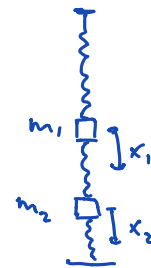
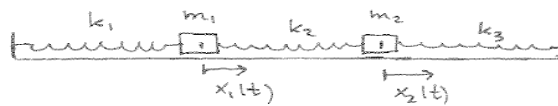


Fri Apr 21

7.4 Mass-spring systems and untethered mass-spring trains.

In your homework and lab for this week you study special cases of the spring systems below, with no damping. Although we draw the pictures horizontally, they would also hold in vertical configuration if we measure displacements from equilibrium in the underlying gravitational field.



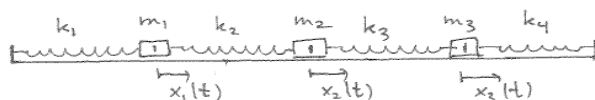
Let's make sure we understand why the natural system of DEs and IVP for this system is

$$\begin{cases} m_1 x_1''(t) = -k_1 x_1 + k_2(x_2 - x_1) \\ m_2 x_2''(t) = -k_2(x_2 - x_1) - k_3 x_2 \\ x_1(0) = a_1, \quad x_1'(0) = a_2 \\ x_2(0) = b_1, \quad x_2'(0) = b_2 \end{cases}$$

Exercise 1a) What is the dimension of the solution space to this homogeneous linear system of differential equations? Why?

$$\begin{aligned} 0) \quad m_1 x_1''(t) &= F_{\text{spring } 1} + F_{\text{spring } 2} = -k_1 x_1 + k_2(x_2 - x_1) \\ m_2 x_2'' &= F_{\text{spring } 2} + F_{\text{spring } 3} = -k_2(x_2 - x_1) - k_3 x_2 \\ \dim &= 4 \quad (\text{4 free parameters}) \\ &\quad (\text{equiv to system of 4 1st order homog. DE's}) \end{aligned}$$

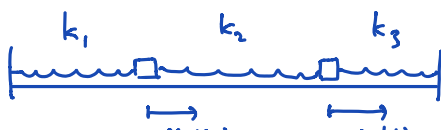
1b) What if one had a configuration of n masses in series, rather than just 2 masses? What would the dimension of the homogeneous solution space be in this case? Why? Examples:



$$\dim = 6$$



$$\dim = 4$$



$$\begin{aligned}
 m_1 x_1'' &= -k_1 x_1 + k_2 (x_2 - x_1) = -(k_1 + k_2) x_1 + k_2 x_2 \\
 m_2 x_2'' &= -k_2 (x_2 - x_1) - k_3 x_2 = k_2 x_1 - (k_2 + k_3) x_2
 \end{aligned}$$

We can write the system of DEs for the system at the top of page 1 in matrix-vector form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 - k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We denote the diagonal matrix on the left as the "mass matrix" M , and the matrix on the right as the spring constant matrix K (although to be completely in sync with Chapter 5 it would be better to call the spring matrix $-K$). All of these configurations of masses in series with springs can be written as

$$M \mathbf{x}''(t) = K \mathbf{x}.$$

If we divide each equation by the reciprocal of the corresponding mass, we can solve for the vector of accelerations:

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{k_1 + k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2 + k_3}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

which we write as

$$\mathbf{x}''(t) = A \mathbf{x}.$$

(You can think of A as the "acceleration" matrix.)

Notice that the simplification above is mathematically identical to the algebraic operation of multiplying the first matrix equation by the (diagonal) inverse of the diagonal mass matrix M . In all cases:

$$M \mathbf{x}''(t) = K \mathbf{x} \Rightarrow \mathbf{x}''(t) = A \mathbf{x}, \text{ with } A = M^{-1}K.$$

How to find a basis for the solution space to conserved-energy mass-spring systems of DEs

$$\mathbf{x}''(t) = A \mathbf{x} \quad (*)$$

Based on our previous experiences, the natural thing for this homogeneous system of linear differential equations is to try and find a basis of solutions of the form

$$\mathbf{x}(t) = f(t) \mathbf{v} \quad (**)$$

You might guess that $f(t) = e^{\lambda t}$ but that turns out to not be the best way to go. Let's see what $f(t)$ should equal by substituting in our guess! (We would maybe also think about first converting the second order system to an equivalent first order system of twice as many DE's, one for each position function and one for each velocity function, and then the exponential guess would work, but they'd end up being complex exponentials.) Substituting (**) into (*) yields

$$f''(t) \mathbf{v} = f'(t) A \mathbf{v} = f(t) A \mathbf{v}.$$

Since for each t , the left side is a scalar multiple of the constant vector \mathbf{v} , so must be the right side. So \mathbf{v} must be an eigenvector of A ,

$$A \mathbf{v} = \lambda \mathbf{v},$$

and if $f(t)$ is a real function and if \mathbf{v} is a real (as opposed to complex) vector, then λ is also real. Then

$$f'(t) \mathbf{v} = A (f(t) \mathbf{v}) = f(t) \lambda \mathbf{v}$$

$$f''(t) = f(t) \lambda$$

So we must have

$$f''(t) - \lambda f(t) = 0.$$

So possible $f(t)$'s are (depending on λ)

Case 1)

$$f''(t) = 0 \Rightarrow f(t) = c_1 + c_2 t \quad \text{if } \lambda = 0$$

$$f(t) = (c_1 + c_2 t) \vec{v}$$

Case 2)

$$f(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) \quad \text{if } \lambda < 0, \lambda = -\omega^2 \quad \omega = \sqrt{-\lambda}$$

$$f''(t) + \omega^2 f(t) = 0$$

Case 3)

$$f(t) = c_1 e^{\sqrt{\lambda} t} + c_2 e^{-\sqrt{\lambda} t} \quad \text{if } \lambda > 0.$$

$$f''(t) - \lambda f(t) = 0$$

Case 3 will never happen for our mass-spring configurations, because of conservation of energy!

This leads to the

Solution space algorithm: Consider a very special case of a homogeneous system of linear differential equations,

$$\mathbf{x}''(t) = A \mathbf{x}.$$

If $A_{n \times n}$ is a diagonalizable matrix and if all of its eigenvalues are non-positive then for each eigenpair $(\lambda_j, \mathbf{v}_j)$ with $\lambda_j < 0$ there are two linearly independent sinusoidal solutions to $\mathbf{x}''(t) = A \mathbf{x}$ given by

$$\mathbf{x}_j(t) = \cos(\omega_j t) \mathbf{v}_j \quad \mathbf{y}_j(t) = \sin(\omega_j t) \mathbf{v}_j$$

with

$$\omega_j = \sqrt{-\lambda_j}.$$

And for an eigenpair $(\lambda_j, \mathbf{v}_j)$ with $\lambda_j = 0$ there are two independent solutions given by constant and linear functions

$$\mathbf{x}_j(t) = \mathbf{v}_j \quad \mathbf{y}_j(t) = t \mathbf{v}_j$$

This procedure constructs $2n$ independent solutions to the system $\mathbf{x}''(t) = A \mathbf{x}$, i.e. a basis for the solution space.

Remark: What's amazing is that the fact that if the system is conservative, the acceleration matrix will always be diagonalizable, and all of its eigenvalues will be non-positive. In fact, if the system is tethered to at least one wall (as in the first two diagrams on page 1), all of the eigenvalues will be strictly negative, and the algorithm above will always yield a basis for the solution space. (If the system is not tethered and is free to move as a train, like the third diagram on page 1, then $\lambda = 0$ will be one of the eigenvalues, and will yield the constant velocity and displacement contribution to the solution space, $(c_1 + c_2 t) \mathbf{v}$, where \mathbf{v} is the corresponding eigenvector. Together with the solutions from strictly negative eigenvalues this will still lead to the general homogeneous solution.)

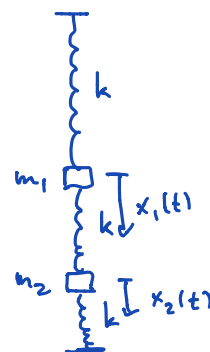
Exercise 2) Consider the special case of the configuration on page one for which $m_1 = m_2 = m$ and

$k_1 = k_2 = k_3 = k$ In this case, the equation for the vector of the two mass accelerations reduces to

if $\ddot{\vec{x}} = A \vec{x}$.

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{k}{m} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



a) Find the eigendata for the matrix

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

b) Deduce the eigendata for the acceleration matrix A which is $\frac{k}{m}$ times this matrix.

c) Find the 4- dimensional solution space to this two-mass, three-spring system.

a). $\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda+2)^2 - 1 = (\lambda+3)(\lambda+1)$
 $\lambda = -3, -1.$

$E_{\lambda=-1} \begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}$
 $E_{\lambda=-1} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$E_{\lambda=-3} \begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}$
 $E_{\lambda=-3} = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

b). If $A\vec{v} = \lambda\vec{v}$

what is $(sA)\vec{v} = s(A\vec{v}) = (s\lambda)\vec{v}$

if mult matrix by constant, eigenvectors stays same.
 eigenvalues get mult by const

for $\frac{k}{m} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

$\lambda = -3 \frac{k}{m}, -1 \cdot \frac{k}{m}$

$\omega = \sqrt{\frac{3k}{m}}, \sqrt{\frac{k}{m}}$
 $\omega_2 \quad \omega_1$

c) solns:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\left[c_1 \cos \sqrt{\frac{k}{m}} t + c_2 \sin \sqrt{\frac{k}{m}} t \right]}_{\text{mode 1}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underbrace{\left[c_3 \cos \sqrt{\frac{3k}{m}} t + c_4 \sin \sqrt{\frac{3k}{m}} t \right]}_{\text{mode 2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

"slow"
in-phase

amplitude
- phase for

"fast"
out of phase

solution The general solution is a superposition of two "fundamental modes". In the slower mode both masses oscillate "in phase", with equal amplitudes, and with angular frequency $\omega_1 = \sqrt{\frac{k}{m}}$. In the faster mode, both masses oscillate "out of phase" with equal amplitudes, and with angular frequency

$\omega_2 = \sqrt{\frac{3k}{m}}$. The general solution can be written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1 \cos(\omega_1 t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\omega_2 t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ = (c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (c_3 \cos(\omega_2 t) + c_4 \sin(\omega_2 t)) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Exercise 3) Show that the general solution above lets you uniquely solve each IVP uniquely. This should reinforce the idea that the solution space to these two second order linear homogeneous DE's is four dimensional.

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

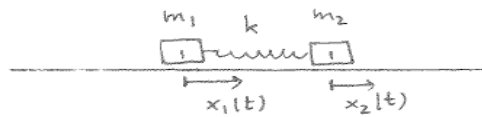
$$x_1(0) = a_1, \quad x_1'(0) = a_2$$

$$x_2(0) = b_1, \quad x_2'(0) = b_2$$

units for k $\frac{N}{m}$ $\frac{kg}{s^2}$

$\frac{k}{m} = \frac{1}{s^2}$

Exercise 4) Consider a train with two cars connected by a spring:



4a) Verify that the linear system of DEs that governs the dynamics of this configuration (it's actually a special case of what we did before, with two of the spring constants equal to zero) is

$$x_1'' = \frac{k}{m_1} (x_2 - x_1)$$

$$x_2'' = -\frac{k}{m_2} (x_2 - x_1)$$

4b) Use the eigenvalues and eigenvectors computed below to find the general solution. For $\lambda = 0$ and its corresponding eigenvector \underline{v} remember that you get two solutions

$$\underline{x}(t) = \underline{v} \text{ and } \underline{x}(t) = t \underline{v},$$

rather than the expected $\cos(\omega t) \underline{v}$, $\sin(\omega t) \underline{v}$. Interpret these solutions in terms of train motions. You will use these ideas in some of your homework problems and in your lab exercise about molecular vibrations.

$$\left[\begin{array}{l} \text{Eigenvectors} \left(\begin{bmatrix} -\frac{k}{m_1} & \frac{k}{m_1} \\ \frac{k}{m_2} & -\frac{k}{m_2} \end{bmatrix} \right); \\ \begin{bmatrix} 0 \\ -\frac{k(m_1 + m_2)}{m_2 m_1} \end{bmatrix}, \begin{bmatrix} 1 & -\frac{m_2}{m_1} \\ 1 & 1 \end{bmatrix} \end{array} \right] \quad (3)$$

Math 2250-004

Week 15: Section 7.4, mass-spring systems.

These are notes for Monday (and possibly part of Tuesday). There will also be course review notes for Tuesday, posted later.

Mon Apr 24

7.4 Mass-spring systems and untethered mass-spring trains, continued. First do the warmup exercise, then finish Friday's notes about unforced oscillations and natural frequencies/modes. Today's notes are about forced oscillation problems.

Warm up exercise: Here are two systems of differential equations, and the eigendata is as shown. The first order system could arise from an input-output model, and the second one could arise from an undamped two mass, three spring model. Write down the general solution to each system.

1a)

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1b)

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

eigendata: For the matrix

$$\begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix}$$

for the eigenvalue $\lambda = -5$, $\mathbf{v} = [-2, 1]^T$ is an eigenvector; for the eigenvalue $\lambda = -1$, $\mathbf{v} = [2, 1]^T$ is an eigenvector

Forced oscillations (still undamped):

$$M \mathbf{x}''(t) = K \mathbf{x} + \mathbf{F}(t) \\ \Rightarrow \mathbf{x}''(t) = A \mathbf{x} + M^{-1} \mathbf{F}(t) .$$

If the forcing is sinusoidal,

$$M \mathbf{x}''(t) = K \mathbf{x} + \cos(\omega t) \mathbf{G}_0 \\ \Rightarrow \mathbf{x}''(t) = A \mathbf{x} + \cos(\omega t) \mathbf{E}_0$$

with $\mathbf{E}_0 = M^{-1} \mathbf{G}_0$.

From the fundamental theorem for linear transformations we know that the general solution to this inhomogeneous linear problem is of the form

$$\mathbf{x}(t) = \mathbf{x}_p(t) + \mathbf{x}_H(t) ,$$

and we've been discussing how to find the homogeneous solutions $\mathbf{x}_H(t)$.

As long as the driving frequency ω is NOT one of the natural frequencies, we don't expect resonance; the method of undetermined coefficients predicts there should be a particular solution of the form

$$\mathbf{x}_p(t) = \cos(\omega t) \mathbf{c}$$

where the vector \mathbf{c} is what we need to find.

Exercise 1) Substitute the guess $\mathbf{x}_p(t) = \cos(\omega t) \mathbf{c}$ into the DE system

$$\mathbf{x}''(t) = A \mathbf{x} + \cos(\omega t) \mathbf{E}_0$$

to find a matrix algebra formula for $\mathbf{c} = \mathbf{c}(\omega)$. Notice that this formula makes sense precisely when ω is NOT one of the natural frequencies of the system.

Solution:

$$\mathbf{c}(\omega) = -(A + \omega^2 I)^{-1} \mathbf{E}_0 .$$

Note, matrix inverse exists precisely if $-\omega^2$ is not an eigenvalue.

Exercise 2) Continuing with the configuration from page 1 of Monday's notes, but now for an inhomogeneous forced problem, let $k = m$, and force the second mass sinusoidally:

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \cos(\omega t) \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

We know from previous work that the natural frequencies are $\omega_1 = 1$, $\omega_2 = \sqrt{3}$ and that

$$\mathbf{x}_H(t) = C_1 \cos(t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\sqrt{3}t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find the formula for $\mathbf{x}_p(t)$, as on the preceding page. Notice that this steady periodic solution blows up as $\omega \rightarrow 1$ or $\omega \rightarrow \sqrt{3}$. (If we don't have time to work this by hand, we may skip directly to the technology check on the next page. But since we have quick formulas for inverses of 2 by 2 matrices, this is definitely a computation we could do by hand.)

Solution: As long as $\omega \neq 1, \sqrt{3}$, the general solution $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_H$ is given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \cos(\omega t) \begin{bmatrix} \frac{3}{(\omega^2 - 1)(\omega^2 - 3)} \\ \frac{6 - 3\omega^2}{(\omega^2 - 1)(\omega^2 - 3)} \end{bmatrix} + C_1 \cos(t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\sqrt{3}t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Interpretation as far as inferred practical resonance for slightly damped problems: If there was even a small amount of damping, the homogeneous solution would actually be transient (it would be exponentially decaying and oscillating - underdamped). There would still be a sinusoidal particular solution, which would have a formula close to our particular solution, the first term above, as long as $\omega \neq 1, \sqrt{3}$. (There would also be a relatively smaller $\sin(\omega t)\mathbf{d}$ term as well.) So we can infer the practical resonance behavior for different ω values with slight damping, by looking at the size of the $\mathbf{c}(\omega)$ term for the undamped problem....see next page for visualizations.

```

> restart :
> with(LinearAlgebra) :
> A := Matrix(2, 2, [-2, 1, 1, -2]) :
> F0 := Vector([0, 3]) :
> Iden := IdentityMatrix(2) :
> c := ω → (A + ω2 · Iden)-1 · (-F0) : # the formula we worked out by hand
> c(ω);

```

$$\begin{bmatrix} \frac{3}{\omega^4 - 4\omega^2 + 3} \\ -\frac{3(\omega^2 - 2)}{\omega^4 - 4\omega^2 + 3} \end{bmatrix}$$

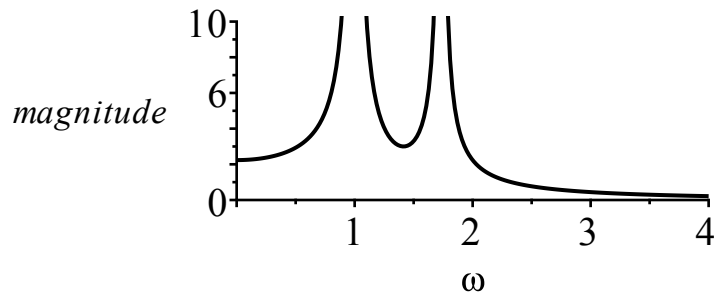
(1)

```

> with(plots) :
> with(LinearAlgebra) :
> plot(Norm(c(ω), 2), ω = 0..4, magnitude = 0..10, color = black, title
= `undamped particular solution amplitudes as proxy for practical resonance`);
# Norm(c(ω), 2) is the magnitude of the c(ω) vector

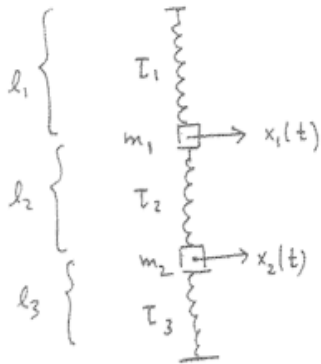
```

*undamped particular solution amplitudes
as proxy for practical resonance*



There are strong connections between our discussion here and the modeling of how earthquakes can shake buildings:

- Transverse oscillations! (i.e. directions \perp to the mass-spring configuration)



T_1, T_2, T_3 are the tensions (^{pulling} forces) of the stretched springs

By linearization, a good model would be

$$m_1 x_1'' = -K_1 x_1 + K_2 (x_2 - x_1) = -(K_1 + K_2) x_1 + K_2 x_2$$

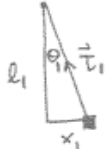
$$m_2 x_2'' = K_2 (x_1 - x_2) - K_3 x_2 = K_2 x_1 - (K_2 + K_3) x_2$$

where K_1, K_2, K_3 are positive constants as before

→ but in general not the Hooke's constants, because to first order the springs are not being stretched beyond their equilibrium lengths in this model.

- upshot: transverse oscillations satisfy analogous systems of 2nd order linear DE's; forcing and resonance will also be analogous to longitudinal vibrations, but probably with different resonant frequencies & ~~the~~ fundamental modes.

As it turns out, for our physics lab springs, the modes and frequencies are almost identical:

[force picture, e.g. ]

horiz force from top spring on mass 1

$$= -T_1 \sin \theta_1 = -T_1 \frac{x_1}{\sqrt{l_1^2 + x_1^2}} \approx -T_1 \frac{x_1}{l_1} = -\frac{T_1}{l_1} x_1$$

So $K_1 = \frac{T_1}{l_1}$

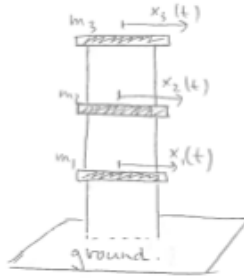
similarly, $K_2 = \frac{T_2}{l_2}, K_3 = \frac{T_3}{l_3}$

for our physics demo springs,
 equilibrium length ≈ 0 , very Hookean so $T \approx k l$; $\frac{T}{l} \approx k$, so
 actually almost recover same $\frac{k}{l}$ fundamental modes !!]

- An interesting shake-table demonstration:

http://www.youtube.com/watch?v=M_x2jOKAhZM

Below is a discussion of how to model the unforced "three-story" building shown shaking in the video above, from which we can see which modes will be excited. There is also a "two-story" building model in the video, and its matrix and eigendata follow. Here's a schematic of the three-story building:



For the unforced (homogeneous) problem, the accelerations of the three massive floors (the top one is the roof) above ground and of mass m , are given by

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \\ x_3''(t) \end{bmatrix} = \frac{k}{m} \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

Note the -1 value in the last diagonal entry of the matrix. This is because $x_3(t)$ is measuring displacements for the top floor (roof), which has nothing above it. The "k" is just the linearization proportionality factor, and depends on the tension in the walls, and the height between floors, etc, as discussed on the previous page.

Here is eigendata for the unscaled matrix $\left(\frac{k}{m} = 1\right)$. For the scaled matrix you'd have the same eigenvectors, but the eigenvalues would all be multiplied by the scaling factor $\frac{k}{m}$ and the natural frequencies would all be scaled by $\sqrt{\frac{k}{m}}$ but the eigenvectors describing the modes would stay the same.

eigenvalues({{-2,0,1,0},{1,-2,1},{0,1,-1}});

Web Apps Examples Random

Input:

eigenvalues $\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

Open code

Results:

$\lambda_1 \approx -3.24698$

$\lambda_2 \approx -1.55496$

$\lambda_3 \approx -0.198062$

Corresponding eigenvectors:

$v_1 \approx (0.591009, -0.736976, 0.327985)$

$v_2 \approx (-0.736976, -0.327985, 0.591009)$

$v_3 \approx (0.327985, 0.591009, 0.736976)$

eigenvalues({{-2,0,1},{1,-1}});

Web Apps Examples Random

Input:

eigenvalues $\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$

Open code

Results:

$\lambda_1 \approx -2.61803$

$\lambda_2 \approx -0.381966$

Corresponding eigenvectors:

$v_1 \approx (-0.850651, 0.525731)$

$v_2 \approx (-0.525731, -0.850651)$

Exercise 3) Interpret the data above, in terms of the natural modes for the 3-story and 2-story shaking buildings. In the youtube video the first mode to appear is the slow and dangerous "sloshing mode", where all three floors oscillate in phase, with amplitude ratios .33 : .59 : .74 from the first to the third floor. What's the second mode that gets excited? The third mode? (They don't show the third mode in the video.)

Remark) All of the ideas we've discussed in section 7.4 also apply to molecular vibrations. The eigendata in these cases is related to the "spectrum" of light frequencies that correspond to the natural fundamental modes for molecular vibrations. See your lab question about the fundamental modes of carbon dioxide.

Math 2250-004
Tuesday, April 25
Course review

Final exam: Thursday April 27, 10:30 a.m. -12:30 p.m. This is the official University time and location – in our MWF lecture room WEB L105. (I will let you start working at 10:15 and stay until 12:45 if you wish and if the room is available.) As usual the exam is closed book and closed note, and the only sort of calculator that is allowed is a simple scientific one. Your cell phones need to be put away for the entire exam. You will be provided the Laplace Transform table from the front cover of our text. The algebra and math on the exam should all be doable by hand.

Review of previous final exam practice exams: Patrick Webb and Kevin Childers will announce their office hours and review sessions. Two old final exams and solutions are posted in CANVAS. I will hold my usual Tuesday, Wednesday office hours in LCB 218.

The final exam will be comprehensive, but weighted to more recent material. Rough percentage ranges per chapter are below – these percentage ranges add up to more than 100% because many topics and problems span several chapters. Also consult the “course learning objectives” from our syllabus (next page).

Chapters

- 1-2: 10-20% first order DEs
- 3-4: 20-40% matrix algebra and vector spaces
- 5: 15-30% linear differential equations and applications
- 6.1-6.2: 15-30% eigenvalues and eigenvectors, including complex case
- 7.1-7.4: 20-40% linear systems of differential equations and applications
- 10.4-10.5, EP 7.6: 20-40% Laplace transform

On the third page is a more detailed list of the topics we’ve investigated this semester. They are more inter-related than you may have realized at the time, so we’ll discuss the connections in class. Then we’ll look at an extended problem that highlights these connections and connects perhaps 70% of the key ideas in this course. We won’t have time to work out everything in class, but filling in the details might help you consolidate these ideas.

Learning Objectives for 2250

The goal of Math 2250 is to master the basic tools and problem solving techniques important in differential equations and linear algebra. These basic tools and problem solving skills are described below.

The essential topics

Be able to model dynamical systems that arise in science and engineering, by using general principles to derive the governing differential equations or systems of differential equations. These principles include linearization, compartmental analysis, Newton's laws, conservation of energy and Kirchoff's law.

Learn solution techniques for first order separable and linear differential equations. Solve initial value problems in these cases, with applications to problems in science and engineering. Understand how to approximate solutions even when exact formulas do not exist. Visualize solution graphs and numerical approximations to initial value problems via slope fields. Understand phase diagram analysis for autonomous first order differential equations.

Become fluent in matrix algebra techniques, in order to be able to compute the solution space to linear systems and understand its structure; by hand for small problems and with technology for large problems.

Be able to use the basic concepts of linear algebra such as linear combinations, span, independence, basis and dimension, to understand the solution space to linear equations, linear differential equations, and linear systems of differential equations.

Understand the natural initial value problems for first order systems of differential equations, and how they encompass the natural initial value problems for higher order differential equations and general systems of differential equations.

Learn how to solve constant coefficient linear differential equations via superposition, particular solutions, and homogeneous solutions found via characteristic equation analysis. Apply these techniques to understand the solutions to the basic unforced and forced mechanical and electrical oscillation problems.

Learn how to use Laplace transform techniques to solve linear differential equations, with an emphasis on the initial value problems of mechanical systems, electrical circuits, and related problems.

Be able to find eigenvalues and eigenvectors for square matrices. Apply these matrix algebra concepts to find the general solution space to first and second order constant coefficient homogeneous linear systems of differential equations, especially those arising from compartmental analysis and mechanical systems.

Understand and be able to use linearization as a technique to understand the behavior of nonlinear dynamical systems near equilibrium solutions. Apply these techniques to non-linear mechanical oscillation problems. (Additional material, subject to time availability: Apply linearization to autonomous systems of two first order differential equations, including interacting populations. Relate the phase portraits of non-linear systems near equilibria to the linearized data, in particular to understand stability.)

Develop your ability to communicate modeling and mathematical explanations and solutions, using technology and software such as Maple, Matlab or internet-based tools as appropriate.

Problem solving fluency

Students will be able to read and understand problem descriptions, then be able to formulate equations modeling the problem usually by applying geometric or physical principles. Solving a problem often requires specific solution methods listed above. Students will be able to select the appropriate operations, execute them accurately, and interpret the results using numerical and graphical computational aids.

Students will also gain experience with problem solving in groups. Students should be able to effectively transform problem objectives into appropriate problem solving methods through collaborative discussion. Students will also learn how to articulate questions effectively with both the instructor and TA, and be able to effectively convey how problem solutions meet the problem objectives.

1-2: first order DEs

slope fields, Euler approximation
phase diagrams for autonomous DEs
equilibrium solutions
stability
existence-uniqueness thm for IVPs
methods:
separable
linear
applications
populations
velocity-acceleration models
input-output models

3-4 matrix algebra and vector spaces

linear systems and matrices
reduced row echelon form
matrix and vector algebra
manipulating and solving matrix-
vector equations for unknown
vectors or matrices.
matrix inverses
determinants
vector space concepts
vector spaces and subspaces
linear combinations
linear dependence/independence
span
basis and dimension
linear transformations
aka superposition
fundamental theorem for solution
space to $L(y)=f$ when L is linear

5 Linear differential equations

IVP existence and uniqueness

Linear DEs

Homogeneous solution space,
its dimension, and why
superposition, $\underline{x}(t) = \underline{x}_p + \underline{x}_h$

Constant coefficient linear DEs

\underline{x}_h via characteristic polynomial

Euler's formula, complex roots

\underline{x}_p via undetermined coefficients
solving IVPs

applications:

mechanical configurations

unforced: undamped and damped
cos and sin addition angle formulas
and amplitude-phase form

forced undamped: beating, resonance

forced damped: $\underline{x}_{sp} + \underline{x}_{tr}$, practical
resonance

RLC circuits

Using conservation of total energy
(=KE+PE) to derive equations of
motion, especially for mass-spring and
pendulum

6.1-6.1 eigenvalues, eigenvectors (eigenspaces),
diagonalizable matrices; real+complex eigendata.

7.1-7.4 linear systems of DEs

first order systems of DEs and tangent vector
fields.

existence-uniqueness thm for first order IVPs

superposition, $\underline{x} = \underline{x}_p + \underline{x}_h$

dimension of solution space for \underline{x}_h .

equivalence of DE IVPs or systems to first
order system IVPs.

Constant coefficient systems and methods for

$\underline{x} = \underline{x}_p + \underline{x}_h$:

$\underline{x}'(t) = A\underline{x}$

$\underline{x}'(t) = A\underline{x} + \underline{f}(t)$

$\underline{x}''(t) = A\underline{x}$ (from conservative systems)

$\underline{x}''(t) = A\underline{x} + \underline{f}(t)$

applications: phase portrait interpretation of
unforced oscillation problems; input-output
modeling; forced and unforced mass-spring
systems and phenomena.

10.1-10.5, EP7.6: Laplace transform

definition, for direct computation

using table for Laplace and inverse Laplace

transforms ... including for topics before/after
the second midterm, i.e. on/off and impulse,
forcing, convolution solutions

Solving linear DE (or system of DE) IVPs with
Laplace transform.

We can illustrate many ideas in this course, and how they are tied together by studying the following two differential equations in as many ways as we can think of.

$$x''(t) + 5x'(t) + 4x(t) = 0$$

$$x''(t) + 5x'(t) + 4x(t) = 3\cos(2t)$$