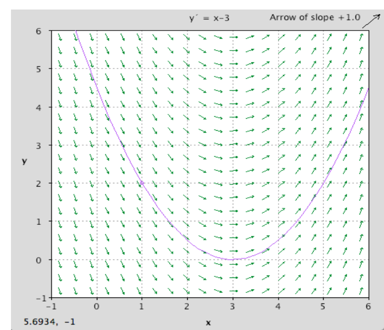
Math 2250-004 : Week 2, Jan 17-20; material from sections 1.3, 1.4, 1.5, EP 3.7
Tues Jan 17
We will mostly use last Friday's notes. Our goals today are

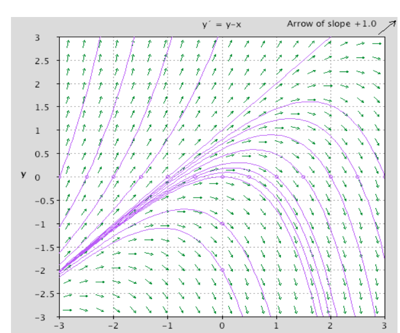
(1) understand what makes a first order differential equation separable.
(2) understand the algorithm based on differentials that solves separable differential equations: why it works, and how it sometimes misses "singular solutions"
(3) understand and apply the existence-uniqueness theorem for first order DE initial value problems.

When discussing the existence-uniqueness theorem at the end of Friday's notes today, we'll refer to examples from Wednesday's notes that we discussed on Friday. Those were:

<u>Exercise 1</u> (Wednesday notes, discussed Friday):
$$\frac{dy}{dx} = x - 3$$
$$y(1) = 2\,.$$
We found the solution $y(x) = \frac{x^2}{2} - 3\,x + \frac{9}{2} = \frac{(x-3)^2}{2}$. Is this consistent with the existence-uniqueness theorem?



<u>Exercise 2</u>: (Wednesday notes, discussed Friday)
$$\frac{dy}{dx} = y - x$$
$$y(0) = 0$$
From a family of solutions that was given to us, we found a solution
$$y(x) = x + 1 - e^x\,.$$
Is this the only possible solution? Hint: use the existence-uniqueness theorem.



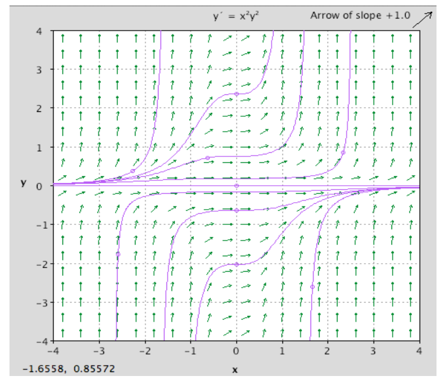
Exercise 1 (today): Here's another example of using a separable DE to illustrate the existence-uniqueness theorem.

a) Does each IVP

$$y' = x^2 y^2$$
$$y(x_0) = y_0$$

have a unique solution?

b) Find all solutions to this differential equation.



Maple check (notice it misses the singular solution):

```
> with(DEtools):
  dsolve(y'(x) = x^2·y(x)^2, y(x));
```

$$y(x) = \frac{3}{-x^3 + 3\_C1} \tag{1}$$

```
>
```

Exercise 2: Do the initial value problems below always have unique solutions? Can you find them? (Notice these are NOT separable differential equations.) Can Maple find formulas for the solution functions?

a)

$$y' = x^2 + y^2$$
$$y(x_0) = y_0$$

b)

$$y' = x^4 + y^4$$
$$y(x_0) = y_0$$

Math 2250-004
Wed Jan 18   Quiz at end of class!
1.4: separable DEs, examples and experiment.

For your section 1.4 hw this week I assigned a selection of separable DE's - some applications will be familiar with from last week, e.g. exponential growth/decay and Newton's Law of cooling. Below is an application that might be new to you, and that illustrates conservation of energy as a tool for modeling differential equations in physics.

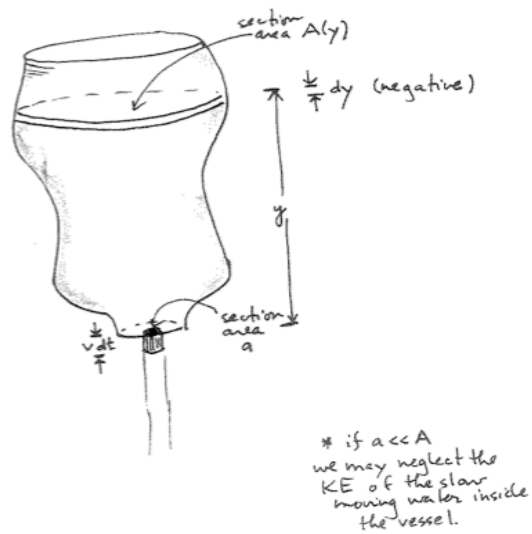
Toricelli's Law, for draining water tanks. Refer to the figure below.
Exercise 1:
a) Neglect friction, use conservation of energy, and assume the water still in the tank is moving with negligable velocity $(a << A)$. Equate the lost potential energy from the top in time dt to the gained kinetic energy in the water streaming out of the hole in the tank to deduce that the speed v with which the water exits the tank is given by

$$v = \sqrt{2gy}$$

when the water depth above the hole is $y(t)$ (and g is accel of gravity).

b) Use part (a) to derive the separable DE for water depth

$$A(y)\frac{dy}{dt} = -k\sqrt{y} \quad (k = a\sqrt{2g}).$$



Experiment fun! I've brought a leaky nalgene canteen so we can test the Toricelli model. For a cylindrical tank of height h as below, the cross-sectional area $A(y)$ is a constant A , so the Toricelli DE and IVP becomes

$$\frac{dy}{dt} = -k\, y^{\frac{1}{2}}$$
$$y(0) = h$$

(different k).

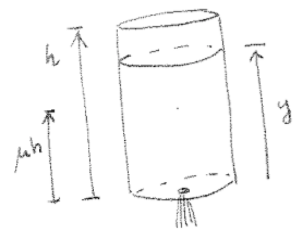
Exercise 2a) Solve the differential equation IVP, and IVP. Note that $y \geq 0$, and that $y = 0$ is a singular solution that separation of variables misses. We may choose our units of length so that $h = 1$ is the maximum water height in the tank. Show that in this case the solution to the IVP is given by

$$y(t) = \left(1 - \frac{k}{2}t\right)^2$$

(until the tank runs empty).

Exercise 2b: (We will use this calculation in our experiment) Setting the height $h = 1$ as in part 2a, let $T(\mu)$ be the time it takes the the water to go from height 1 (full) to height μ , where the fraction μ is between 0 and 1. Note, $T(1) = 0$ and $T(0)$ is the time it takes for the tank to empty completely. Show that $T(0)$ is related to $T(\mu)$ by

$$T(0)\left(1 - \sqrt{\mu}\right) = T(\mu), \quad \text{i.e. } T(0) = \frac{T(\mu)}{1 - \sqrt{\mu}}.$$



Experiment! We'll time how long it takes to half-empty the canteen, and predict how long it will take to completely empty it when we rerun the experiment. Here are numbers I once got in my office, let's see how ours compare.

```
> Digits := 5 : # that should be enough significant digits
> 1/(1 - sqrt(.5)); # the factor from above, when mu is 0.5
                                3.4143                                  (2)
> Thalf := 35;   # seconds to half-empty canteen
  Tpredict := 3.4143·Thalf;  #prediction
                              Thalf := 35
                           Tpredict := 119.50                           (3)
```

Remark: Maple can draw direction fields, although they're not as easy to create as in "dfield". On the other hand, Maple can do any undergraduate mathematics computation, including solving pretty much any differential equation that has a closed form solution. Let's see what the commands below produce. We'll get some error messages related to the existence-uniqueness theorem!

> $with(DEtools):$ # this loads a library of DE commands

> $deqtn1 := y'(t) = -\sqrt{y(t)}$; # took k=1
$ics1 := y(0) = 1$; # took initial height=1

$$deqtn1 := D(y)(t) = -\sqrt{y(t)}$$

$$ics1 := y(0) = 1 \tag{4}$$

> $dsolve(\{deqtn1, ics1\}, y(t))$; # DE IVP sol!

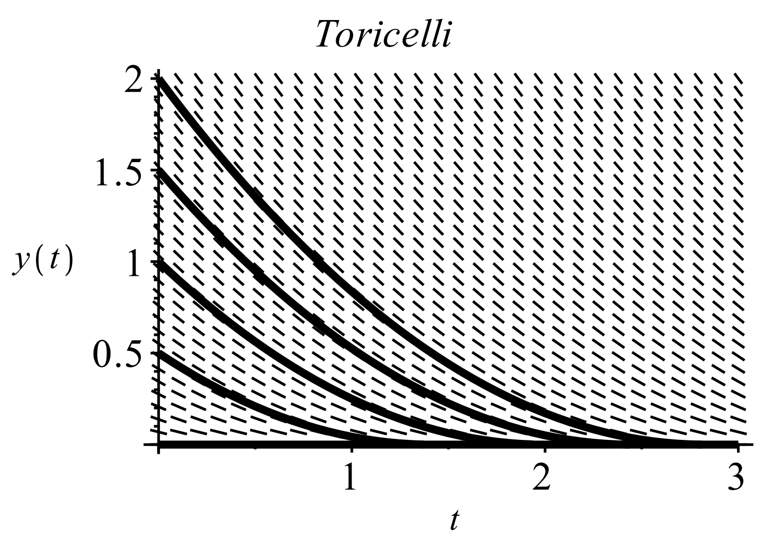
$$y(t) = \frac{1}{4}t^2 - t + 1 \tag{5}$$

> $factor(\%)$; #how we wrote it

$$y(t) = \frac{1}{4}(t-2)^2 \tag{6}$$

> $DEplot(deqtn1, y(t), 0..3, \{[y(0)=0], [y(0)=1.], [y(0)=2.], [y(0)=0.5], [y(0)=1.5]\}, arrows = line, color = black, linecolor = black, dirgrid = [30, 30], stepsize = .1, title = `Toricelli`)$;

```
Warning, plot may be incomplete, the following errors(s) were issued:
   cannot evaluate the solution further right of 1.4141924, probably a singularity
Warning, plot may be incomplete, the following errors(s) were issued:
   cannot evaluate the solution further right of 1.9999776, probably a singularity
Warning, plot may be incomplete, the following errors(s) were issued:
   cannot evaluate the solution further right of 2.4494671, probably a singularity
Warning, plot may be incomplete, the following errors(s) were issued:
   cannot evaluate the solution further right of 2.8283979, probably a singularity
```



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Math 2250-004
Fri Jan 20
1.5: linear DEs, and applications.

Section 1.5, linear differential equations:

A first order linear DE for $y(x)$ is one that can be written as

$$y' + P(x)y = Q(x)$$

Exercise 1: Classify the differential equations below as linear, separable, both, or neither. Justify your answers.

a) $y'(x) = -2y + 4x^2$
b) $y'(x) = x - y^2 + 1$
c) $y'(x) = x^2 - x^2 y + 1$
d) $y'(x) = \dfrac{6x - 3xy}{x^2 + 1}$
e) $y'(x) = x^2 + y^2$
f) $y'(x) = x^2 e^{x^3}$.

Algorithm for solving linear DEs is a method to use the differentiation product rule backwards:

$$y' + P(x)y = Q(x)$$

Let $\int P(x)dx$ be any antiderivative of P. Multiply both sides of the DE by its exponential to yield an equivalent DE:

$$e^{\int P(x)dx}(y' + P(x)y) = e^{\int P(x)dx} Q(x)$$

because the left side is a derivative (product rule):

$$\frac{d}{dx}\left(e^{\int P(x)dx} y\right) = e^{\int P(x)dx} Q(x) .$$

So you can antidifferentiate both sides with respect to x :

$$e^{\int P(x)dx} y = \int e^{\int P(x)dx} Q(x)dx \quad + C.$$

Dividing by the positive function $e^{\int P(x)dx}$ yields a formula for $y(x)$. Notice, if you look carefully at this formula for the solution, that if $P(x), Q(x)$ are defined and continuous on any interval I, then the resulting formula for $y(x)$ can be used to find a solution to any IVP with initial point in that interval, defined on the entire interval. This is in contrast to what can happen with separable differential equations.

Exercise 2: Solve the differential equation

$$y'(x) = -2\,y + 4\,x^2\,,$$

and compare your solutions to the dfield plot below.

```
> with(DEtools):    # load differential equations library
> deqtn2 := y'(x) = -2·y(x) + 4·x^2 :  #notice you must use · for multiplication in Maple,
                                        # and write y(x) rather than y.
  dsolve(deqtn2, y(x)); # Maple check
```

$$y(x) = 2\,x^2 - 2\,x + 1 + e^{-2\,x}\,\_C1 \qquad \textbf{(7)}$$

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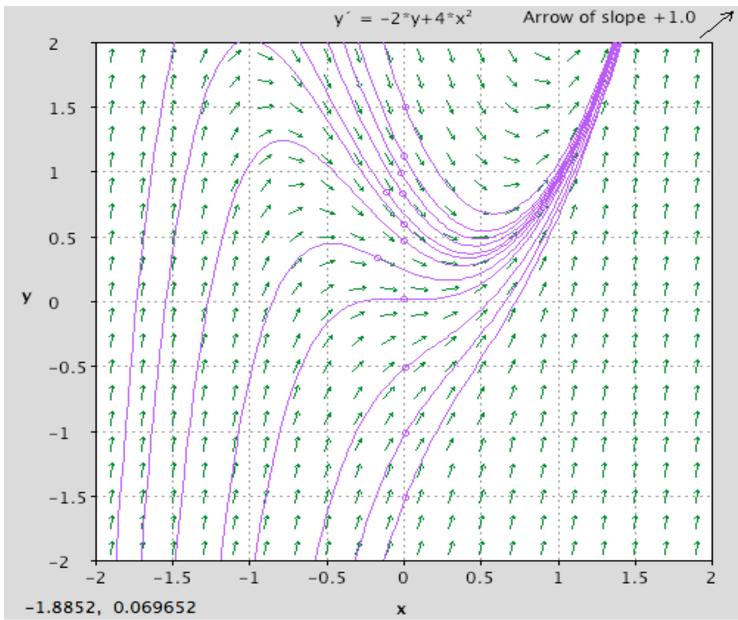


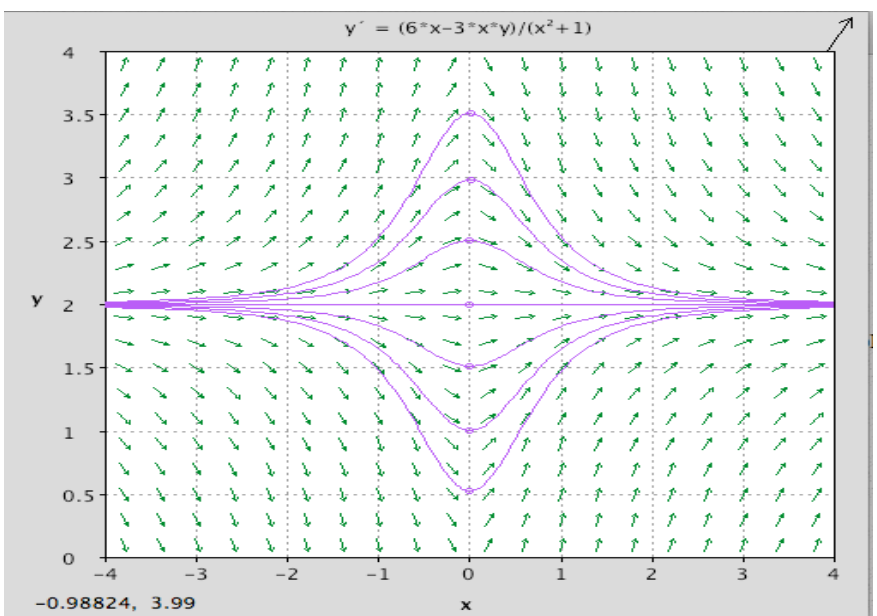
Exercise 3: Find all solutions to the linear (and also separable) DE

$$y'(x) = \frac{6x - 3xy}{x^2 + 1}$$

Hint: as you can verify below, the general solution is $y(x) = 2 + C\left(x^2 + 1\right)^{-\frac{3}{2}}$.

> *with(DEtools)* :
> $dsolve\left(y'(x) = \frac{(6\cdot x - 3\cdot x\cdot y(x))}{x^2+1}, y(x)\right)$; *#Maple check*

$$y(x) = 2 + \frac{\_C1}{\left(x^2+1\right)^{3/2}} \quad \textbf{(8)}$$

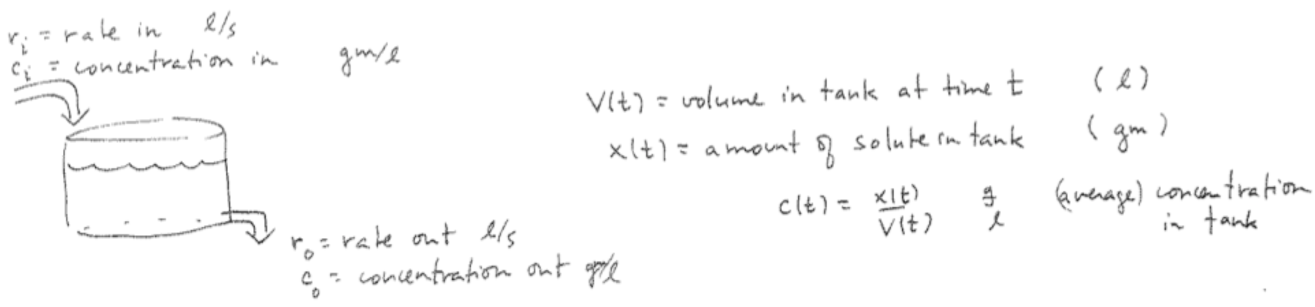


An extremely important class of modeling problems that lead to linear DE's involve input-output models. These have diverse applications ranging from bioengineering to environmental science. For example, The "tank" below could actually be a human body, a lake, or a pollution basin, in different applications.

For the present considerations, consider a tank holding liquid, with volume $V(t)$ (e.g units l). Liquid flows in at a rate r_i (e.g. units $\frac{l}{s}$), and with solute concentration c_i (e.g. units $\frac{gm}{l}$). Liquid flows out at a rate r_o , and with concentration c_0 . We are attempting to model the volume $V(t)$ of liquid and the amount of solute $x(t)$ (e.g. units gm) in the tank at time t , given $V(0) = V_0,\ x(0) = x_0$. We assume the solution in the tank is well-mixed, so that we can treat the concentration as uniform throughout the tank, i.e.

$$c_o = \frac{x(t)}{V(t)} \ \frac{gm}{l} .$$

See the diagram below.



<u>Exercise 4:</u> Under these assumptions use your modeling ability and Calculus to derive the following differential equations for $V(t)$ and $x(t)$:

a) The DE for $V(t)$, which we can just integrate:

$$V'(t) = r_i - r_0$$

so $V(t) = V_0 + \int_0^t r_i(\tau) - r_0(\tau)\, d\tau$

b) The linear DE for $x(t)$.

$$x'(t) = r_i c_i - r_o c_o = r_i c_i - r_o \frac{x}{V}$$

$$x'(t) + \frac{r_o}{V} x(t) = r_i c_i$$

Often (but not always) the tank volume remains constant, i.e. $r_i = r_o$. If the incoming concentration c_i is also constant, then the IVP for solute amount is

$$x' + a\,x = b$$
$$x(0) = x_0$$

where a, b are constants. This differential equation is separable and linear, and it is recommended that you become good at solving it. Notice that it includes the exponential growth/decay and Newton's law of cooling DE's as special cases.