Fri Apr 13

7.1 Systems of differential equations - to model multi-component systems via compartmental analysis:
http://en.wikipedia.org/wiki/Multi-compartment_model

Announcements:

- You'll be computing eigenvalues & eigenvectors for the rest of course — purpose will be to solve systems of linear differential eqtns
- Today is sort of an overview of how that happens (Chapter 7, last chapter of course)

(warm-up for Lab #3)

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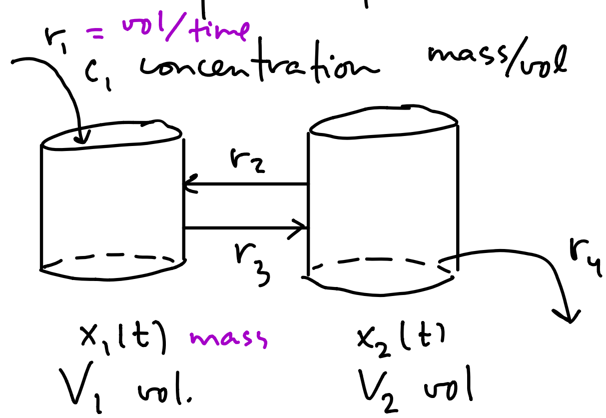
Warm-up Exercise: Let $x_1(t)$, $x_2(t)$ be solute amounts in this 2-component input-output model



$\frac{\text{vol}}{\text{time}}$

$V_1'(t) = 0$.

$V_1'(t) = r_1 + r_2 - r_3 = 0$

$V_2'(t) = 0$

$V_2'(t) = -r_2 - r_4 + r_3 = 0$

a) What condition on the rates r_1, r_2, r_3, r_4 (vol/time) keeps V_1, V_2 constant?

b) What are the DE's for $x_1(t)$ & $x_2(t)$? (assume V_1, V_2 const)

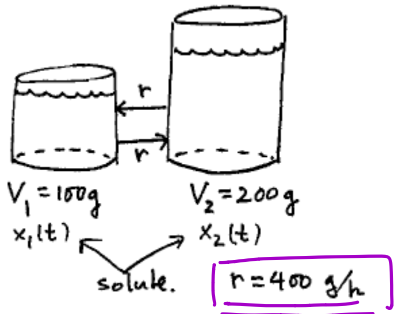
$$x_1'(t) = r_1 c_1 + r_2 \frac{x_2}{V_2} - r_3 \frac{x_1}{V_1}$$

$\frac{\text{mass}}{\text{time}}$ $\quad \frac{\cancel{\text{vol}}}{\text{time}} \frac{\text{mass}}{\cancel{\text{vol}}}$

$\frac{x_2}{V_2}$ ↑ average concentration in tank (at time t)

$$x_2'(t) = r_3 \frac{x_1}{V_1} - (r_2 + r_4) \frac{x_2}{V_2}$$

Here's a relatively simple 2-tank problem to illustrate the ideas:



g = gallons
h = hours

Exercise 1) Find differential equations for solute amounts $x_1(t), x_2(t)$ above, using input-output modeling. Assume solute concentration is uniform in each tank. If $x_1(0) = b_1, x_2(0) = b_2$, write down the initial value problem that you expect would have a unique solution.

$$x_1'(t) = r \cdot \frac{x_2}{200} - r\frac{x_1}{100} = 400\frac{x_2}{200} - \frac{400\,x_1}{100} = -4x_1 + 2x_2$$

$$x_2' = 400\frac{x_1}{100} - 400\frac{x_2}{200} = 4x_1 - 2x_2$$

IVP
$$\begin{cases} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \begin{bmatrix} x_1(0) \\ x_1'(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \end{cases}$$

answer (in matrix-vector form):

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Geometric interpretation of first order systems of differential equations.

The example on page 1 is a special case of the general initial value problem for a first order system of differential equations:

vector version of Chapter 1 existence-unique.

IVP $\left\{ \begin{aligned} \underline{\boldsymbol{x}}'(t) &= \underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t)) \\ \underline{\boldsymbol{x}}(t_0) &= \underline{\boldsymbol{x}}_0 \end{aligned} \right.$

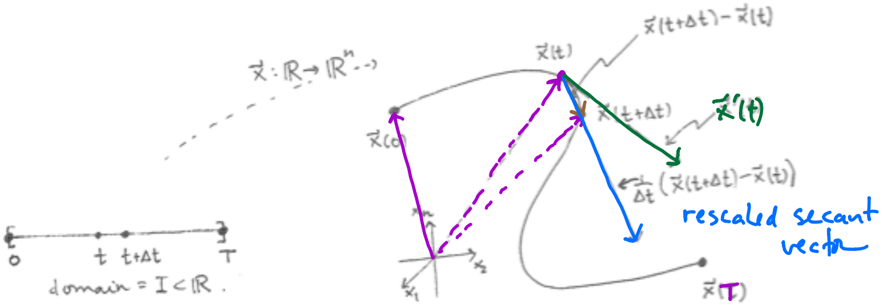
know "veloc." vect. in terms time & current location

know where we start

• We will see how any single differential equation (of any order), or any system of differential equations (of any order) is equivalent to a larger first order system of differential equations. And we will discuss how the natural initial value problems correspond.

Why we expect IVP's for first order systems of DE's to have unique solutions $\underline{\boldsymbol{x}}(t)$:

• From either a multivariable calculus course, or from physics, recall the geometric/physical interpretation of $\underline{\boldsymbol{x}}'(t)$ as the tangent/velocity vector to the parametric curve of points with position vector $\underline{\boldsymbol{x}}(t)$, as t varies. This picture should remind you of the discussion, but ask questions if this is new to you:



Analytically, the reason that the vector of derivatives $\underline{\boldsymbol{x}}'(t)$ computed component by component is actually a limit of scaled secant vectors (and therefore a tangent/velocity vector) is:

$$\underline{\boldsymbol{x}}'(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\begin{bmatrix} x_1(t+\Delta t) \\ x_2(t+\Delta t) \\ \vdots \\ x_n(t+\Delta t) \end{bmatrix} - \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \right)$$

$$= \lim_{\Delta t \to 0} \begin{bmatrix} \frac{1}{\Delta t}\left(x_1(t+\Delta t) - x_1(t)\right) \\ \frac{1}{\Delta t}\left(x_2(t+\Delta t) - x_2(t)\right) \\ \vdots \\ \frac{1}{\Delta t}\left(x_n(t+\Delta t) - x_n(t)\right) \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix},$$

provided each component function is differentiable. Therefore, the reason you expect a unique solution to the IVP for a first order system is that you know where you start ($\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$), and you know your "velocity" vector (depending on time and current location) $\Rightarrow$ you expect a unique solution! (Plus, you could use something like a vector version of Euler's method or the Runge-Kutta method to approximate it! You just convert the scalar quantities in the code into vector quantities. And this is what numerical solvers do.)

Exercise 2) Return to the page 1 tank example

$$x_1'(t) = -4x_1 + 2x_2$$
$$x_2'(t) = 4x_1 - 2x_2$$
$$x_1(0) = 9 \leftarrow \text{9 lbs in tank 1}$$
$$x_2(0) = 0 \leftarrow \text{0 lbs in tank 2.}$$

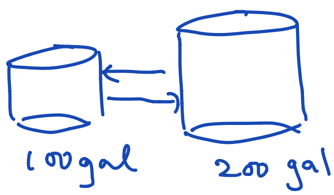
2a) Interpret the parametric solution curve $[x_1(t), x_2(t)]^T$ to this IVP, as indicated in the pplane screen shot below. ("pplane" is the sister program to "dfield", that we were using in Chapters 1-2.) Notice how it follows the "velocity" vector field (which is time-independent in this example), and how the "particle motion" location $[x_1(t), x_2(t)]^T$ is actually the vector of solute amounts in each tank, at time t. If your system involved ten coupled tanks rather than two, then this "particle" is moving around in $\mathbb{R}^{10}$.

2b) What are the apparent limiting solute amounts in each tank?

2c) How could your smart-alec younger sibling have told you the answer to 2b without considering any differential equations or "velocity vector fields" at all?

$$\lim_{t\to\infty} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

2c)



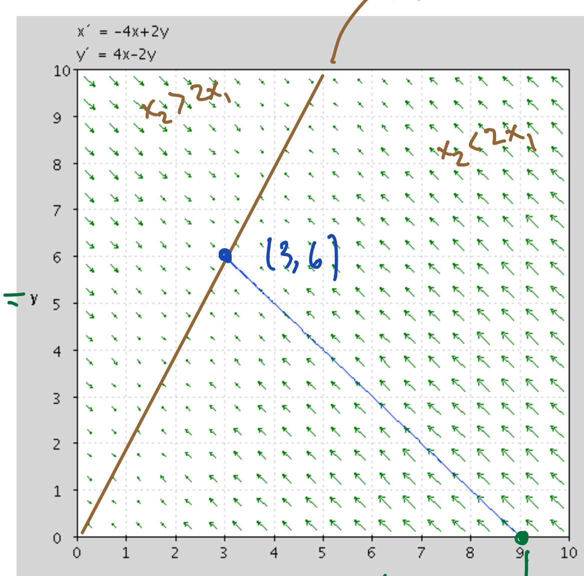
expect same concentrations in each tank as $t \to \infty$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

& expect total of 9 lbs & twice as much in 2nd tank because twice the vol.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{(2x_1 - x_2)}_{0} \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$x_2 = 2x_1$



$x_2 = y$

$x = x_1$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

Chapter scalar eqtns

First order systems of differential equations of the form

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

are called linear homogeneous systems of DE's. (Think of rewriting the system as

$$\underline{\boldsymbol{x}}'(t) - A\,\underline{\boldsymbol{x}} = \underline{\mathbf{0}}$$

$x'(t) + p(t)x = 0$

in analogy with how we wrote linear scalar differential equations.) Then the inhomogeneous system of first order DE's would be written as

$$\boxed{\underline{\boldsymbol{x}}'(t) - A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{f}}(t)}$$

$x'(t) + p(t)x(t) = q(t)$

or

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}} + \underline{\boldsymbol{f}}(t)$$

Notice that the operator on vector-valued functions $\underline{\boldsymbol{x}}(t)$ defined by

$$L(\underline{\boldsymbol{x}}(t)) := \underline{\boldsymbol{x}}'(t) - A\,\underline{\boldsymbol{x}}(t)$$

is linear, i.e.

$$L(\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{y}}(t)) = L(\underline{\boldsymbol{x}}(t)) + L(\underline{\boldsymbol{y}}(t))$$
$$L(c\,\underline{\boldsymbol{x}}(t)) = c\,L(\underline{\boldsymbol{x}}(t)).$$

$$L(\vec{x}(t)+\vec{y}(t)) = (\vec{x}'+\vec{y}') - A(\vec{x}+\vec{y}) = (\vec{x}'-A\vec{x}) + (\vec{y}'-A\vec{y}) = L(\vec{x}) + L(\vec{y})$$

SO! The space of solutions to the homogeneous first order system of differential equations

$$\underline{\boldsymbol{x}}'(t) - A\,\underline{\boldsymbol{x}} = \underline{\mathbf{0}}$$

is a subspace. AND the general solution to the inhomogeneous system

$$\underline{\boldsymbol{x}}'(t) - A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{f}}(t)$$

will be of the form

$$\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}_P + \underline{\boldsymbol{x}}_H$$

where $\underline{\boldsymbol{x}}_P$ is any single particular solution and $\underline{\boldsymbol{x}}_H$ is the general homogeneous solution.

Exercise 3) In the case that A is a constant matrix (i.e. entries don't depend on t), consider the homogeneous problem

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}.$$

Look for solutions of the form

$$\boxed{\underline{\boldsymbol{x}}(t) = e^{\lambda t}\underline{\boldsymbol{v}},}$$

where $\underline{\boldsymbol{v}}$ is a constant vector. Show that $\underline{\boldsymbol{x}}(t) = e^{\lambda t}\underline{\boldsymbol{v}}$ solves the homogeneous DE system if and only if v is an eigenvector of A, with eigenvalue λ, i.e. $A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$.

Hint: In order for such an $\underline{\boldsymbol{x}}(t)$ to solve the DE it must be true that

$$\underline{\boldsymbol{x}}'(t) = \lambda\,e^{\lambda t}\underline{\boldsymbol{v}}$$

and

$$A\,\underline{\boldsymbol{x}}(t) = A\,e^{\lambda t}\underline{\boldsymbol{v}} = e^{\lambda t}A\,\underline{\boldsymbol{v}}$$

Set these two expressions equal.

$$\vec{x}'(t) = A\vec{x}$$

look for soltns of form $e^{\lambda t}\vec{v} = \vec{x}$ $\quad \vec{v}$ const. vector

$$\Rightarrow \quad \lambda e^{\lambda t}\vec{v} = \vec{x}'$$

$$A\vec{x} = A(e^{\lambda t}\vec{v}) = e^{\lambda t}(A\vec{v})$$

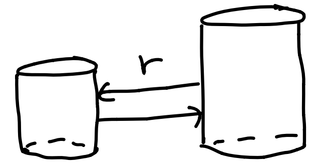
$$\vec{x}' = A\vec{x}$$
$$\lambda \cancel{e^{\lambda t}}\vec{v} = \cancel{e^{\lambda t}}A\vec{v}$$

$$\boxed{A\vec{v} = \lambda\vec{v}}$$

Exercise 4) Use the idea of Exercise 3 to solve the initial value problem of Exercise 2!! Compare your solution $\underline{\boldsymbol{x}}(t)$ to the parametric curve drawn by pplane, that we looked at a couple of pages back.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$



$x_1(t)$ $x_2(t)$

$V_1 = 100\,g$ $V_2 = 200\,g$

$r = 400\ g/h$

$$\begin{vmatrix} -4-\lambda & 2 \\ 4 & -2-\lambda \end{vmatrix} = (-4-\lambda)(-2-\lambda) - 8 = \cancel{8} + 6\lambda + \lambda^2 - \cancel{8} = \lambda(\lambda+6)$$

eigenvalues $\lambda = 0, -6.$

$E_{\lambda=0}$:

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \\ \hline 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$E_{\lambda=0} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

OR

$v_1 = \frac{1}{2}t$

$v_2 = t$

$\vec{v} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

$E_{\lambda=-6}$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$E_{\lambda=-6} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \cancel{e^{0t}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

find c_1 & c_2

find $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

Exercise 5) Lessons learned from tank example: What condition on the matrix $A_{n \times n}$ will allow you to uniquely solve every initial value problem

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0 \in \mathbb{R}^n$$

using the method in Exercise 3-4 ? Hint: Chapter 6. (If that condition fails there are other ways to find the unique solutions.)