Math 2250-004
Week 14 April 16-20: sections 7.1-7.3 first order systems of linear differential equations; 7.4 mass-spring systems.

Fri - Mon

Mon Apr 16
7.1-7.2 Systems of differential equations (7.1), and the vector Calculus we need to study them (7.2).
Every differential equation or system of differential equations can be converted into a first order system of differential equations (7.1).

Announcements: plan: Tuesday next week review (mostly, at least)

- today I'll demo pplane, I ask you to use this for a couple of your hw problems. from same URL as dfield.

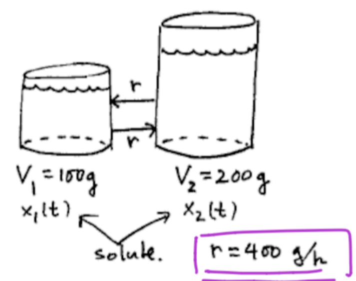
'til 10:45
Warm-up Exercise: Review the discussion of the initial value problem for 1st order systems of DE's (that we had on Friday) – see page 3 of this week's notes

<u>Summary and continuation of Friday overview/introduction to Chapter 7</u>, which is about systems of differential equations. We began with a specific two-tank input-output model, with the goal of tracking the vector

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

of solute amounts in each tank. The initial value problem for this tank system was of the form

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0$$



g = gallons
h = hours

<u>Exercise 1)</u> Find differential equations for solute amounts $x_1(t), x_2(t)$ above, using input-output modeling. Assume solute concentration is uniform in each tank. If $x_1(0) = b_1, x_2(0) = b_2$, write down the initial value problem that you expect would have a unique solution.

$$x_1'(t) = r \cdot \frac{x_2}{200} - r\frac{x_1}{100} = 400\frac{x_2}{200} - \frac{400x_1}{100} = -4x_1 + 2x_2$$

$$x_2' = 400\frac{x_1}{100} - 400\frac{x_2}{200} = 4x_1 - 2x_2$$

IVP
$$\begin{cases} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4x_1 + 2x_2 \\ 4x_1 - 2x_2 \end{bmatrix} \\ \begin{bmatrix} x_1(0) \\ x_1'(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \end{cases}$$

answer (in matrix-vector form):

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

The more general first order system of differential equations, and associated initial value problem is,

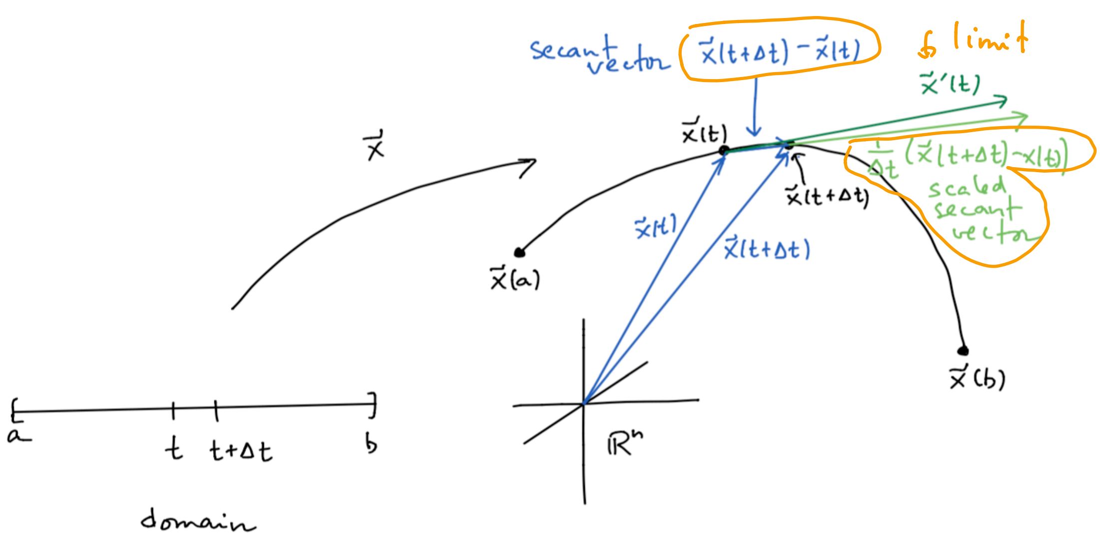
$$\underline{\boldsymbol{x}}'(t) = \underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t))$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

<u>Existence and Uniqueness for solutions to IVP's</u>: The above IVP is a vectorized version of the scalar first order DE IVP that we considered in Chapter 1. In Chapter 1 we understood why (with the right conditions on the right hand side), these IVP's have unique solutions. There is an analogous existence-uniqueness theorem for the vectorized version we study in Chapter 7, and it's believable for the same reasons the Chapter 1 theorem seemed reasonable. We just have to remember the geometric meaning of the *tangent* vector $\underline{\boldsymbol{x}}'(t)$ to a parametric curve in $\mathbb{R}^n$ (which is also called the *velocity* vector in physics, when you study particle motion):

Algebra:

$$\underline{\boldsymbol{x}}'(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\begin{bmatrix} x_1(t+\Delta t) \\ x_2(t+\Delta t) \\ : \\ x_n(t+\Delta t)\end{bmatrix} - \begin{bmatrix} x_1(t) \\ x_2(t) \\ : \\ x_n(t)\end{bmatrix}\right) = \lim_{\Delta t \to 0}\begin{bmatrix} \frac{1}{\Delta t}\left(x_1(t+\Delta t) - x_1(t)\right) \\ \frac{1}{\Delta t}\left(x_2(t+\Delta t) - x_2(t)\right) \\ : \\ \frac{1}{\Delta t}\left(x_n(t+\Delta t) - x_n(t)\right)\end{bmatrix} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ : \\ x_n'(t)\end{bmatrix}.$$

Geometric interpretation in terms of displacement vectors along a parametric curve:



So the existence-uniquess theorem for first order systems of DE's is true because if you know where you start at time t_0, namely $\underline{\boldsymbol{x}}_0$; and if you know your tangent vector $\underline{\boldsymbol{x}}'(t)$ at every later time -in terms of your location $\underline{\boldsymbol{x}}(t)$ and what time t it is, as specified by the vector function $\underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t))$; then there should only be one way the parametric curve $\underline{\boldsymbol{x}}(t)$ can develop. This is analogous to our reasoning in Chapter 1 that there should only be one way to follow a slope field, given the initial point one starts at.

For the two-tank example, we used output from *pplane*, the sister program to *dfield*, to illustrate how solutions follow tangent vector fields, and tracked the solution to

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -4\,x_1(t) + 2\,x_2(t) \\ 4\,x_1(t) - 2\,x_2(t) \end{bmatrix} = \left(4\,x_1(t) - 2\,x_2(t)\right) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

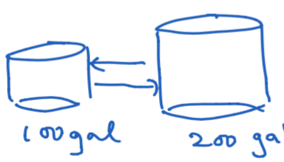
$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

We noticed that the limiting solute amounts appeared to be $\begin{bmatrix} x_1(\infty) \\ x_2(\infty) \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, which makes complete sense.

Note: This system of DE's is *autonomous*, since the formula $\underline{\boldsymbol{x}}'(t)$ only depends on the value of $\underline{\boldsymbol{x}}$ and not on the value of t, so the tangent field is not changing in time; the pplane *phase portrait* is analgous to the *phase diagram* lines we drew for autonomous first order differential equations in Chapter 2.

$$\lim_{t\to\infty} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

2c)



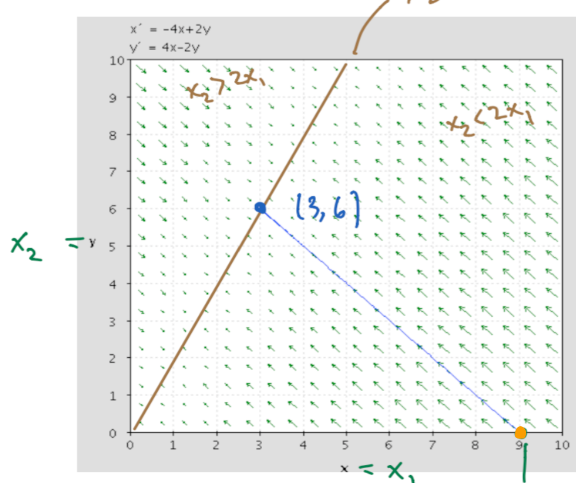
expect same concentrations in each tank as $t \to \infty$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

& expect total of 9 lbs & twice as much in 2nd tank because twice the vol.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = (\underbrace{2x_1 - x_2}_{0}) \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$x_2 = 2x_1$



$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

On Friday, we began to solve the two-tank IVP example analytically, using eigenvalues and eigenvectors from the matrix A in that problem (!). That is typical of what we will do in section 7.3, to solve the first order system of DE's

$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}}$$
$$\underline{\mathbf{x}}(0) = \underline{\mathbf{x}}_0 .$$

But before we finish that computation, it's a better idea to review and extend some differentiation rules you probably learned in multivariable Calculus, when you studied the calculus of parametric curves. This is related to material in section 7.2 of the text.

1) If $\underline{\mathbf{x}}(t) = \underline{\boldsymbol{b}}$ is a constant vector, then $\underline{\mathbf{x}}'(t) = \underline{\mathbf{0}}$ for all t, and vise-verse. (Because all of the entries in the vector $\underline{\boldsymbol{b}}$ are constants, and their derivatives are zero. And if the derivatives of all entries of a vector are identically zero, then the entries are constants.)

2) Sum rule for differentiation:

$$\frac{d}{dt}\left(\underline{\mathbf{x}}(t) + \underline{\mathbf{y}}(t)\right) = \underline{\mathbf{x}}'(t) + \underline{\mathbf{y}}'(t): \qquad \text{Both sides simplify to} \begin{bmatrix} x_1'(t) + y_1'(t) \\ x_2'(t) + y_2'(t) \\ \vdots \\ x_n'(t) + y_n'(t) \end{bmatrix}$$

3) Constant multiple rule for differentiation:

$$\frac{d}{dt}\left(c\,\underline{\mathbf{x}}(t)\right) = c\,\underline{\mathbf{x}}'(t): \qquad \text{Both sides simplify to} \quad c \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}.$$

4) Matrix-valued functions sometimes show up and need to be differentiated. This is done with the limit definition, and amounts to differentiating each entry of the matrix. For example, if $A(t)$ is a 2×2 matrix, then

$$\frac{d}{dt}\begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\begin{bmatrix} a_{11}(t+\Delta t) & a_{12}(t+\Delta t) \\ a_{21}(t+\Delta t) & a_{22}(t+\Delta t) \end{bmatrix} - \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}\right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\begin{bmatrix} a_{11}(t+\Delta t)-a_{11}(t) & a_{12}(t+\Delta t)-a_{12}(t) \\ a_{21}(t+\Delta t)-a_{21}(t) & a_{22}(t+\Delta t)-a_{22}(t) \end{bmatrix}$$

$$= \lim_{\Delta t \to 0}\begin{bmatrix} \dfrac{a_{11}(t+\Delta t)-a_{11}(t)}{\Delta t} & \dfrac{a_{12}(t+\Delta t)-a_{12}(t)}{\Delta t} \\ \dfrac{a_{21}(t+\Delta t)-a_{21}(t)}{\Delta t} & \dfrac{a_{22}(t+\Delta t)-a_{22}(t)}{\Delta t} \end{bmatrix} = \begin{bmatrix} a_{11}'(t) & a_{12}'(t) \\ a_{21}'(t) & a_{22}'(t) \end{bmatrix}.$$

5) The constant rule (1), sum rule (2), and constant multiple rule (3) also hold for matrix derivatives.

Universal product rule: Shortcut to take the derivatives of

$f(t)\underline{\mathbf{x}}(t)$ (scalar function times vector function),
$f(t)A(t)$ (scalar function times matrix function),
$A(t)\underline{\mathbf{x}}(t)$ (matrix function times vector function),
$\underline{\mathbf{x}}(t) \cdot \underline{\mathbf{y}}(t)$ (vector function dot product with vector function),
$\underline{\mathbf{x}}(t) \times \underline{\mathbf{y}}(t)$ (cross product of two vector functions),
$A(t)B(t)$ (matrix function times matrix function).

it's any one of these

As long as the "product" operation distributes over addition, and scalars times the product equal the products where the scalar is paired with either one of the terms, there is a product rule. Since the product operation is not assumed to be commutative you need to be careful about the order in which you write down the terms in the product rule, though.

Theorem. Let $A(t)$, $B(t)$ be differentiable scalar, matrix or vector-valued functions of t, and let $*$ be a product operation as above. Then

Not a convolution

$$\boxed{\frac{d}{dt}\left(A(t) * B(t)\right) = A'(t) * B(t) + A(t) * B'(t).}$$

The explanation just rewrites the limit definition explanation for the scalar function product rule that you learned in Calculus, and assumes the product distributes over sums and that scalars can pass through the product to either one of the terms, as is true for all the examples above. It also uses the fact that differentiable functions are continuous, that you learned in Calculus. Here is one explanation that proves all of those product rules at once:

$$\frac{d}{dt}\left(A(t) * B(t)\right) := \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(A(t+\Delta t) * B(t+\Delta t) - A(t) * B(t)\right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(A(t+\Delta t) * B(t+\Delta t) \overbrace{- A(t+\Delta t) * B(t) + A(t+\Delta t) * B(t)}^{=0} - A(t) * B(t)\right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(A(t+\Delta t) * B(t+\Delta t) - A(t+\Delta t) * B(t)\right) + \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(A(t+\Delta t) * B(t) - A(t) * B(t)\right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Big(A(t+\Delta t) * \left(B(t+\Delta t) - B(t)\right) + \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(A(t+\Delta t) - A(t)\right) * B(t)$$

$$= \lim_{\Delta t \to 0} \left(A(t+\Delta t) * \left(\frac{1}{\Delta t}\left(B(t+\Delta t) - B(t)\right)\right)\right) + \lim_{\Delta t \to 0}\left(\frac{1}{\Delta t}\left(A(t+\Delta t) - A(t)\right)\right) * B(t)$$

$$= A(t) * B'(t) + A'(t) * B(t).$$

examples

$$\frac{d}{dt}\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}\begin{bmatrix}e^{2t}\\ 3+4t^2\end{bmatrix} = \cancel{A'}^{\,0}\,\tilde{x} + A\tilde{x}'$$

$$A \qquad \tilde{x}$$

$$= \begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}\begin{bmatrix}2e^{2t}\\ 8t\end{bmatrix}$$

much better

long way would be

$$A\tilde{x} = \begin{bmatrix}e^{2t}+6+8t^2\\ -2e^{2t}+3+4t^2\end{bmatrix}$$

$$\frac{d}{dt}: \begin{bmatrix}2e^{2t}+16t\\ -4e^{2t}+8t\end{bmatrix}$$

§7.1

It is always the case that an initial value problem for single differential equation, or for a system of differential equations is equivalent to an initial value problem for a larger system of first order differential equations, ~~as in the previous example~~. (See examples and homework problems in section 7.1) This gives us a new perspective on e.g. homogeneous differential equations from Chapter 5.

For example, consider this overdamped problem from Chapter 5:

$$(1)\qquad \begin{aligned} x''(t) + 7x'(t) + 6x(t) &= 0 \\ x(0) &= 1 \\ x'(0) &= 4. \end{aligned}$$

Exercise 3a) Solve the IVP above, using Chapter 5 and characteristic polynomial.

$p(r) = r^2 + 7r + 6 = (r+6)(r+1)$ $\qquad \{e^{-t}, e^{-6t}\}$ basis for solutions

$x_H(t) = c_1 e^{-t} + c_2 e^{-6t}$

$x(0) = 1 = c_1 + c_2$

$x'(0) = 4 = -c_1 - 6c_2$

$E_1 + E_2 \Rightarrow 5 = -5c_2 \Rightarrow c_2 = -1$

$1 = c_1 - 1 \Rightarrow c_1 = 2$

$\boxed{x(t) = 2e^{-t} - e^{-6t}}$ $\qquad x(0)=1$, $x'(0)=4$ ✓

3b) Show that if $x(t)$ solves the IVP above (1), then $[x(t), x'(t)]^T$ solves the first order system of DE's IVP

$$(2)\qquad \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -6x_1 - 7x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

Use your work to write down the solution to the IVP in 3b.

If x solves 1: $\begin{bmatrix} x \\ x' \end{bmatrix}' = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} x' \\ -6x - 7x' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix}\begin{bmatrix} x \\ x' \end{bmatrix}$ $\qquad$ IC's: $\begin{bmatrix} x(0) \\ x'(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

(1) $x'' = -6x - 7x'$

3c) Show that if $[x_1(t), x_2(t)]^T$ solves the IVP in 3b then the first entry $x_1(t)$ solves the original second order DE IVP. So converting a second order DE to a first order system is a reversible procedure.

∴ Soln to (2) is $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-6t} \\ -2e^{-t} + 6e^{-6t} \end{bmatrix}$ $\qquad$ (you could check directly!)

$$(2)\ \begin{cases} x_1' = x_2 \\ x_2' = -6x_1 - 7x_2 \\ x_1(0) = 1 \\ x_2(0) = 4 \end{cases}$$

If $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ solves (2), then $\boxed{x_1(t) \text{ solves } (1)}$

$x_1' = x_2$

$\frac{d}{dt}$: $x_1'' = x_2' = -6x_1 - 7x_2 = -6x_1 - 7x_1'$

so $x_1'' + 7x_1' + 6x_1 = 0$

& $x_1(0) = 1$, $x_2(0) = x_1'(0) = 4$

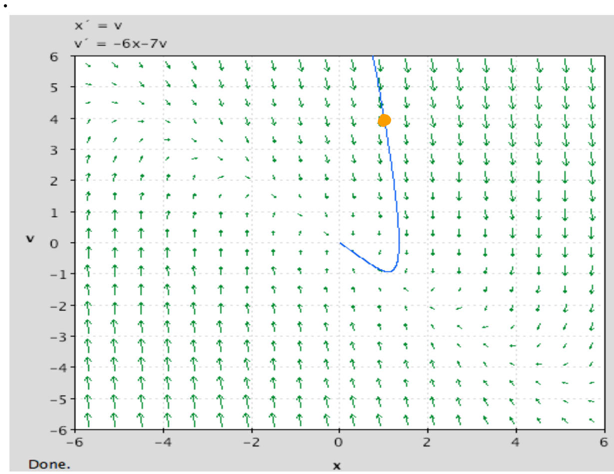
3d) Compare the characteristic polynomial for the homogeneous DE in 3a, to the one for the matrix in the first order system in 3b. It's a mystery (for now)!

$$x''(t) + 7x'(t) + 6x(t) = 0 \qquad p(r) = r^2 + 7r + 6$$

$$\begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \quad |A-\lambda I| = \begin{vmatrix} -\lambda & 1 \\ -6 & -7-\lambda \end{vmatrix} = \lambda(7+\lambda) + 6$$
$$= \lambda^2 + 7\lambda + 6$$

Pictures of the phase portrait for the system in 3b, which is tracking position and velocity of the solution to 3a.

From pplane, for the system:



$$\begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

overdamped

From Wolfram alpha, for the underdamped second order DE in 3a.

"same"

Input:

$\{x''(t) + 7x'(t) + 6x(t) = 0,\ x(0) = 1,\ x'(0) = 4\}$

Differential equation solution:

$x(t) = e^{-6t}(2e^{5t} - 1)$

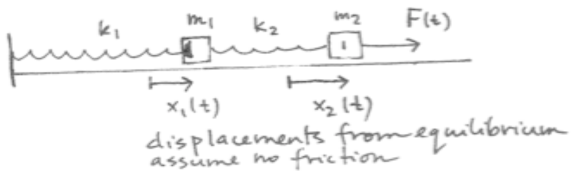
Plots of the solution:

A larger example of converting higher order DE's and systems of DE's into first-order ones....

like §7.1

Example:

Consider this configuration of two coupled masses and springs:



Exercise 4) Use Newton's second law to derive a system of two second order differential equations for $x_1(t), x_2(t)$, the displacements of the respective masses from the equilibrium configuration. What initial value problem do you expect yields unique solutions in this case? (See homework in section 7.1)

$m_1 x_1'' =$ net forces

$m_2 x_2'' =$ net forces