Math 2250-004
Week 9: March 6-10 5.2-5.4

Mon Mar 6: Section 5.2 from last week.

The two main goals in Chapter 5 are to learn the structure of solution sets to n^{th} order linear DE's, including how to solve the IVPs

$$
\begin{aligned}
y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f \\
y(x_0) = b_0 \\
y'(x_0) = b_1 \\
y''(x_0) = b_2 \\
\vdots \\
y^{(n-1)}(x_0) = b_{n-1}
\end{aligned}
$$

<u>and</u> to learn important physics/engineering applications of these general techniques.

Finish Wednesday March 1 notes from last week which discuss the case $n = 2$, and continue into the Friday March 3 notes which discuss the analogous ideas for general n. This is section 5.2 of the text.

Tues Mar 7

5.3: Algorithms for the basis and general (homogeneous) solution to

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

when the coefficients $a_{n-1}, a_{n-2}, \ldots a_1, a_0$ are all constant.

step 1) Try to find a basis made of exponential functions....try $y(x) = e^{r x}$. In this case

$$L(y) = e^{r x}\left(r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0\right) = e^{r x} p(r) .$$

We call this polynomial $p(r)$ the characteristic polynomial for the differential equation, and can read off what it is directly from the expression for $L(y)$. For each root r_j of $p(r)$, we get a solution $e^{r_j x}$ to the homogeneous DE.

Case 1) If $p(r)$ has n distinct (i.e. different) real roots $r_1, r_2, \ldots, r_n$, then

$$e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}$$

is a basis for the solution space; i.e. the general solution is given by

$$y_H(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_n e^{r_n x} .$$

Example: Exercise 1 from last Friday's notes: The differential equation

$$y''' + 3y'' - y' - 3y = 0$$

has characteristic polynomial

$$p(r) = r^3 + 3r^2 - r - 3 = (r+3)\cdot\left(r^2 - 1\right) = (r+3)(r+1)\cdot(r-1)$$

so the general solution to

$$y''' + 3y'' - y' - 3y = 0$$

is

$$y_H(x) = c_1 e^x + c_2 e^{-x} + c_3 e^{-3x} .$$

Exercise 1) By construction, $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}$ all solve the differential equation. Show that they're linearly independent. This will be enough to verify that they're a basis for the solution space, since we know the solution space is n-dimensional. Hint: The easiest way to show this is to list your roots so that $r_1 < r_2 < \ldots < r_n$ and to use a limiting argument.

Case 2) Repeated real roots. In this case $p(r)$ has all real roots $r_1, r_2, \ldots r_m$ $(m < n)$ with the r_j all different, but some of the factors $(r - r_j)$ in $p(r)$ appear with powers bigger than 1 . In other words, $p(r)$ factors as

$$p(r) = (r - r_1)^{k_1} (r - r_2)^{k_2} \ldots (r - r_m)^{k_m}$$

with some of the $k_j > 1$, and $k_1 + k_2 + \ldots + k_m = n$.

Start with a small example: The case of a second order DE for which the characteristic polynomial has a double root.

Exercise 2) Let r_1 be any real number. Consider the homogeneous DE

$$L(y) := y'' - 2\,r_1\, y' + r_1^2\, y = 0\,.$$

with $p(r) = r^2 - 2\,r_1\, r + r_1^2 = (r - r_1)^2$, i.e. r_1 is a double root for $p(r)$. Show that $e^{r_1 x}, x\, e^{r_1 x}$ are a basis for the solution space to $L(y) = 0$, so the general homogeneous solution is

$y_H(x) = c_1 e^{r_1 x} + c_2\, x\, e^{r_1 x}$. Start by checking that $x\, e^{r_1 x}$ actually (magically?) solves the DE.

(We may wish to study a special case $y'' + 6\,y' + 9\,y = 0$.)

Here's the general algorithm: If

$$p(r) = (r - r_1)^{k_1} (r - r_2)^{k_2} \cdots (r - r_m)^{k_m}$$

then (as before) $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_m x}$ are independent solutions, but since $m < n$ there aren't enough of them to be a basis. Here's how you get the rest: For each $k_j > 1$, you actually get independent solutions

$$e^{r_j x}, x e^{r_j x}, x^2 e^{r_j x}, \ldots, x^{k_j - 1} e^{r_j x}.$$

This yields k_j solutions for each root r_j, so since $k_1 + k_2 + \ldots + k_m = n$ you get a total of n solutions to the differential equation. There's a good explanation in the text as to why these additional functions actually do solve the differential equation, see pages 316-318 and the discussion of "polynomial differential operators". I've also made a homework problem in which you can explore these ideas. Using the limiting method we discussed earlier, it's not too hard to show that all n of these solutions are indeed linearly independent, so they are in fact a basis for the solution space to $L(y) = 0$.

Exercise 3) Explicitly antidifferentiate to show that the solution space to the differential equation for $y(x)$

$$y^{(4)} - y^{(3)} = 0$$

agrees with what you would get using the repeated roots algorithm in Case 2 above. Hint: first find $v = y'''$, using $v' - v = 0$, then antidifferentiate three times to find y_H. When you compare to the repeated roots algorithm, note that it includes the possibility $r = 0$ and that $e^{0x} = 1$.

Case 3) Complex number roots - this will be our surprising and fun topic on Wednesday. Our analysis will explain exactly how and why trig functions and mixed exponential-trig-polynomial functions show up as solutions for some of the homogeneous DE's you worked with in your homework for this week. This analysis depends on Euler's formula, one of the most beautiful and useful formulas in mathematics:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

for $i^2 = -1$.

Math 2250-004
Wed Mar 8

5.3 continued. How to find the solution space for n^{th} order linear homogeneous DE's with constant coefficients, and why the algorithms work.

Strategy: In all cases we first try to find a basis for the n-dimensional solution space made of or related to exponential functions....trying $y(x) = e^{r\,x}$ yields

$$L(y) = e^{r\,x}\left(r^n + a_{n-1}r^{n-1} + ... + a_1 r + a_0\right) = e^{r\,x}\,p(r)\,.$$

The characteristic polynomial $p(r)$ and how it factors are the keys to finding the solution space to $L(y) = 0$. There are three cases, of which the first two (distinct and repeated real roots) are in yesterday's notes.

Case 3) $p(r)$ has complex number roots. This is the hardest, but also most interesting case.The punch line is that exponential functions $e^{r\,x}$ still work, except that $r = a \pm b\,i$; but, rather than use those complex exponential functions to construct solution space bases we decompose them into real-valued solutions that are products of exponential and trigonometric functions.

To understand how this all comes about, we need to learn Euler's formula . This also lets us review some important Taylor's series facts from Calc 2. As it turns out, complex number arithmetic and complex exponential functions actually are important in many engineering and science applications.

Recall the Taylor-Maclaurin formula from Calculus

$$f(x) \sim f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + ... + \frac{1}{n!}f^{(n)}(0)x^n + ...$$

(Recall that the partial sum polynomial through order n matches f and its first n derivatives at $x_0 = 0$. When you studied Taylor series in Calculus you sometimes expanded about points other than $x_0 = 0$. You also needed error estimates to figure out on which intervals the Taylor polynomials actually coverged back to f.)

Exercise 1) Use the formula above to recall the three very important Taylor series for

1a) $e^x =$
1b) $\cos(x) =$
1c) $\sin(x) =$
In Calculus you checked that these series actually converge and equal the given functions, for all real numbers x.

Exercise 2) Let $x = i\,\theta$ and use the Taylor series for e^x as the definition of $e^{i\,\theta}$ in order to derive Euler's formula:

$$e^{i\,\theta} = \cos(\theta) + i\sin(\theta)\,.$$

From Euler's formula it makes sense to define
$$e^{a+b\,i} := e^a e^{b\,i} = e^a\left(\cos(b) + i\sin(b)\right)$$
for $a, b \in \mathbb{R}$. So for $x \in \mathbb{R}$ we also get
$$e^{(a+b\,i)x} = e^{a\,x}\left(\cos(b\,x) + i\sin(b\,x)\right) = e^{a\,x}\cos(b\,x) + i\,e^{a\,x}\sin(b\,x)\,.$$

For a complex function $f(x) + i\,g(x)$ we define the derivative by
$$D_x\left(f(x) + i\,g(x)\right) := f'(x) + i\,g'(x)\,.$$
It's straightforward to verify (but would take some time to check all of them) that the usual differentiation rules, i.e. sum rule, product rule, quotient rule, constant multiple rule, all hold for derivatives of complex functions. The following rule pertains most specifically to our discussion and we should check it:

<u>Exercise 3)</u> Check that $D_x\left(e^{(a+b\,i)x}\right) = (a+b\,i)e^{(a+b\,i)x}$, i.e.
$$D_x\,e^{r\,x} = r\,e^{r\,x}$$
even if r is complex. (So also $D_x^2\,e^{r\,x} = D_x r\,e^{r\,x} = r^2\,e^{r\,x}$, $D_x^3\,e^{r\,x} = r^3\,e^{r\,x}$, etc.)

Now return to our differential equation questions, with
$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1\,y' + a_0\,y\,.$$
Then even for complex $r = a + b\,i$ $(a, b \in \mathbb{R})$, our work above shows that
$$L\left(e^{r\,x}\right) = e^{r\,x}\left(r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0\right) = e^{r\,x}\,p(r)\,.$$
So if $r = a + b\,i$ is a complex root of $p(r)$ then $e^{r\,x}$ is a complex-valued function solution to $L(y) = 0$. But L is linear, and because of how we take derivatives of complex functions, we can compute in this case that
$$\begin{aligned} 0 + 0\,i = L\left(e^{r\,x}\right) &= L\left(e^{a\,x}\cos(b\,x) + i\,e^{a\,x}\sin(b\,x)\right) \\ &= L\left(e^{a\,x}\cos(b\,x)\right) + i\,L\left(e^{a\,x}\sin(b\,x)\right)\,. \end{aligned}$$
Equating the real and imaginary parts in the first expression to those in the final expression (because that's what it means for complex numbers to be equal) we deduce
$$0 = L\left(e^{a\,x}\cos(b\,x)\right)$$
$$0 = L\left(e^{a\,x}\sin(b\,x)\right)\,.$$
<u>Upshot:</u> If $r = a + b\,i$ is a complex root of the characteristic polynomial $p(r)$ then
$$y_1 = e^{a\,x}\cos(b\,x)$$
$$y_2 = e^{a\,x}\sin(b\,x)$$
are two solutions to $L(y) = 0$. (The conjugate root $a - b\,i$ would give rise to y_1, $-y_2$, which have the same span.

Case 3) Let L have characteristic polynomial

$$p(r) = r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0$$

with real constant coefficients $a_{n-1}, \ldots, a_1, a_0$. If $(r - (a + b\,i))^k$ is a factor of $p(r)$ then so is the conjugate factor $(r - (a - b\,i))^k$. Associated to these two factors are $2\,k$ real and independent solutions to $L(y) = 0$, namely

$$\begin{array}{cc} e^{a x}\cos(b\,x), & e^{a x}\sin(b\,x) \\ x\,e^{a x}\cos(b\,x), & x\,e^{a x}\sin(b\,x) \\ \vdots & \vdots \\ x^{k-1} e^{a x}\cos(b\,x), & x^{k-1} e^{a x}\sin(b\,x) \end{array}$$

Combining cases 1,2,3, yields a complete algorithm for finding the general solution to $L(y) = 0$, as long as you are able to figure out the factorization of the characteristic polynomial $p(r)$.

Exercise 4) Find a basis for the solution space of functions $y(x)$ that solve

$$y'' + 4\,y = 0\,.$$

(You were told a basis in the last problem of last week's hw....now you know where it came from.)

Exercise 5) Find a basis for the solution space of functions $y(x)$ that solve

$$y'' + 6\,y' + 13\,y = 0\,.$$

Exercise 6) Suppose a 7^{th} order linear homogeneous DE has characteristic polynomial

$$p(r) = \left(r^2 + 6\,r + 13\right)^2 (r - 2)^3\,.$$

What is the general solution to the corresponding homogeneous DE?

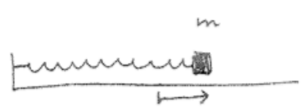
Friday Mar 10

5.4: Applications of 2^{nd} order linear homogeneous DE's with constant coefficients, to unforced spring (and related) configurations.

In this section we study the differential equation below for functions $x(t)$:

$$m\,x'' + c\,x' + k\,x = 0\,.$$

model



$x(t)$ displacement from equilibrium.

equilibrium location

possible external forcing function

Newton's 2^{nd} Law: $m\,x''(t)$ = net forces

$$= -c\,x' - k\,x + F(t)$$

linear drag, proportional to velocity

force from compression or stretching of spring, relative to equilibrium $x=0$

In section 5.4 we assume the time dependent external forcing function $F(t) \equiv 0$. The expression for internal forces $-c\,x' - k\,x$ is a linearization model, about the constant solution $x = 0$, $x' = 0$, for which the net forces must be zero. Notice that $c \geq 0$, $k > 0$. The actual internal forces are probably not exactly linear, but this model is usually effective when $x(t)$, $x'(t)$ are sufficiently small. k is called the <u>Hooke's constant</u>, and c is called the <u>damping coefficient</u>.

This is a constant coefficient linear homogeneous DE, so we try $x(t) = e^{r\,t}$ and compute

$$L(x) := m\,x'' + c\,x' + k\,x = e^{r\,t}\left(m\,r^2 + c\,r + k\right) = e^{r\,t}\,p(r).$$

The different behaviors exhibited by solutions to this mass-spring configuration depend on what sorts of roots the characteristic polynomial $p(r)$ pocesses...

Case 1) no damping ($c = 0$).

$$m\,x'' + k\,x = 0$$
$$x'' + \frac{k}{m}\,x = 0\,.$$
$$p(r) = r^2 + \frac{k}{m}\,,$$

has roots

$$r^2 = -\frac{k}{m} \qquad \text{i.e.} \quad r = \pm\, i\sqrt{\frac{k}{m}}\,.$$

So the general solution is

$$x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}\,t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\,t\right).$$

We write $\sqrt{\frac{k}{m}} := \omega_0$ and call ω_0 the natural angular frequency. Notice that its units are radians per time. We also replace the linear combination coefficients c_1, c_2 by A, B. So, using the alternate letters, the general solution to

$$x'' + \omega_0^{\,2}\,x = 0$$

is

$$x(t) = A\cos\left(\omega_0\,t\right) + B\sin\left(\omega_0\,t\right).$$

This motion is called simple harmonic motion. The reason for this is that $x(t)$ can be rewritten as

$$x(t) = C\cos\left(\omega_0\,t - \alpha\right) = C\cos\left(\omega_0\left(t - \delta\right)\right)$$

in terms of an amplitude $C > 0$ and a phase angle α (or in terms of a time delay δ).

To see why functions of the form

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

are equal (for appropriate choices of constants) to ones of the form

$$x(t) = C\cos(\omega_0 t - \alpha)$$

we use the very important the addition angle trigonometry identities, in this case the addition angle for *cosine* : Consider the possible equality of functions

$$A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \alpha) .$$

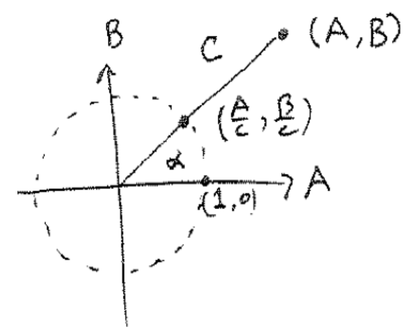
Exercise 1) Use the addition angle formula $\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$ to show that the two functions above are equal provided

$$A = C\cos\alpha$$
$$B = C\sin\alpha \ .$$

So if C, α are given, the formulas above determine A, B. Conversely, if A, B are given then

$$C = \sqrt{A^2 + B^2}$$
$$\frac{A}{C} = \cos(\alpha),\ \frac{B}{C} = \sin(\alpha)$$

determine C, α. These correspondences are best remembered using a diagram in the $A - B$ plane:



It is important to understand the behavior of the functions

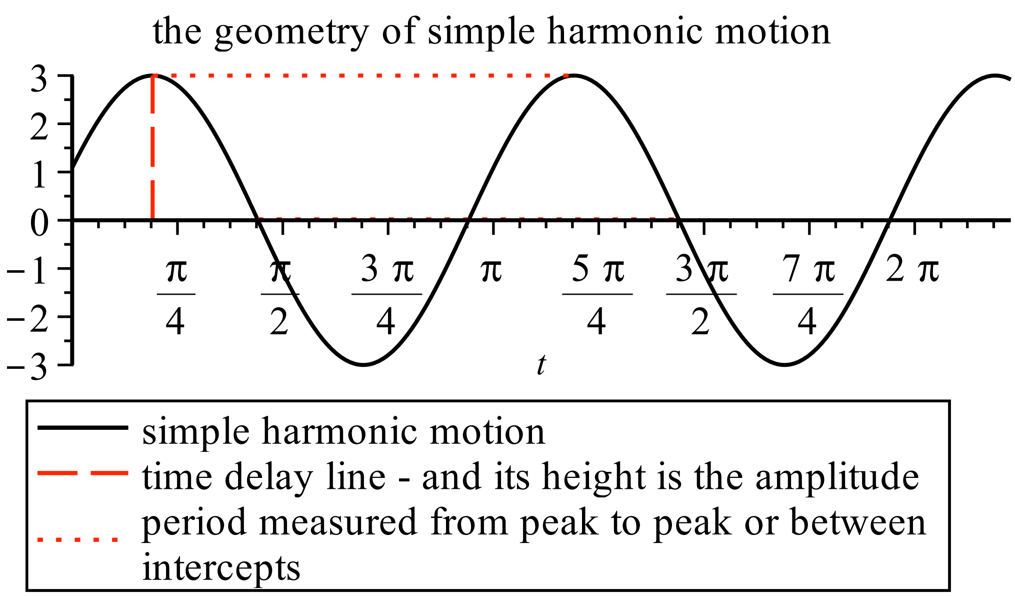
$$A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \alpha) = C\cos(\omega_0 (t - \delta))$$

and the standard terminology:

The <u>amplitude</u> C is the maximum absolute value of $x(t)$. The time delay δ is how much the graph of $C\cos(\omega_0 t)$ is shifted to the right in order to obtain the graph of $x(t)$. Other important data is

$$f = \text{frequency} \;=\; \frac{\omega_0}{2\pi} \quad \text{cycles/time}$$

$$T = \text{period} \;=\; \frac{2\pi}{\omega_0} \;=\; \text{time/cycle.}$$



(I made that plot above with these commands...and then added a title and a legend, from the plot options.)

```
> with(plots) :
> plot1 := plot(3·cos(2(t − .6)), t = 0 ..7, color = black) :
  plot2 := plot([.6, t, t = 0 ..3.], linestyle = dash) :
  plot3 := plot(3, t = .6 ..(.6) + Pi, linestyle = dot) :
  plot4 := plot(0.02, t = .6 + Pi/4 ...6 + 5·Pi/4, linestyle = dot) :
> display({plot1, plot2, plot3, plot4});
>
```

Exercise 2) A mass of 2 kg oscillates without damping on a spring with Hooke's constant $k = 18\ \dfrac{N}{m}$. It is initially stretched 1 m from equilibrium, and released with a velocity of $\dfrac{3}{2}\ \dfrac{m}{s}$.

2a) Show that the mass' motion is described by $x(t)$ solving the initial value problem

$$x'' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

2b) Solve the IVP in a, and convert $x(t)$ into amplitude-phase and amplitude-time delay form. Sketch the solution, indicating amplitude, period, and time delay. Check your work with the commands below.

```
> unassign(`x`);
> with(plots):
  with(DEtools):
> dsolve({x''(t) + 9·x(t) = 0, x(0) = 1, x'(0) = 3/2});
> plot(rhs(%), t = 0..5, color = green);
>
```

• Then, if time, discuss the possibilities that arise when the damping coefficient $c > 0$. There are three cases, depending on the roots of the characteristic polynomial:

Case 2: damping

$$m\,x'' + c\,x' + k\,x = 0$$
$$x'' + \frac{c}{m}\,x' + \frac{k}{m}\,x = 0$$

rewrite as

$$x'' + 2\,p\,x' + \omega_0^2\,x = 0.$$

$\left(p = \frac{c}{2\,m},\ \omega_0^2 = \frac{k}{m}\right)$. The characteristic polynomial is

$$r^2 + 2\,p\,r + \omega_0^2 = 0$$

which has roots

$$r = -\frac{2\,p \pm \sqrt{4\,p^2 - 4\omega_0^2}}{2} = -p \pm \sqrt{p^2 - \omega_0^2}.$$

2a) $(p^2 > \omega_0^2$, or $c^2 > 4\,m\,k)$. overdamped. In this case we have two negative real roots

$$r_1 < r_2 < 0$$

and

$$x(t) = c_1 e^{r_1 t} + c_2\,e^{r_2 t} = e^{r_2 t}\left(c_1 e^{(r_1 - r_2)\,t} + c_2\right).$$

• solution converges to zero exponentially fast; solution passes through equilibrium location $x = 0$ at most once.

2b) ($p^2 = \omega_0^2$, or $c^2 = 4\,m\,k$) critically damped. Double real root $r_1 = r_2 = -p = -\dfrac{c}{2\,m}$.

$$x(t) = e^{-p\,t}\left(c_1 + c_2\,t\right).$$

- solution converges to zero exponentially fast, passing through $x = 0$ at most once, just like in the overdamped case. The critically damped case is the transition between overdamped and underdamped:

2c) ($p^2 < \omega_0^2$, or $c^2 < 4\,m\,k$) underdamped. Complex roots

$$r = -p \pm \sqrt{p^2 - \omega_0^2} = -p \pm i\,\omega_1$$

with $\omega_1 = \sqrt{\omega_0^2 - p^2} < \omega_0$.

$$x(t) = e^{-p\,t}\left(A\cos(\omega_1 t) + B\sin(\omega_1 t)\right) = e^{-p\,t}C\cos(\omega_1 t - \alpha_1).$$

- solution decays exponentially to zero, but oscillates infinitely often, with exponentially decaying pseudo-amplitude $e^{-p\,t}C$ and pseudo-angular frequency ω_1, and pseudo-phase angle α_1 .

Exercise 3) Classify by finding the roots of the characteristic polynomial. Then solve for $x(t)$:
3a)

$$x'' + 6\,x' + 9\,x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}\,.$$

> $with(DEtools):$

> $dsolve\left(\left\{x''(t) + 6\cdot x'(t) + 9\cdot x(t) = 0, x(0) = 1, x'(0) = \frac{3}{2}\right\}\right);$

>

3b)

$$x'' + 10\,x' + 9\,x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}\,.$$

> $dsolve\left(\left\{x''(t) + 10\cdot x'(t) + 9\cdot x(t) = 0, x(0) = 1, x'(0) = \frac{3}{2}\right\}\right);$

>

3c)

$$x'' + 2\,x' + 9\,x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}\,.$$

> $dsolve\left(\left\{x''(t) + 2\cdot x'(t) + 9\cdot x(t) = 0, x(0) = 1, x'(0) = \frac{3}{2}\right\}\right);$

>

> *with*(*plots*) :

> $plot0 := plot\left(\cos(3\cdot t) + \frac{1}{2}\cdot\sin(3\cdot t),\ t = 0\,..4,\ color = red\right)$:

$plot1a := plot\left(\exp(-3\cdot t)\cdot\left(1 + \frac{9}{2}\cdot t\right),\ t = 0\,..4,\ color = green\right)$:

$plot1b := plot\left(\frac{21}{16}\cdot\exp(-t) - \frac{5}{16}\cdot\exp(-9\cdot t),\ t = 0\,..4,\ color = blue\right)$:

$plot1c := plot\left(\frac{5}{8}\cdot\sqrt{2}\,\mathrm{e}^{-t}\cdot\sin\left(2\sqrt{2}\cdot t\right) + \mathrm{e}^{-t}\cdot\cos\left(2\sqrt{2}\cdot t\right),\ t = 0\,..4,\ color = black\right)$:

display({*plot0*, *plot1a*, *plot1b*, *plot1c*}, *title* = \`*IVP with all damping possibilities*\`);