e.

$$A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$$

universal product rule.

then

$$\underline{\boldsymbol{X}}(t) = e^{\lambda t}\underline{\boldsymbol{v}}$$

$$\vec{X}'(t) = \lambda e^{\lambda t}\vec{v} + e^{\lambda t}\cancel{\frac{d}{dt}\vec{v}}^{\,0}$$

satisifies

$$\underline{\boldsymbol{X}}'(t) = \lambda\, e^{\lambda t}\underline{\boldsymbol{v}}$$

and

$$A\,\underline{\boldsymbol{X}}(t) = A\left(e^{\lambda t}\underline{\boldsymbol{v}}\right) = e^{\lambda t}A\,\underline{\boldsymbol{v}} = e^{\lambda t}\lambda\,\underline{\boldsymbol{v}}\ .$$

So $\underline{\boldsymbol{X}}(t) = e^{\lambda t}\underline{\boldsymbol{v}}$ is a solution to the system of differential equations

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}.$$

If A is diagonalizable then there is a basis of $\mathbb{R}^n$ made out of its eigenvectors, and we will have a corresponding basis for the solution space to the first order system:

$$\left\{ e^{\lambda_1 t}\underline{\boldsymbol{v}}_1,\ e^{\lambda_2 t}\underline{\boldsymbol{v}}_2,\ \ldots\ e^{\lambda_n t}\underline{\boldsymbol{v}}_n \right\}.$$

In other words, we can solve each initial value problem

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0$$

with a unique linear combination of the basis solutions,

$$\underline{\boldsymbol{x}}(t) = c_1 e^{\lambda_1 t}\underline{\boldsymbol{v}}_1 + c_2 e^{\lambda_2 t}\underline{\boldsymbol{v}}_2 + c_n e^{\lambda_n t}\underline{\boldsymbol{v}}_n\ .$$

at $t=0$: $\quad c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{x}_0$

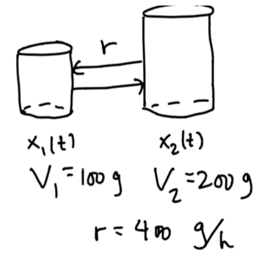
$$\left[\begin{array}{c|c|c|c} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{array}\right]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{x}_0.$$

unique $\vec{c}$ vectors for each $\vec{x}_0$.

Exercise 1 For the tank problem last Friday,

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$



We found the eigendata for the matrix *A*. But it's been a while, so let's recompute. Then write down the general solution to the first order system and solve the initial value problem. Compare the solution to the pplane analysis we did on Friday, on the next page.

1. find eigendata for A.
2. get solns (basis) of form $e^{\lambda t}\vec{v}$.

$$|A-\lambda I| = \begin{vmatrix} -4-\lambda & 2 \\ 4 & -2-\lambda \end{vmatrix} = (\lambda+4)(\lambda+2) - 8$$
$$= \lambda^2 + 6\lambda + \cancel{8} - \cancel{8}$$
$$= \lambda(\lambda+6) = 0$$
$$\lambda = 0, -6.$$

$E_{\lambda=0}$

$(A - 0I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right]$$

$R_2/{-2}$

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$E_{\lambda=0} = \text{span}\left\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$$

long way:

$$\left[\begin{array}{cc|c} 1 & -.5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$v_1 = .5t$
$v_2 = t$
$\vec{v} = t\begin{bmatrix} .5 \\ 1 \end{bmatrix}$

$\vec{x}' = A\vec{x}$ soln

$$e^{0t}\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$E_{\lambda=-6}$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$E_{\lambda=-6} = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$$

shortcut.

soln $\vec{x}' = A\vec{x}$

$$e^{-6t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$\vec{x}(t) = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-6t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$\vec{x}(0) = \begin{bmatrix} 9 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 9 \\ -18 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}.$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{x}(t) = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 6e^{-6t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

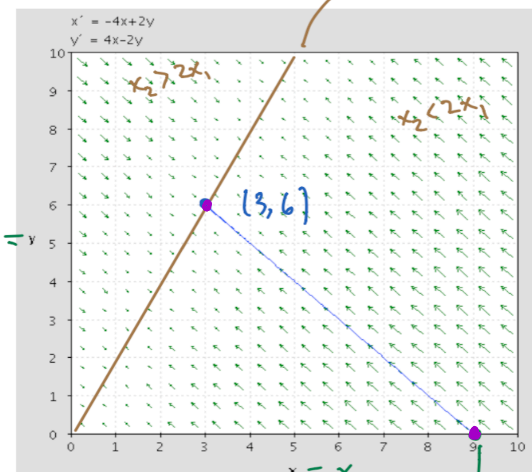
$$\boxed{\begin{aligned} x_1(t) &= 3 + 6e^{-6t} \\ x_2(t) &= 6 - 6e^{-6t} \end{aligned}}$$

$x_1(\infty) = 3$
$x_2(\infty) = 6$ ✓

$x_1(0) = 9$
$x_2(0) = 0$ ✓

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{(2x_1 - x_2)}_{0} \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$x_2 = 2x_1$



$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

$$\vec{x}'(t) = 36e^{-6t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad ✓$$

Exercise 2) For the overdamped mass-spring IVP on Monday

$$x''(t) + 7x'(t) + 6x(t) = 0$$
$$x(0) = 1$$
$$x'(0) = 4$$

$x'' = -6x - 7x'$

$$\begin{bmatrix} x \\ x' \end{bmatrix}' = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} x' \\ -6x-7x' \end{bmatrix}$$

we found the solution

$$\boxed{x(t) = 2\,e^{-t} - e^{-6t}.}$$

$$\begin{bmatrix} x \\ x' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix}\begin{bmatrix} x \\ x' \end{bmatrix}$$

We noted that the equivalent first order system IVP for $[x(t), x'(t)]^T$ is

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

Hw $\quad x'' + 4x = 0$

$$\begin{bmatrix} x \\ x' \end{bmatrix}' = \begin{bmatrix} x' \\ -4x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} x \\ x' \end{bmatrix}$$

2a) Solve this IVP using the algorithm for first order systems of DE's, and verify that $x_1(t)$ is the solution to the original second order DE IVP.

① eigendata.

$$\begin{vmatrix} -\lambda & 1 \\ -6 & -7-\lambda \end{vmatrix} = \lambda(\lambda+7) + 6 = \lambda^2 + 7\lambda + 6 = (\lambda+6)(\lambda+1)$$

$E_{\lambda=-1}$: $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -6 & -6 & 0 \end{array}\right]$

$E_{\lambda=-1} = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

$e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$E_{\lambda=-6}$ $\left[\begin{array}{cc|c} 6 & 1 & 0 \\ -6 & -1 & 0 \end{array}\right]$

$E_{\lambda=-6} = \text{span}\left\{\begin{bmatrix} 1 \\ -6 \end{bmatrix}\right\}$

$e^{-6t}\begin{bmatrix} 1 \\ -6 \end{bmatrix}.$

$$\vec{x}(t) = c_1 e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{-6t}\begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

$x_1(0) = 1 = -c_1 + c_2 \qquad E_1$

$x_2(0) = 4 = c_1 - 6c_2 \qquad E_2$

$E_1 + E_2 \Rightarrow 5 = -5c_2 \Rightarrow c_2 = -1$

$E_1 \Rightarrow 1 = -c_1 - 1$

$2 = -c_1 \quad c_1 = -2$

$$\vec{x}(t) = -2e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} - e^{-6t}\begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

$\boxed{x_1(t) = 2e^{-t} - e^{-6t}} = x(t)$ from yesterday.

2b) Compare the Chapter 5 "Wronskian matrix" for the second order DE, to the Chapter 7 "Wronskian matrix" for the system. (We already noted that the characteristic polynomials were the same.)

$$x''(t) + 7x'(t) + 6x(t) = 0$$

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$