Solution To solve the equation in (14), we separate the variables and integrate. We get

$$\int \frac{dP}{P(P-150)} = \int 0.0004\,dt,$$

$$-\frac{1}{150}\int\left(\frac{1}{P} - \frac{1}{P-150}\right)dP = \int 0.0004\,dt \quad \text{[partial fractions]},$$

$$\ln|P| - \ln|P-150| = -0.06t + C,$$

$$\frac{P}{P-150} = \pm e^{C}e^{-0.06t} = Be^{-0.06t} \qquad [\text{where } B = \pm e^{C}]. \tag{15}$$

(a) Substitution of $t = 0$ and $P = 200$ into (15) gives $B = 4$. With this value of B we solve Eq. (15) for

$$P(t) = \frac{600e^{-0.06t}}{4e^{-0.06t} - 1}. \tag{16}$$

Note that, as t increases and approaches $T = \ln(4)/0.06 \approx 23.105$, the positive denominator on the right in (16) decreases and approaches 0. Consequently $P(t) \to +\infty$ as $t \to T^-$. This is a *doomsday* situation—a real population *explosion*.

(b) Substitution of $t = 0$ and $P = 100$ into (15) gives $B = -2$. With this value of B we solve Eq. (15) for

$$P(t) = \frac{300e^{-0.06t}}{2e^{-0.06t} + 1} = \frac{300}{2 + e^{0.06t}}. \tag{17}$$

Note that, as t increases without bound, the positive denominator on the right in (16) approaches $+\infty$. Consequently, $P(t) \to 0$ as $t \to +\infty$. This is an (eventual) *extinction* situation. ■

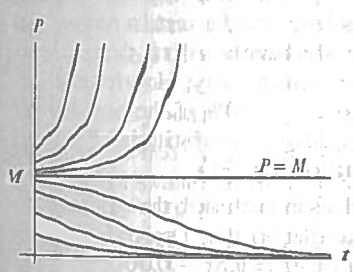

FIGURE 2.1.6. Typical solution curves for the explosion/extinction equation $P' = kP(P - M)$.

Thus the population in Example 7 either explodes or is an endangered species threatened with extinction, depending on whether or not its initial size exceeds the threshold population $M = 150$. An approximation to this phenomenon is sometimes observed with animal populations, such as the alligator population in certain areas of the southern United States.

Figure 2.1.6 shows typical solution curves that illustrate the two possibilities for a population $P(t)$ satisfying Eq. (13). If $P_0 = M$ (exactly!), then the population remains constant. However, this equilibrium situation is very unstable. If P_0 exceeds M (even slightly), then $P(t)$ rapidly increases without bound, whereas if the initial (positive) population is less than M (however slightly), then it decreases (more gradually) toward zero as $t \to +\infty$. See Problem 33.

2.1 Problems

Separate variables and use partial fractions to solve the initial value problems in Problems 1 through 8. Use either the exact solution or a computer-generated slope field to sketch the graphs of several solutions of the given differential equation, and highlight the indicated particular solution.

1. $\dfrac{dx}{dt} = x - x^2,\ x(0) = 2$ **2.** $\dfrac{dx}{dt} = 10x - x^2,\ x(0) = 1$

3. $\dfrac{dx}{dt} = 1 - x^2,\ x(0) = 3$ **4.** $\dfrac{dx}{dt} = 9 - 4x^2,\ x(0) = 0$

5. $\dfrac{dx}{dt} = 3x(5-x), x(0) = 8$

6. $\dfrac{dx}{dt} = 3x(x-5), x(0) = 2$

7. $\dfrac{dx}{dt} = 4x(7-x), x(0) = 11$

8. $\dfrac{dx}{dt} = 7x(x-13), x(0) = 17$

9. The time rate of change of a rabbit population P is proportional to the square root of P. At time $t = 0$ (months) the population numbers 100 rabbits and is increasing at the rate of 20 rabbits per month. How many rabbits will there be one year later?

10. Suppose that the fish population $P(t)$ in a lake is attacked by a disease at time $t = 0$, with the result that the fish cease to reproduce (so that the birth rate is $\beta = 0$) and the death rate δ (deaths per week per fish) is thereafter proportional to $1/\sqrt{P}$. If there were initially 900 fish in the lake and 441 were left after 6 weeks, how long did it take all the fish in the lake to die?

11. Suppose that when a certain lake is stocked with fish, the birth and death rates β and δ are both inversely proportional to $\sqrt{P}$. (a) Show that

$$P(t) = \left(\tfrac{1}{2}kt + \sqrt{P_0}\right)^2,$$

where k is a constant. (b) If $P_0 = 100$ and after 6 months there are 169 fish in the lake, how many will there be after 1 year?

12. The time rate of change of an alligator population P in a swamp is proportional to the square of P. The swamp contained a dozen alligators in 1988, two dozen in 1998. When will there be four dozen alligators in the swamp? What happens thereafter?

13. Consider a prolific breed of rabbits whose birth and death rates, β and δ, are each proportional to the rabbit population $P = P(t)$, with $\beta > \delta$. (a) Show that

$$P(t) = \frac{P_0}{1 - kP_0t}, \quad k \text{ constant.}$$

Note that $P(t) \to +\infty$ as $t \to 1/(kP_0)$. This is doomsday. (b) Suppose that $P_0 = 6$ and that there are nine rabbits after ten months. When does doomsday occur?

14. Repeat part (a) of Problem 13 in the case $\beta < \delta$. What now happens to the rabbit population in the long run?

15. Consider a population $P(t)$ satisfying the logistic equation $dP/dt = aP - bP^2$, where $B = aP$ is the time rate at which births occur and $D = bP^2$ is the rate at which deaths occur. If the initial population is $P(0) = P_0$, and B_0 births per month and D_0 deaths per month are occurring at time $t = 0$, show that the limiting population is $M = B_0P_0/D_0$.

16. Consider a rabbit population $P(t)$ satisfying the logistic equation as in Problem 15. If the initial population is 120 rabbits and there are 8 births per month and 6 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 95% of the limiting population M?

17. Consider a rabbit population $P(t)$ satisfying the logistic equation as in Problem 15. If the initial population is 240 rabbits and there are 9 births per month and 12 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 105% of the limiting population M?

18. Consider a population $P(t)$ satisfying the extinction-explosion equation $dP/dt = aP^2 - bP$, where $B = aP^2$ is the time rate at which births occur and $D = bP$ is the rate at which deaths occur. If the initial population is $P(0) = P_0$ and B_0 births per month and D_0 deaths per month are occurring at time $t = 0$, show that the threshold population is $M = D_0P_0/B_0$.

19. Consider an alligator population $P(t)$ satisfying the extinction-explosion equation as in Problem 18. If the initial population is 100 alligators and there are 10 births per month and 9 deaths per months occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 10 times the threshold population M?

20. Consider an alligator population $P(t)$ satisfying the extinction/explosion equation as in Problem 18. If the initial population is 110 alligators and there are 11 births per month and 12 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 10% of the threshold population M?

21. Suppose that the population $P(t)$ of a country satisfies the differential equation $dP/dt = kP(200 - P)$ with k constant. Its population in 1940 was 100 million and was then growing at the rate of 1 million per year. Predict this country's population for the year 2000.

22. Suppose that at time $t = 0$, half of a "logistic" population of 100,000 persons have heard a certain rumor, and that the number of those who have heard it is then increasing at the rate of 1000 persons per day. How long will it take for this rumor to spread to 80% of the population? (*Suggestion*: Find the value of k by substituting $P(0)$ and $P'(0)$ in the logistic equation, Eq. (3).)

23. As the salt KNO_3 dissolves in methanol, the number $x(t)$ of grams of the salt in a solution after t seconds satisfies the differential equation $dx/dt = 0.8x - 0.004x^2$.

    (a) What is the maximum amount of the salt that will ever dissolve in the methanol?

    (b) If $x = 50$ when $t = 0$, how long will it take for an additional 50 g of salt to dissolve?

24. Suppose that a community contains 15,000 people who are susceptible to Michaud's syndrome, a contagious disease. At time $t = 0$ the number $N(t)$ of people who have developed Michaud's syndrome is 5000 and is increasing at the rate of 500 per day. Assume that $N'(t)$ is proportional to the product of the numbers of those who have caught the disease and of those who have not. How long will it take for another 5000 people to develop Michaud's syndrome?

25. The data in the table in Fig. 2.1.7 are given for a certain population $P(t)$ that satisfies the logistic equation in (3). **(a)** What is the limiting population M? (*Suggestion*: Use the approximation

$$P'(t) \approx \frac{P(t+h) - P(t-h)}{2h}$$

with $h = 1$ to estimate the values of $P'(t)$ when $P = 25.00$ and when $P = 47.54$. Then substitute these values in the logistic equation and solve for k and M.) **(b)** Use the values of k and M found in part (a) to determine when $P = 75$. (*Suggestion*: Take $t = 0$ to correspond to the year 1925.)

Year	P (millions)
1924	24.63
1925	25.00
1926	25.38
⋮	⋮
1974	47.04
1975	47.54
1976	48.04

FIGURE 2.1.7. Population data for Problem 25.

26. A population $P(t)$ of small rodents has birth rate $\beta = (0.001)P$ (births per month per rodent) and *constant* death rate δ. If $P(0) = 100$ and $P'(0) = 8$, how long (in months) will it take this population to double to 200 rodents? (*Suggestion*: First find the value of δ.)

27. Consider an animal population $P(t)$ with constant death rate $\delta = 0.01$ (deaths per animal per month) and with birth rate β proportional to P. Suppose that $P(0) = 200$ and $P'(0) = 2$. **(a)** When is $P = 1000$? **(b)** When does doomsday occur?

28. Suppose that the number $x(t)$ (with t in months) of alligators in a swamp satisfies the differential equation $dx/dt = 0.0001x^2 - 0.01x$.

(a) If initially there are 25 alligators in the swamp, solve this differential equation to determine what happens to the alligator population in the long run.

(b) Repeat part (a), except with 150 alligators initially.

29. During the period from 1790 to 1930, the U.S. population $P(t)$ (t in years) grew from 3.9 million to 123.2 million. Throughout this period, $P(t)$ remained close to the solution of the initial value problem

$$\frac{dP}{dt} = 0.03135P - 0.0001489P^2, \quad P(0) = 3.9.$$

(a) What 1930 population does this logistic equation predict?

(b) What limiting population does it predict?

(c) Has this logistic equation continued since 1930 to accurately model the U.S. population?

[This problem is based on a computation by Verhulst, who in 1845 used the 1790–1840 U.S. population data to predict accurately the U.S. population through the year 1930 (long after his own death, of course).]

30. A tumor may be regarded as a population of multiplying cells. It is found empirically that the "birth rate" of the cells in a tumor decreases exponentially with time, so that $\beta(t) = \beta_0 e^{-\alpha t}$ (where α and β_0 are positive constants), and hence

$$\frac{dP}{dt} = \beta_0 e^{-\alpha t} P, \quad P(0) = P_0.$$

Solve this initial value problem for

$$P(t) = P_0 \exp\left(\frac{\beta_0}{\alpha}(1 - e^{-\alpha t})\right).$$

Observe that $P(t)$ approaches the finite limiting population $P_0 \exp(\beta_0/\alpha)$ as $t \to +\infty$.

31. For the tumor of Problem 30, suppose that at time $t = 0$ there are $P_0 = 10^6$ cells and that $P(t)$ is then increasing at the rate of 3×10^5 cells per month. After 6 months the tumor has doubled (in size and in number of cells). Solve numerically for α, and then find the limiting population of the tumor.

32. Derive the solution

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}$$

of the logistic initial value problem $P' = kP(M - P)$, $P(0) = P_0$. Make it clear how your derivation depends on whether $0 < P_0 < M$ or $P_0 > M$.

33. **(a)** Derive the solution

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{kMt}}$$

of the extinction-explosion initial value problem $P' = kP(P - M)$, $P(0) = P_0$.

(b) How does the behavior of $P(t)$ as t increases depend on whether $0 < P_0 < M$ or $P_0 > M$?

34. If $P(t)$ satisfies the logistic equation in (3), use the chain rule to show that

$$P''(t) = 2k^2 P(P - \tfrac{1}{2}M)(P - M).$$

Conclude that $P'' > 0$ if $0 < P < \frac{1}{2}M$; $P'' = 0$ if $P = \frac{1}{2}M$; $P'' < 0$ if $\frac{1}{2}M < P < M$; and $P'' > 0$ if $P > M$. In particular, it follows that any solution curve that crosses the line $P = \frac{1}{2}M$ has an inflection point where it crosses that line, and therefore resembles one of the lower S-shaped curves in Fig. 2.1.3.

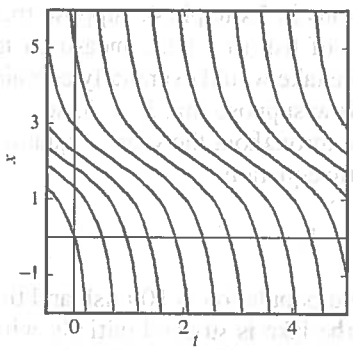

FIGURE 2.2.11. Solution curves of the equation $x' = x(4-x) - h$ with excessive harvesting $h = 5$.

If, finally, $h > 4$, then the quadratic equation corresponding to (20) has no real solutions and the differential equation in (19) has no equilibrium solutions. The solution curves then look like those illustrated in Fig. 2.2.11, and (whatever the initial number of fish) the population dies out as a result of the excessive harvesting. ■

If we imagine turning a dial to gradually increase the value of the parameter h in Eq. (19), then the picture of the solution curves changes from one like Fig. 2.2.8 with $h < 4$, to Fig. 2.2.10 with $h = 4$, to one like Fig. 2.2.11 with $h > 4$. Thus the differential equation has

- two critical points if $h < 4$;
- one critical point if $h = 4$;
- no critical point if $h > 4$.

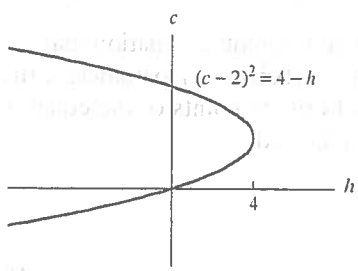

FIGURE 2.2.12. The parabola $(c-2)^2 = 4 - h$ is the bifurcation diagram for the differential equation $x' = x(4-x) - h$.

The value $h = 4$—for which the qualitative nature of the solutions changes as h increases—is called a **bifurcation point** for the differential equation containing the parameter h. A common way to visualize the corresponding "bifurcation" in the solutions is to plot the **bifurcation diagram** consisting of all points (h, c), where c is a critical point of the equation $x' = x(4-x) + h$. For instance, if we rewrite Eq. (20) as

$$c = 2 \pm \sqrt{4-h},$$
$$(c-2)^2 = 4 - h,$$

where either $c = N$ or $c = H$, then we get the equation of the parabola that is shown in Fig. 2.2.12. This parabola is then the bifurcation diagram for our differential equation that models a logistic fish population with harvesting at the level specified by the parameter h.

2.2 Problems

In Problems 1 through 12, first solve the equation $f(x) = 0$ to find the critical points of the given autonomous differential equation $dx/dt = f(x)$. Then analyze the sign of $f(x)$ to determine whether each critical point is stable or unstable, and construct the corresponding phase diagram for the differential equation. Next, solve the differential equation explicitly for $x(t)$ in terms of t. Finally, use either the exact solution or a computer-generated slope field to sketch typical solution curves for the given differential equation, and verify visually the stability of each critical point.

1. $\dfrac{dx}{dt} = x - 4$
2. $\dfrac{dx}{dt} = 3 - x$
3. $\dfrac{dx}{dt} = x^2 - 4x$
4. $\dfrac{dx}{dt} = 3x - x^2$
5. $\dfrac{dx}{dt} = x^2 - 4$
6. $\dfrac{dx}{dt} = 9 - x^2$
7. $\dfrac{dx}{dt} = (x-2)^2$
8. $\dfrac{dx}{dt} = -(3-x)^2$
9. $\dfrac{dx}{dt} = x^2 - 5x + 4$
10. $\dfrac{dx}{dt} = 7x - x^2 - 10$
11. $\dfrac{dx}{dt} = (x-1)^3$
12. $\dfrac{dx}{dt} = (2-x)^3$

In Problems 13 through 18, use a computer system or graphing calculator to plot a slope field and/or enough solution curves to indicate the stability or instability of each critical point of the given differential equation. (Some of these critical points may be semistable in the sense mentioned in Example 6.)

13. $\dfrac{dx}{dt} = (x+2)(x-2)^2$
14. $\dfrac{dx}{dt} = x(x^2 - 4)$
15. $\dfrac{dx}{dt} = (x^2 - 4)^2$
16. $\dfrac{dx}{dt} = (x^2 - 4)^3$
17. $\dfrac{dx}{dt} = x^2(x^2 - 4)$
18. $\dfrac{dx}{dt} = x^3(x^2 - 4)$
19. The differential equation $dx/dt = \frac{1}{10}x(10-x) - h$ models a logistic population with harvesting at rate h. Determine (as in Example 6) the dependence of the number of critical points on the parameter h, and then construct a bifurcation diagram like Fig. 2.2.12.
20. The differential equation $dx/dt = \frac{1}{100}x(x-5) + s$ models a population with stocking at rate s. Determine the depen-

dence of the number of critical points c on the parameter s, and then construct the corresponding bifurcation diagram in the sc-plane.

21. Consider the differential equation $dx/dt = kx - x^3$. **(a)** If $k \leqq 0$, show that the only critical value $c = 0$ of x is stable. **(b)** If $k > 0$, show that the critical point $c = 0$ is now unstable, but that the critical points $c = \pm\sqrt{k}$ are stable. Thus the qualitative nature of the solutions changes at $k = 0$ as the parameter k increases, and so $k = 0$ is a bifurcation point for the differential equation with parameter k. The plot of all points of the form (k, c), where c is a critical point of the equation $x' = kx - x^3$, is the "pitchfork diagram" shown in Fig. 2.2.13.

22. Consider the differential equation $dx/dt = x + kx^3$ containing the parameter k. Analyze (as in Problem 21) the dependence of the number and nature of the critical points on the value of k, and construct the corresponding bifurcation diagram.

23. Suppose that the logistic equation $dx/dt = kx(M - x)$ models a population $x(t)$ of fish in a lake after t months during which no fishing occurs. Now suppose that, because of fishing, fish are removed from the lake at the rate of hx fish per month (with h a positive constant). Thus fish are "harvested" at a rate proportional to the existing fish population, rather than at the constant rate of Example 4. **(a)** If $0 < h < kM$, show that the population is still logistic. What is the new limiting population? **(b)** If $h \geqq kM$, show that $x(t) \to 0$ are $t \to +\infty$, so the lake is eventually fished out.

24. Separate variables in the logistic harvesting equation $dx/dt = k(N - x)(x - H)$ and then use partial fractions to derive the solution given in Eq. (15).

25. Use the alternative forms

$$x(t) = \frac{N(x_0 - H) + H(N - x_0)e^{-k(N-H)t}}{(x_0 - H) + (N - x_0)e^{-k(N-H)t}}$$

$$= \frac{H(N - x_0)e^{-k(N-H)t} - N(H - x_0)}{(N - x_0)e^{-k(N-H)t} - (H - x_0)}$$

of the solution in (15) to establish the conclusions stated in (17) and (18).

Example 4 dealt with the case $4h > kM^2$ in the equation $dx/dt = kx(M - x) - h$ that describes constant-rate harvesting of a logistic population. Problems 26 and 27 deal with the other cases.

26. If $4h = kM^2$, show that typical solution curves look as illustrated in Fig. 2.2.14. Thus if $x_0 \geqq M/2$, then $x(t) \to M/2$ as $t \to +\infty$. But if $x_0 < M/2$, then $x(t) = 0$ after a finite period of time, so the lake is fished out. The critical point $x = M/2$ might be called *semistable*, because it looks stable from one side, unstable from the other.

27. If $4h > kM^2$, show that $x(t) = 0$ after a finite period of time, so the lake is fished out (whatever the initial population). [*Suggestion*: Complete the square to rewrite the differential equation in the form $dx/dt = -k[(x - a)^2 + b^2]$. Then solve explicitly by separation of variables.] The results of this and the previous problem (together with Example 4) show that $h = \frac{1}{4}kM^2$ is a critical harvesting rate for a logistic population. At any lesser harvesting rate the population approaches a limiting population N that is less than M (why?), whereas at any greater harvesting rate the population reaches extinction.

28. This problem deals with the differential equation $dx/dt = kx(x - M) - h$ that models the harvesting of an unsophisticated population (such as alligators). Show that this equation can be rewritten in the form $dx/dt = k(x - H)(x - K)$, where

$$H = \tfrac{1}{2}\left(M + \sqrt{M^2 + 4h/k}\right) > 0,$$

$$K = \tfrac{1}{2}\left(M - \sqrt{M^2 + 4h/k}\right) < 0.$$

Show that typical solution curves look as illustrated in Fig. 2.2.15.

29. Consider the two differential equations

$$\frac{dx}{dt} = (x - a)(x - b)(x - c) \tag{21}$$

and

$$\frac{dx}{dt} = (a - x)(b - x)(c - x), \tag{22}$$

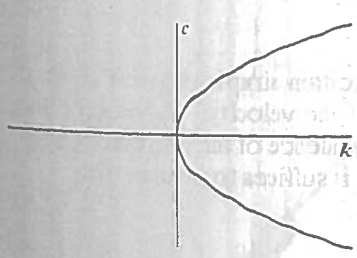

FIGURE 2.2.13. Bifurcation diagram for $dx/dt = kx - x^3$.

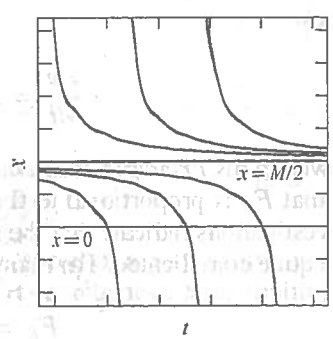

FIGURE 2.2.14. Solution curves for harvesting a logistic population with $4h = kM^2$.

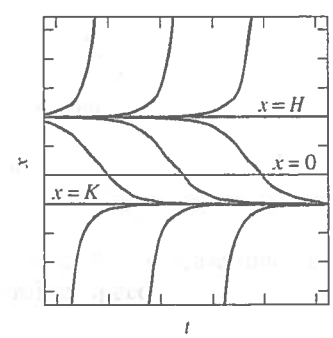

FIGURE 2.2.15. Solution curves for harvesting a population of alligators.

2.3 Problems

1. The acceleration of a Maserati is proportional to the difference between 250 km/h and the velocity of this sports car. If this machine can accelerate from rest to 100 km/h in 10 s, how long will it take for the car to accelerate from rest to 200 km/h?

2. Suppose that a body moves through a resisting medium with resistance proportional to its velocity v, so that $dv/dt = -kv$. (a) Show that its velocity and position at time t are given by

$$v(t) = v_0 e^{-kt}$$

and

$$x(t) = x_0 + \left(\frac{v_0}{k}\right)(1 - e^{-kt}).$$

(b) Conclude that the body travels only a finite distance, and find that distance.

3. Suppose that a motorboat is moving at 40 ft/s when its motor suddenly quits, and that 10 s later the boat has slowed to 20 ft/s. Assume, as in Problem 2, that the resistance it encounters while coasting is proportional to its velocity. How far will the boat coast in all?

4. Consider a body that moves horizontally through a medium whose resistance is proportional to the *square* of the velocity v, so that $dv/dt = -kv^2$. Show that

$$v(t) = \frac{v_0}{1 + v_0 kt}$$

and that

$$x(t) = x_0 + \frac{1}{k}\ln(1 + v_0 kt).$$

Note that, in contrast with the result of Problem 2, $x(t) \to +\infty$ as $t \to +\infty$. Which offers less resistance when the body is moving fairly slowly—the medium in this problem or the one in Problem 2? Does your answer seem consistent with the observed behaviors of $x(t)$ as $t \to \infty$?

5. Assuming resistance proportional to the square of the velocity (as in Problem 4), how far does the motorboat of Problem 3 coast in the first minute after its motor quits?

6. Assume that a body moving with velocity v encounters resistance of the form $dv/dt = -kv^{3/2}$. Show that

$$v(t) = \frac{4v_0}{\left(kt\sqrt{v_0} + 2\right)^2}$$

and that

$$x(t) = x_0 + \frac{2}{k}\sqrt{v_0}\left(1 - \frac{2}{kt\sqrt{v_0} + 2}\right).$$

Conclude that under a $\frac{3}{2}$-power resistance a body coasts only a finite distance before coming to a stop.

7. Suppose that a car starts from rest, its engine providing an acceleration of 10 ft/s^2, while air resistance provides 0.1 ft/s^2 of deceleration for each foot per second of the car's velocity. (a) Find the car's maximum possible (limiting) velocity. (b) Find how long it takes the car to attain 90% of its limiting velocity, and how far it travels while doing so.

8. Rework both parts of Problem 7, with the sole difference that the deceleration due to air resistance now is $(0.001)v^2$ ft/s^2 when the car's velocity is v feet per second.

9. A motorboat weighs 32,000 lb and its motor provides a thrust of 5000 lb. Assume that the water resistance is 100 pounds for each foot per second of the speed v of the boat. Then

$$1000\frac{dv}{dt} = 5000 - 100v.$$

If the boat starts from rest, what is the maximum velocity that it can attain?

10. A woman bails out of an airplane at an altitude of 10,000 ft, falls freely for 20 s, then opens her parachute. How long will it take her to reach the ground? Assume linear air resistance ρv ft/s^2, taking $\rho = 0.15$ without the parachute and $\rho = 1.5$ with the parachute. (*Suggestion*: First determine her height above the ground and velocity when the parachute opens.)

11. According to a newspaper account, a paratrooper survived a training jump from 1200 ft when his parachute failed to open but provided some resistance by flapping unopened in the wind. Allegedly he hit the ground at 100 mi/h after falling for 8 s. Test the accuracy of this account. (*Suggestion*: Find ρ in Eq. (4) by assuming a terminal velocity of 100 mi/h. Then calculate the time required to fall 1200 ft.)

12. It is proposed to dispose of nuclear wastes—in drums with weight $W = 640$ lb and volume 8 ft^3—by dropping them into the ocean ($v_0 = 0$). The force equation for a drum falling through water is

$$m\frac{dv}{dt} = -W + B + F_R,$$

where the buoyant force B is equal to the weight (at 62.5 lb/ft^3) of the volume of water displaced by the drum (Archimedes' principle) and F_R is the force of water resistance, found empirically to be 1 lb for each foot per second of the velocity of a drum. If the drums are likely to burst upon an impact of more than 75 ft/s, what is the maximum depth to which they can be dropped in the ocean without likelihood of bursting?

13. Separate variables in Eq. (12) and substitute $u = v\sqrt{\rho/g}$ to obtain the upward-motion velocity function given in Eq. (13) with initial condition $v(0) = v_0$.

14. Integrate the velocity function in Eq. (13) to obtain the upward-motion position function given in Eq. (14) with initial condition $y(0) = y_0$.

15. Separate variables in Eq. (15) and substitute $u = v\sqrt{\rho/g}$ to obtain the downward-motion velocity function given in Eq. (16) with initial condition $v(0) = v_0$.

16. Integrate the velocity function in Eq. (16) to obtain the downward-motion position function given in Eq. (17) with initial condition $y(0) = y_0$.

17. Consider the crossbow bolt of Example 3, shot straight upward from the ground ($y = 0$) at time $t = 0$ with initial velocity $v_0 = 49$ m/s. Take $g = 9.8$ m/s^2 and $\rho = 0.0011$ in Eq. (12). Then use Eqs. (13) and (14) to show that the bolt reaches its maximum height of about 108.47 m in about 4.61 s.

18. Continuing Problem 17, suppose that the bolt is now dropped ($v_0 = 0$) from a height of $y_0 = 108.47$ m. Then use Eqs. (16) and (17) to show that it hits the ground about 4.80 s later with an impact speed of about 43.49 m/s.

19. A motorboat starts from rest (initial velocity $v(0) = v_0 = 0$). Its motor provides a constant acceleration of 4 ft/s^2, but water resistance causes a deceleration of $v^2/400$ ft/s^2. Find v when $t = 10$ s, and also find the *limiting velocity* as $t \to +\infty$ (that is, the maximum possible speed of the boat).

20. An arrow is shot straight upward from the ground with an initial velocity of 160 ft/s. It experiences both the deceleration of gravity and deceleration $v^2/800$ due to air resistance. How high in the air does it go?

21. If a ball is projected upward from the ground with initial velocity v_0 and resistance proportional to v^2, deduce from Eq. (14) that the maximum height it attains is

$$y_{\max} = \frac{1}{2\rho}\ln\left(1 + \frac{\rho v_0^2}{g}\right).$$

22. Suppose that $\rho = 0.075$ (in fps units, with $g = 32$ ft/s^2) in Eq. (15) for a paratrooper falling with parachute open. If he jumps from an altitude of 10,000 ft and opens his parachute immediately, what will be his terminal speed? How long will it take him to reach the ground?

23. Suppose that the paratrooper of Problem 22 falls freely for 30 s with $\rho = 0.00075$ before opening his parachute. How long will it now take him to reach the ground?

24. The mass of the sun is 329,320 times that of the earth and its radius is 109 times the radius of the earth. (a) To what radius (in meters) would the earth have to be compressed in order for it to become a *black hole*—the escape velocity from its surface equal to the velocity $c = 3 \times 10^8$ m/s of light? (b) Repeat part (a) with the sun in place of the earth.

25. (a) Show that if a projectile is launched straight upward from the surface of the earth with initial velocity v_0 less than escape velocity $\sqrt{2GM/R}$, then the maximum distance from the center of the earth attained by the projectile is

$$r_{\max} = \frac{2GMR}{2GM - Rv_0^2},$$

where M and R are the mass and radius of the earth, respectively. (b) With what initial velocity v_0 must such a projectile be launched to yield a maximum altitude of 100 kilometers above the surface of the earth? (c) Find the maximum distance from the center of the earth, expressed in terms of earth radii, attained by a projectile launched from the surface of the earth with 90% of escape velocity.

26. Suppose that you are stranded—your rocket engine has failed—on an asteroid of diameter 3 miles, with density equal to that of the earth with radius 3960 miles. If you have enough spring in your legs to jump 4 feet straight up on earth while wearing your space suit, can you blast off from this asteroid using leg power alone?

27. (a) Suppose a projectile is launched vertically from the surface $r = R$ of the earth with initial velocity $v_0 = \sqrt{2GM/R}$ so $v_0^2 = k^2/R$, where $k^2 = 2GM$. Then solve the differential equation $dr/dt = k/\sqrt{r}$ (from Eq. (23) in this section) explicitly to deduce that $r(t) \to \infty$ as $t \to \infty$.

(b) If the projectile is launched vertically with initial velocity $v_0 > \sqrt{2GM/R}$, deduce that

$$\frac{dr}{dt} = \sqrt{\frac{k^2}{r} + \alpha} > \frac{k}{\sqrt{r}}.$$

Why does it again follow that $r(t) \to \infty$ as $t \to \infty$?

28. (a) Suppose that a body is dropped ($v_0 = 0$) from a distance $r_0 > R$ from the earth's center, so its acceleration is $dv/dt = -GM/r^2$. Ignoring air resistance, show that it reaches the height $r < r_0$ at time

$$t = \sqrt{\frac{r_0}{2GM}}\left(\sqrt{rr_0 - r^2} + r_0\cos^{-1}\sqrt{\frac{r}{r_0}}\right).$$

(*Suggestion*: Substitute $r = r_0\cos^2\theta$ to evaluate $\int\sqrt{r/(r_0 - r)}\,dr$.) (b) If a body is dropped from a height of 1000 km above the earth's surface and air resistance is neglected, how long does it take to fall and with what speed will it strike the earth's surface?

29. Suppose that a projectile is fired straight upward from the surface of the earth with initial velocity $v_0 < \sqrt{2GM/R}$. Then its height $y(t)$ above the surface satisfies the initial value problem

$$\frac{d^2y}{dt^2} = -\frac{GM}{(y+R)^2}; \quad y(0) = 0, \quad y'(0) = v_0.$$

Substitute $dv/dt = v(dv/dy)$ and then integrate to obtain

$$v^2 = v_0^2 - \frac{2GMy}{R(R+y)}$$

for the velocity v of the projectile at height y. What maximum altitude does it reach if its initial velocity is 1 km/s?

30. In Jules Verne's original problem, the projectile launched from the surface of the earth is attracted by both the earth and the moon, so its distance $r(t)$ from the center of the earth satisfies the initial value problem

$$\frac{d^2r}{dt^2} = -\frac{GM_e}{r^2} + \frac{GM_m}{(S-r)^2}; \quad r(0) = R, \quad r'(0) = v_0,$$

where M_e and M_m denote the masses of the earth and the moon, respectively; R is the radius of the earth; and $S = 384{,}400$ km is the distance between the centers of the earth and the moon. To reach the moon, the projectile must only just pass the point between the moon and earth where its net acceleration vanishes. Thereafter it is "under the control" of the moon, and falls from there to the lunar surface. Find the *minimal* launch velocity v_0 that suffices for the projectile to make it "From the Earth to the Moon."