Math 2250-004
Week 6 notes. Midterm exam this Friday Feb 17!

Mon Feb 13
3.5-3.6 Matrix inverses; matrix determinants.

- Finish last Friday's notes: how to find matrix inverses when they exist, and how to figure out when they don't exist. This will lead naturally into determinants, section 3.6, in this week's notes.

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Determinants are scalars defined for square matrices $A_{n \times n}$ and they always <u>determine</u> whether or not the inverse matrix A^{-1} exists, (i.e. whether the reduced row echelon form of A is the identity matrix). It turns out that the determinant of A is non-zero if and only if A^{-1} exists. The determinant of a 1×1 matrix $\left[a_{11}\right]$ is defined to be the number a_{11}; determinants of 2×2 matrices are defined as in Friday's notes; and in general determinants for $n \times n$ matrices are defined recursively, in terms of determinants of $(n-1) \times (n-1)$ submatrices:

<u>Definition:</u> Let $A_{n \times n} = \left[a_{ij}\right]$. Then the determinant of A, written $det(A)$ or $|A|$, is defined by

$$det(A) := \sum_{j=1}^{n} a_{1j}(-1)^{1+j} M_{1j} = \sum_{j=1}^{n} a_{1j} C_{1j}\,.$$

Here M_{1j} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and the j^{th} column, and C_{1j} is simply $(-1)^{1+j} M_{1j}$.

More generally, the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A is called the <u>$i\,j$ Minor</u> M_{ij} of A, and $C_{ij} := (-1)^{i+j} M_{ij}$ is called the <u>$i\,j$ Cofactor</u> of A .

<u>Theorem:</u> (proof is in text appendix) $det(A)$ can be computed by expanding across any row, say row i:

$$det(A) := \sum_{j=1}^{n} a_{ij}(-1)^{i+j} M_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij}$$

or by expanding down any column, say column j:

$$det(A) := \sum_{i=1}^{n} a_{ij}(-1)^{i+j} M_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij}\,.$$

Exercise 1a) Let $A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$. Compute $det(A)$ using the definition.

1b) Verify that the matrix of all the cofactors of A is given by $[C_{ij}] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$. Then expand $det(A)$ down various columns and rows using the a_{ij} factors and C_{ij} cofactors. Verify that you always get the same value for $det(A)$. Notice that in each case you are taking the dot product of a row (or column) of A with the corresponding row (or column) of the cofactor matrix.

1c) What happens if you take dot products between a row of A and a *different* row of $[C_{ij}]$? A column of A and a *different* column of $[C_{ij}]$? The answer may seem magic.

Exercise 2) Compute the following determinants by being clever about which rows or columns to use:

2a) $\begin{vmatrix} 1 & 38 & 106 & 3 \\ 0 & 2 & 92 & -72 \\ 0 & 0 & 3 & 45 \\ 0 & 0 & 0 & -2 \end{vmatrix}$; 2b) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ \pi^2 & 2 & 0 & 0 \\ 0.476 & 88 & 3 & 0 \\ 1 & 22 & 33 & -2 \end{vmatrix}$.

Exercise 3) Explain why it is always true that for an upper triangular matrix (as in 2a), or for a lower triangular matrix (as in 2b), the determinant is always just the product of the diagonal entries.

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Continue with section 3.6 Determinants

The effective way to compute determinants for larger-sized matrices without lots of zeroes is to not use the definition, but rather to use the following facts, which track how elementary row operations affect determinants:

- (1a) Swapping any two rows changes the sign of the determinant.
  proof: This is clear for 2×2 matrices, since
  $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \qquad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad\,.$$
  For 3×3 determinants, expand across the row *not* being swapped, and use the 2×2 swap property to deduce the result. Prove the general result by induction: once it's true for $n \times n$ matrices you can prove it for any $(n+1) \times (n+1)$ matrix, by expanding across a row that wasn't swapped, and applying the $n \times n$ result.
  (1b) Thus, if two rows in a matrix are the same, the determinant of the matrix must be zero: on the one hand, swapping those two rows leaves the matrix and its determinant unchanged; on the other hand, by (1a) the determinant changes its sign. The only way this is possible is if the determinant is zero.
- (2a) If you factor a constant out of a row, then you factor the same constant out of the determinant. Precisely, using $\mathcal{R}_i$ for i^{th} row of A, and writing $\mathcal{R}_i = c\,\mathcal{R}_i^*$
  $$\begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_i \\ \mathcal{R}_n \end{vmatrix} = \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ c\,\mathcal{R}_i^* \\ \mathcal{R}_n \end{vmatrix} = c \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_i^* \\ \mathcal{R}_n \end{vmatrix}.$$
  proof: expand across the i^{th} row, noting that the corresponding cofactors don't change, since they're computed by deleting the i^{th} row to get the corresponding minors:
  $$det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} c\, a_{ij}^* C_{ij} = c \sum_{j=1}^{n} a_{ij}^* C_{ij} = c\, det\left(A^*\right).$$
  (2b) Combining (2a) with (1b), we see that if one row in A is a scalar multiple of another, then $det(A) = 0$.

- (3) If you replace row i of A by its sum with a multiple of another row, then the determinant is unchanged! Expand across the i^{th} row:

$$\begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_k \\ \mathcal{R}_i + c\,\mathcal{R}_k \\ \mathcal{R}_n \end{vmatrix} = \sum_{j=1}^{n} \left(a_{ij} + c\,a_{kj}\right) C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij} + c \sum_{j=1}^{n} a_{kj} C_{ij} = det(A) + c \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_k \\ \mathcal{R}_k \\ \mathcal{R}_n \end{vmatrix} = det(A) + 0 \, .$$

<u>Remark:</u> The analogous properties hold for corresponding "elementary column operations". In fact, the proofs are almost identical, except you use column expansions.

Exercise 1) Recompute $\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix}$ from yesterday (using row and column expansions we always got an answer of 15 then.) This time use elementary row operations (and/or elementary column operations).

Exercise 2) Compute $\begin{vmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ -1 & 0 & -2 & 1 \end{vmatrix}$.

Maple check:

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> with(LinearAlgebra):
> A := Matrix(4, 4, [1, 0, -1, 2, 2, 1, 1, 0, 2, 0, 1, 1, -1, 0, -2, 1]);
  Determinant(A);
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$$A := \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ -1 & 0 & -2 & 1 \end{bmatrix}$$

$$0 \qquad (1)$$

Theorem: Let $A_{n \times n}$. Then A^{-1} exists if and only if $det(A) \neq 0$.

proof: We already know that A^{-1} exists if and only if the reduced row echelon form of A is the identity matrix. Now, consider reducing A to its reduced row echelon form, and keep track of how the determinants of the corresponding matrices change: As we do elementary row operations,

- if we swap rows, the sign of the determinant switches.
- if we factor non-zero factors out of rows, we factor the same factors out of the determinants.
- if we replace a row by its sum with a multiple of another row, the determinant is unchanged.

Thus,

$$|A| = c_1|A_1| = c_1c_2|A_2| = ... = c_1c_2 ... c_N|rref(A)|$$

where the nonzero c_k's arise from the three types of elementary row operations. If $rref(A) = I$ its determinant is 1, and $|A| = c_1c_2 ... c_N \neq 0$. If $rref(A) \neq I$ then its bottom row is all zeroes and its determinant is zero, so $|A| = c_1c_2 ... c_N(0) = 0$. Thus $|A| \neq 0$ if and only if $rref(A) = I$ if and only if A^{-1} exists.

Remark: Using the same ideas as above, you can show that $det(A\,B) = det(A)det(B)$. This is an important identity that gets used, for example, in multivariable change of variables formulas for integration, using the Jacobian matrix. (It is not true that $det(A + B) = det(A) + det(B)$.) Here's how to show $det(A\,B) = det(A)det(B)$: The key point is that if you do an elementary row operation to AB, that's the same as doing the elementary row operation to A, and then multiplying by B. With that in mind, if you do exactly the same elementary row operations as you did for A in the theorem above, you get

$$|A\,B| = c_1|A_1B| = c_1c_2|A_2B| = ... = c_1c_2 ... c_N|rref(A)B|.$$

If $rref(A) = I$, then from the theorem above, $|A| = c_1c_2 ... c_N$, and we deduce $|A\,B| = |A||B|$. If $rref(A) \neq I$, then its bottom row is zeroes, and so is the bottom row of $rref(A)B$. Thus $|A\,B| = 0$ and also $|A||B| = 0$.

There is a "magic" formula for the inverse of square matrices A (called the "adjoint formula") that uses the determinant of A along with the cofactor matrix of A.
In order to understand the $n \times n$ magic formula for matrix inverses, we first need to talk about matrix *transposes:*

<u>Definition:</u> Let $B_{m \times n} = [b_{ij}]$. Then the <u>transpose</u> of B, denoted by B^T is an $n \times m$ matrix defined by

$$entry_{ij}(B^T) := entry_{ji}(B) = b_{ji}.$$

The effect of this definition is to turn the columns of B into the rows of B^T :

$$entry_i(col_j(B)) = b_{ij}.$$

$$entry_i(row_j(B^T)) = entry_{ji}(B^T) = b_{ij}.$$

And to turn the rows of B into the columns of B^T:

$$entry_j(row_i(B)) = b_{ij}$$

$$entry_j(col_i(B^T)) = entry_{ji}(B^T) = b_{ij}.$$

<u>Exercise 3)</u> explore these properties with the identity

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Theorem: Let $A_{n \times n}$, and denote its cofactor matrix by $cof(A) = [C_{ij}]$, with $C_{ij} = (-1)^{i+j} M_{ij}$, and $M_{ij} =$ the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A.
Define the adjoint matrix to be the transpose of the cofactor matrix:

$$Adj(A) := cof(A)^T$$

Then, when A^{-1} exists it is given by the formula

$$A^{-1} = \frac{1}{det(A)} Adj(A) .$$

Exercise 4) Show that in the 2×2 case this reproduces the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} .$$

Exercise 5) For our friend $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$ we worked out $cof(A) = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$ and $det(A) = 15$.

Use the Theorem to find A^{-1} and check your work. Does the matrix multiplication relate to the dot products we computed between various rows of A and rows of $cof(A)$?

Exercise 6) Continuing with our example,

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad cof(A) = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix} \qquad Adj(A) = \begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix}$$

6a) The $(1, 1)$ entry of $(A)(Adj(A))$ is $15 = 1\cdot5 + 2\cdot2 + (-1)(-6)$. Explain why this is $det(A)$, expanded across the first row.

6b) The $(2, 1)$ entry of $(A)(Adj(A))$ is $0\cdot5 + 3\cdot2 + (1)(-6) = 0$. Notice that you're using the same cofactors as in (4a). What matrix, which is obtained from A by keeping two of the rows, but replacing a third one with one of those two, is this the determinant of?

6c) The $(3, 2)$ entry of $(A)(Adj(A))$ is $2\cdot0 - 2\cdot3 + 1\cdot6 = 0$. What matrix (which uses two rows of A) is this the determinant of?

If you completely understand 6abc, then you have realized why

$$[A][Adj(A)] = det(A)[I]$$

for every square matrix, and so also why

$$A^{-1} = \frac{1}{det(A)} Adj(A) .$$

Precisely,

$$entry_{i\,i}\, A(Adj(A)) = row_i(A) \bullet col_i(Adj(A)) = row_i(A) \bullet row_i(cof(A)) = det(A),$$

expanded across the i^{th} row.

On the other hand, for $i \neq k$,

$$entry_{k\,i}\, A(Adj(A)) = row_k(A) \bullet col_i(Adj(A)) = row_k(A) \bullet row_i(cof(A)) .$$

This last dot produce is zero because it is the determinant of a matrix made from A by replacing the i^{th} row with the k^{th} row, expanding across the i^{th} row, and whenever two rows are equal, the determinant of a matrix is zero.

There's a related formula for solving for individual components of $\underline{\boldsymbol{x}}$ when $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ has a unique solution ($\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$). This can be useful if you only need one or two components of the solution vector, rather than all of it:

Cramer's Rule: Let $\underline{\boldsymbol{x}}$ solve $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$, for invertible A. Then

$$x_k = \frac{det\left(A_k\right)}{det(A)}$$

where A_k is the matrix obtained from A by replacing the k^{th} column with $\underline{\boldsymbol{b}}$.

proof: Since $\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$ the k^{th} component is given by

$$\begin{aligned} x_k &= entry_k\left(A^{-1}\underline{\boldsymbol{b}}\right) \\ &= entry_k\left(\frac{1}{|A|}Adj(A)\underline{\boldsymbol{b}}\right) \\ &= \frac{1}{|A|}row_k(Adj(A)) \cdot \underline{\boldsymbol{b}} \\ &= \frac{1}{|A|}col_k(cof(A)) \cdot \underline{\boldsymbol{b}}. \end{aligned}$$

Notice that $col_k(cof(A)) \cdot \underline{\boldsymbol{b}}$ is the determinant of the matrix obtained from A by replacing the k^{th} column by $\underline{\boldsymbol{b}}$, where we've computed that determinant by expanding down the k^{th} column! This proves the result.

Exercise 7) Solve $\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$.

7a) With Cramer's rule

7b) With A^{-1}, using the adjoint formula.

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Wed Feb 15

We will use the first portion of class to finish the loose ends from section 3.6 determinants, if necessary.

Then we'll go over the logistics for the exam, below.

Finally, the concept-organizing questions for Chapters 1-3 are on the following pages. I've put the Chapter 3 questions first because that's the most recent material.

The exam is this Friday February 17, from 10:40-11:40 a.m. Note that it will start 5 minutes before the official start time for this class, and end 5 minutes afterwards, so you should have one hour to work on the exam. Get to class early, and bring your University I.D. card, which we might ask you to show if we don't recognize you from sections or lecture. You may be asked to sit in alternate seats, especially towards the rear of our classroom WEB L105.

This exam will cover textbook material from 1.1-1.5, 2.1-2.4, 3.1-3.6. The exam is closed book and closed note. You may use a scientific (but not a graphing) calculator, although symbolic answers are accepted for all problems, so no calculator is really needed. (Using a graphing calculator, which can do matrix computations for example, is grounds for receiving grade of 0 on your exam. So please ask before the exam if you're unsure about your calculator. And of course, your cell phones must be put away.)

I recommend trying to study by organizing the conceptual and computational framework of the course so far. Only then, test yourself by making sure you can explain the concepts and do typical problems which illustrate them. The class notes and text should have explanations for the concepts, along with worked examples. Old homework assignments and quizzes are also a good source of problems. It could be helpful to look at quizzes/exams from my previous Math 2250 classes, which go back several years from the link

http://www.math.utah.edu/~korevaar/oldclasses.html.

Your lab meetings tomorrow will be exam review sessions.

Chapter 3:

3a) Can you recognize an algebraic linear system of equations?

3b) Can you interpret the solution set geometrically when there are 2 or three unknowns?

3c) Can you use Gaussian elimination to compute reduced row echelon form for matrices? Can you apply this algorithm to augmented matrices to solve linear systems ?

3d) What does the shape of the reduced row echelon form of a matrix A tell you about the possible solution sets to $A\underline{\mathbf{x}} = \underline{\mathbf{b}}$ *(perhaps depending, and perhaps not depending on* $\underline{\mathbf{b}}$ *)?*

3e) What properties do (and do not) hold for the matrix algebra of addition, scalar multiplication, and matrix multiplication?

3f) What is the matrix inverse, A^{-1} *for a square matrix A ? Does every square matrix have an inverse? How can you tell whether or not a matrix has an inverse, using reduced row echelon form? What's the row operations way of finding* A^{-1}*, when it exists? Can you use matrix algebra to solve matrix equations for unknown vectors* $\underline{\mathbf{x}}$ *or matrices* $\boldsymbol{X}$*, possibly using matrix inverses and other algebra manipulations?*

3g) Can you compute $|A|$ *for a square matrix A using cofactors, row operations, or some combination of those algorithms? What does the value of* $|A|$ *have to do with whether* A^{-1} *exists? What's the magic formula for the inverse of a matrix? Can you work with this formula in the two by two or three by three cases? What's Cramer's rule? Can you use it?*

Chapters 1-2:

1a) What is a differential equation? What is its order? What is an initial value problem, for a first or second order DE?

1b) How do you check whether a function solves a differential equation? An initial value problem?

1c) What is the connection between a first order differential equation and a slope field for that differential equation? The connection between an IVP and the slope field?

1d) Do you expect solutions to IVP's to exist, at least for values of the input variable close to its initial value? Why? Do you expect uniqueness? What can cause solutions to not exist beyond a certain input variable value?

1e) What is Euler's numerical method for approximating solutions to first order IVP's, and how does it relate to slope fields?

1f) What's an autonomous differential equation? What's an equilibrium solution to an autonomous differential equation? What is a phase diagram for an autonomous first order DE, and how do you construct one? How does a phase diagram help you understand stability questions for equilibria? What does the phase diagram for an autonomous first order DE have to do with the slope field?

1g) Can you recognize the first order differential equations for which we've studied solution algorithms, even if the DE is not automatically given to you pre-set up for that algorithm? Do you know the algorithms for solving these particular first order DE's?

2) Can you convert a description of a dynamical system in terms of rates of change, or a geometric configuration in terms of slopes, into a differential equation? What are the models we've studied carefully in Chapters 1-2? What sorts of DE's and IVP's arise? Can you solve these basic application DE's, once you've set up the model as a differential equation and/or IVP?