Math 2250-004
Week 7 notes: Sections 4.1-4.3 vector space concepts
Tues Feb 21

• Finish section 3.6 on Determinants and connections to matrix inverses. Use last week's notes. Then if we have time on Tuesday, begin:

4.1-4.3 The vector space $\mathbb{R}^m$ and its subspaces; concepts related to linear combinations of vectors.

We never wrote it down carefully in Chapter 3, but for any natural number $m = 1, 2, 3$ ... the space $\mathbb{R}^m$ may be thought of in two equivalent ways. In both cases, $\mathbb{R}^m$ consists of all possible $m-tuples$ of numbers:

(i) We can think of those $m-tuples$ as representing points, as we're used to doing for $m = 1, 2, 3$. In this case we can write

$$\mathbb{R}^m = \{ (x_1, x_2, ..., x_m), \ s.t. \ x_1, x_2, ..., x_m \in \mathbb{R} \}.$$

(ii) We can think of those $m-tuples$ as representing vectors that we can add and scalar multiply. In this case we can write

$$\mathbb{R}^m = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_m \end{bmatrix}, \ s.t. \ x_1, x_2, ..., x_m \in \mathbb{R} \right\}.$$

Since algebraic vectors (as above) can be used to measure geometric displacement, one can identify the two models of $\mathbb{R}^m$ as sets by identifying each point $(x_1, x_2, ... x_m)$ in the first model with the displacement vector $\underline{\boldsymbol{x}} = [x_1, x_2, ... x_m]^T$ from the origin to that point, in the second model, i.e. the position vector. (Notice we just used a transpose, writing a column vector as a transpose of a row vector.)

One of the key themes of Chapter 4 is the idea of linear combinations. These have an algebraic definition (that we've seen before in Chapter 3 and repeat here), as well as a geometric interpretation as combinations of displacements, as we will review in our first few exercises.

Definition: If we have a collection of n vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ ... \ \underline{\boldsymbol{v}}_n\}$ in $\mathbb{R}^m$, then any vector $\underline{\boldsymbol{v}} \in \mathbb{R}^m$ that can be expressed as a sum of scalar multiples of these vectors is called a linear combination of them. In other words, if we can write

$$\underline{\boldsymbol{v}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ ... \ + c_n\underline{\boldsymbol{v}}_n ,$$

then $\underline{\boldsymbol{v}}$ is a linear combination of $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ ... \ \underline{\boldsymbol{v}}_n$. The scalars $c_1, c_2, ..., c_n$ are called the linear combination coefficients.

Example You've probably seen linear combinations in previous math/physics classes. For example you might have expressed the position vector $\underline{\mathbf{r}}$ as a linear combination

$$\underline{\mathbf{r}} = x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\mathbf{k}}$$

where $\underline{\mathbf{i}}$, $\underline{\mathbf{j}}$, $\underline{\mathbf{k}}$ represent the unit displacements in the x,y,z directions, respectively. Since we can express these displacements using Math 2250 notation as

$$\underline{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \underline{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \underline{\mathbf{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we have

$$x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\mathbf{k}} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Remarks: When we had free parameters in our explicit solutions to linear systems of equations $A\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$ back in Chapter 3, we sometimes rewrote the explicit solutions using linear combinations, where the scalars were the free parameters (which we often labeled with letters that were t, t_4, t_3 etc., rather than with "c's"). When we return to differential equations in Chapter 5 -studying higher order differential equations - then the explicit solutions will also be expressed using "linear combinations", just as we did in Chapters 1 -2, where we used the letter "C" for the single free parameter in first order differential equation solutions:

Definition: If we have a collection $\{y_1, y_2, \ldots, y_n\}$ of n functions $y(x)$ defined on a common interval I, then any function that can be expressed as a sum of scalar multiples of these functions is called a linear combination of them. In other words, if we can write

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n\,,$$

then y is a linear combination of $y_1, y_2, \ldots, y_n$.

The reason that the same words are used to describe what look like two quite different settings, is that there is a common fabric of mathematics (called vector space theory) that underlies both situations. We shall be exploring these concepts over the next several lectures, using a lot of the matrix algebra theory we've just developed in Chapter 3. This vector space theory will tie in directly to our study of differential equations, in Chapter 5 and subsequent chapters.

Exercise 1) (Linear combinations in $\mathbb{R}^2$... this will also review the geometric meaning of vector addition and scalar multiplication in terms of net displacements.)
Can you get to the point $(-2, 8) \in \mathbb{R}^2$, from the origin $(0, 0)$, by moving only in the $(\pm)$ directions of $\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$? Algebraically, this means we want to solve the linear combination problem

$$c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}.$$

1a) Plot the points $(1, -1)$ and $(1, 3)$, which have position vectors $\underline{\boldsymbol{v}}_1 = [1, -1]^T$ and $\underline{\boldsymbol{v}}_2 = [1, 3]^T$. Compute $\underline{\boldsymbol{v}}_1 + \underline{\boldsymbol{v}}_2$, $\underline{\boldsymbol{v}}_1 - \underline{\boldsymbol{v}}_2$. Plot the points for which these are the position vectors. Plot the line of points having position vectors

$$\{ \underline{\boldsymbol{v}}_1 + t\,\underline{\boldsymbol{v}}_2,\ t \in \mathbb{R} \}.$$

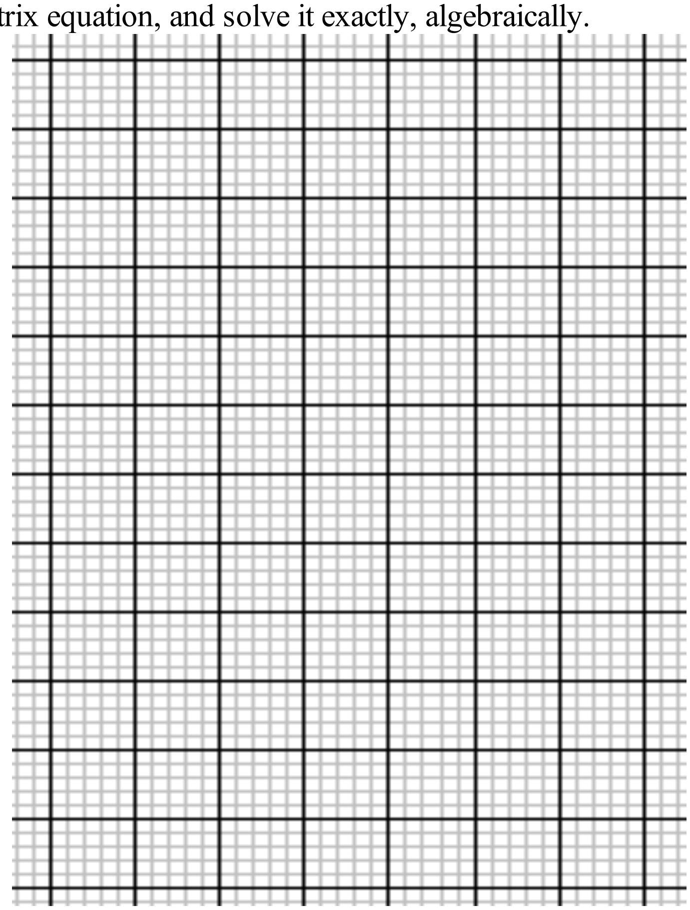
Note: Depending on where you took multivariable calculus you may have written this parametric line in various ways:

$$x = 1 + t$$
$$y = -1 + 3\,t$$

OR

$$\langle x, y \rangle = (1 + t)\underline{\mathbf{i}} + (-1 + 3\,t)\underline{\mathbf{j}}.$$

1b) Superimpose a grid related to the displacement vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2$ onto the graph paper below, and, recalling that vector addition yields net displacement, and scalar multiplication yields scaled displacement, try to approximately solve the linear combination problem above, geometrically.
1c) Rewrite the linear combination problem as a matrix equation, and solve it exactly, algebraically.



1c) Can you get to any point (x, y) in $\mathbb{R}^2$, starting at $(0, 0)$ and moving only in directions parallel to $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2$? Argue geometrically and algebraically. How many ways are there to express $[x, y]^T$ as a linear combination of $\underline{\boldsymbol{v}}_1$ and $\underline{\boldsymbol{v}}_2$?

Definition: The span of a collection of vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ in $\mathbb{R}^m$ is the collection of all vectors $\underline{\boldsymbol{w}}$ which can be expressed as linear combinations of $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$. We denote this collection as

$$span\{ \underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n \}.$$

Remark: The mathematical meaning of the word span is related to the English meaning - as in "wing span" or "span of a bridge", but it's also different. The span of a collection of vectors goes on and on and does not "stop" at the vector or associated endpoint:

Example 1)

- In Exercise 1, consider the $span\{\underline{\boldsymbol{v}}_1\} = span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$. This is the set of all vectors of the form

$\begin{bmatrix} c \\ -c \end{bmatrix}$ with free parameter $c \in \mathbb{R}$. This is a line through the origin of $\mathbb{R}^2$ described parametrically, that we're more used to describing with implicit equation $y = -x$ (which is short for $\{(x, y) \in \mathbb{R}^2 \ s.t.\ y = -x\}$). (More precisely, $span\{\underline{\boldsymbol{v}}_1\}$ is the collection of all position vectors for that line.)

Example 2:

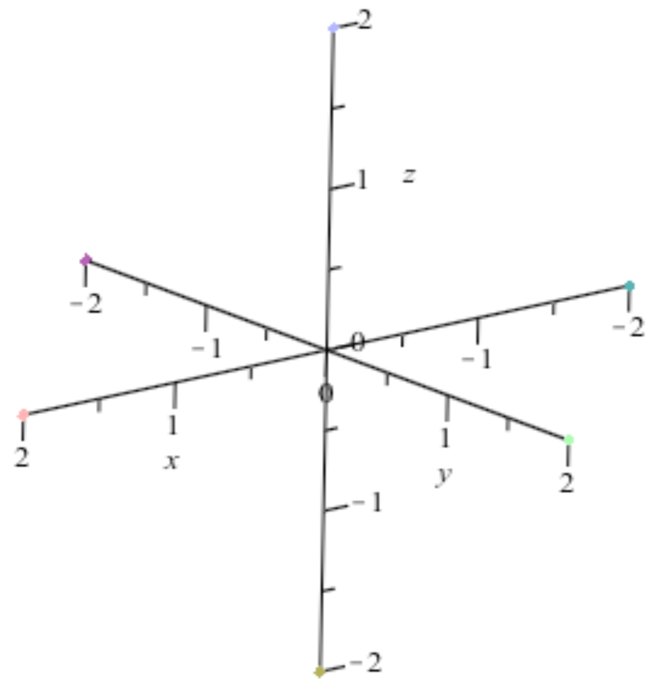
- In Exercise 1 we showed that the span of $\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is all of $\mathbb{R}^2$.

Exercise 2) Consider the two vectors $\underline{v}_1 = [1, 0, 2]^T, \underline{v}_2 = [-1, 2, 0]^T \in \mathbb{R}^3$.

2a) Sketch these two vectors as position vectors in $\mathbb{R}^3$, using the axes below.

2b) What geometric object is $span\{\underline{v}_1\}$? Sketch a portion of this object onto your picture below. Remember though, the "span" continues beyond whatever portion you can draw.

2c) What geometric object is $span\{\underline{v}_1, \underline{v}_2\}$? Sketch a portion of this object onto your picture below. Remember though, the "span" continues beyond whatever portion you can draw.



2d) What implicit equation must vectors $[x, y, z]^T$ satisfy in order to be in $span\{\underline{v}_1, \underline{v}_2\}$? Hint: For what $[x, y, z]^T$ can you solve the system

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

for c_1, c_2 ? Write this an augmented matrix problem and use row operations to reduce it, to see when you get a consistent system for c_1, c_2.

Wednesday Feb. 22

We've been talking about "linear combinations" of vectors. Finish Tuesday's notes and then continue the discussion here.

Exercise 1:
a) What is the definition of "a linear combination" of the vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ "

b) What is the "span" of the vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$?

Yesterday we interpreted linear combinations geometrically. And, we noticed that to answer natural questions we ended up using matrix theory from Chapter 3 This is because

Exercise 2) By carefully expanding the linear combination below, check than in $\mathbb{R}^m$, the linear combination

$$c_1 \begin{bmatrix} a_{11} \\ a_{21} \\ : \\ a_{m1} \end{bmatrix} + c_2 \begin{bmatrix} a_{12} \\ a_{22} \\ : \\ a_{m2} \end{bmatrix} + \ldots + c_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ : \\ a_{mn} \end{bmatrix} = \begin{bmatrix} \\ \\ : \\ \\ \end{bmatrix}$$

is always just the matrix times vector product

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ : & : & : & : \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ : \\ c_n \end{bmatrix}.$$

Thus linear combination problems in $\mathbb{R}^m$ can usually be answered using the linear system and matrix techniques we've just been studying in Chapter 3. This will be the main theme of Chapter 4. We've seen this theme in action, in exercises 1,2 in Tuesday's notes.

When we are discussing the span of a collection of vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ we would like to know that we are being efficient in describing this collection, and not wasting any free parameters because of redundancies. This has to do with the concept of "linear independence":

Definition:
a) The vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly independent if no one of the vectors is a linear combination of (some) of the other vectors. The logically equivalent concise way to say this is that the only way $\underline{\mathbf{0}}$ can be expressed as a linear combination of these vectors,

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n = \underline{\mathbf{0}},$$

is for all the linear combination coefficients $c_1 = c_2 = \ldots = c_n = 0$.

b) $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly dependent if at least one of these vectors *is* a linear combination of (some) of the other vectors. The concise way to say this is that there is some way to write $\underline{\mathbf{0}}$ as a linear combination of these vectors

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n = \underline{\mathbf{0}}$$

where not all of the $c_j = 0$. (We call such an equation a linear dependency. Note that if we have any such linear dependency, then any $\underline{\boldsymbol{v}}_j$ with $c_j \neq 0$ is a linear combination of the remaining $\underline{\boldsymbol{v}}_k$ with $k \neq j$. We say that such a $\underline{\boldsymbol{v}}_j$ is linearly dependent on the remaining $\underline{\boldsymbol{v}}_k$.)

Note: Two non-zero vectors are linearly independent precisely when they are not multiples of each other. For more than two vectors the situation is more complicated.

Example (Refer to Exercise 1 Tuesday):

The vectors $\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \underline{\boldsymbol{v}}_3 = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$ in $\mathbb{R}^2$ are linearly dependent because, as we showed on Tuesday and as we can quickly recheck,

$$-3.5\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1.5\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}.$$

We can also write this linear dependency as

$$-3.5\underline{\boldsymbol{v}}_1 + 1.5\underline{\boldsymbol{v}}_2 - \underline{\boldsymbol{v}}_3 = \underline{\mathbf{0}}$$

(or any non-zero multiple of that equation.)

Exercise 3) Are the vectors $\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ linearly independent? How about $\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\underline{\boldsymbol{v}}_3 = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$?

Exercise 4) For linearly independent vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$, every vector $\underline{\boldsymbol{v}}$ in their span can be written as $\underline{\boldsymbol{v}} = d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + \ldots + d_n\underline{\boldsymbol{v}}_n$ uniquely, i.e. for exactly one choice of linear combination coefficients $d_1, d_2, \ldots d_n$. This is not true if vectors are dependent. Explain these facts. (You can illustrate these facts with the vectors in Exercise 3.)

Exercise 5) (Refer to Exercise 2 in Tuesday's notes):

5a) Are the vectors

$$\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \underline{\boldsymbol{v}}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

linearly independent?

5b) Show that the vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \ \underline{v}_3 = \begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix}$$

are linearly dependent (even though no two of them are scalar multiples of each other). What does this mean geometrically about the span of these three vectors?

Hint: You might find this computation useful:

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> with(LinearAlgebra) :
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$$ReducedRowEchelonForm\left(\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 6 \\ 2 & 0 & 4 \end{bmatrix}\right);$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \textbf{(1)}$$

Exercise 6) Are the vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \ \underline{w}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

linearly independent? What is their span? Hint:

$$> ReducedRowEchelonForm\left(\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}\right);$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \textbf{(2)}$$

Math 2250-004
Fri Feb 24
4.1 - 4.3 Concepts related to linear combinations of vectors.

Exercise 1) Vocabulary review (these need to be memorized!)

A linear combination of the vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ is

The span of $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ is

The vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly independent iff

The vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly dependent iff

• Keep recalling that for vectors in $\mathbb{R}^m$ all linear combination questions can be reduced to matrix questions because any linear combination like the one on the left is actually just the matrix product on the right:

$$c_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + c_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + c_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ 0 \\ a_{mn} \end{bmatrix} = \begin{bmatrix} c_1 a_{11} + c_2 a_{12} + \ldots \\ c_1 a_{21} + c_2 a_{22} + \ldots \\ \vdots \\ c_1 a_{m1} + c_2 a_{m2} + \ldots \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Definition Let W be a subset of $\mathbb{R}^m$ that is closed under addition and scalar multiplication; in other words

α) Whenever $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in W$ then $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} \in W$

β) Whenever $\underline{\boldsymbol{v}} \in W$ and $c \in \mathbb{R}$ then $c\underline{\boldsymbol{v}} \in W$.

Then W is called a subspace of $\mathbb{R}^m$.

Notice that the span of any collection of vectors is a subspace because if you add two linear combinations of vectors, the sum is still a linear combination of the (same) vectors; and if you multiply a linear combination by a constant it is still a linear combination.

Definition Let W be a subspace. If $W = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, ...\underline{\boldsymbol{v}}_n\}$ and if $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, ... \underline{\boldsymbol{v}}_n$ are linearly independent, then we say that they are a basis for W. (The word "basis" makes sense because the entire subspace can be reconstructed by taking linear combinations of the basis vectors, and the linear combinations coefficients for each element in W are unique.

Examples (explain answers!)

Exercise 2:

a) Show $\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a basis for the line in $\mathbb{R}^2$ with implicit equation $y = -x$.

b) $\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ are a basis for $\mathbb{R}^2$.

(The vectors $\underline{\boldsymbol{e}}_1 = \underline{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \underline{\boldsymbol{e}}_2 = \underline{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the so-called "standard basis" for $\mathbb{R}^2$.)

c) $\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \underline{\boldsymbol{v}}_3 = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$ are not a basis for $\mathbb{R}^2$.

Use matrix theory to explain why

d) Fewer than two vectors cannot be a basis for all of $\mathbb{R}^2$.

e) More than two vectors cannot be a basis for all of $\mathbb{R}^2$.

f) A choice of exactly two vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2\}$ will be a a basis of $\mathbb{R}^2$ if and only if the reduced row echelon form of the matrix having those two vectors as columns is the identity matrix.

Exercise 3a) $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ are a basis for the plane with implicit equation $2x + y - z = 0$. (See Exercise 2d in Tuesday's notes.)

b) $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \ \underline{v}_3 = \begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix}$ are not a basis for the plane with implicit equation $2x + y - z = 0$, even though all three vectors lie on the plane. They are also not a basis for $\mathbb{R}^3$.

> $ReducedRowEchelonForm\left(\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 6 \\ 2 & 0 & 4 \end{bmatrix}\right);$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad (3)$$

c) $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \ \underline{w}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ are a basis for $\mathbb{R}^3$.

(The vectors $\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \underline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are the so-called "standard basis" for $\mathbb{R}^3$.)

> $ReducedRowEchelonForm\left(\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}\right);$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4)$$

Exercise 4) Use properties of reduced row echelon form matrices to answer the following questions:
4a) Why must more than 3 vectors in $\mathbb{R}^3$ always be linearly dependent?
4b) Why can fewer than 3 vectors never span $\mathbb{R}^3$?
(So every basis of $\mathbb{R}^3$ must have exactly three vectors.)
4c) If you are given 3 vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \underline{\boldsymbol{v}}_3$ in $\mathbb{R}^3$, what is the condition on the reduced row echelon form of the 3 × 3 matrix $\langle \underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2 | \underline{\boldsymbol{v}}_3 \rangle$ that guarantees they're linearly independent? That guarantees they span $\mathbb{R}^3$? That guarantees they're a basis of $\mathbb{R}^3$?
4d) What is the dimension of $\mathbb{R}^3$?