Math 2250-004
Week 14 April 17-21: sections 7.1-7.3 first order systems of linear differential equations; 7.4 mass-spring systems.
Mon Apr 17

Continue discussing systems of differential equations, 7.1-7.3

• Review exercises 3-5 in last Friday's notes, which foreshadow the eigenvalue-eigenvector method for solving homogeneous first order systems of differential equation IVPs

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

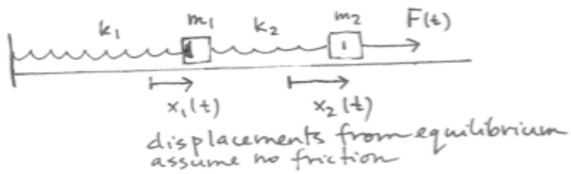
that the book covers in detail in section 7.3.

• Then discuss the important fact that every n^{th} order differential equation or system of differential equations is actually equivalent to a (possibly quite large) system of first order differential equations This discussion is in today's notes and is the content of section 7.1. The existence-uniqueness theorem for first order systems of differential equations will then explain why the natural initial value problems for higher order differential equations, that we studied in Chapter 5, also always have unique solutions. Another mystery that this correspondence will solve is why we used "characteristic polynomial" when looking for basis solutions $e^{r\,t}$ to n^{th} order constant coefficient homogeneous differential equations and then used exactly the same terminology, "characteristic polynomial", when we were looking for matrix eigenvalues (see section 7.3).

• Along the way and tomorrow as well, we will also be discussing how the theory and template for finding solutions to first order systems of linear differential equations precisely mirrors the template we developed for single n^{th}-order linear differential equations in Chapter 5. This is the content of section 7.2.

Converting higher order DE's or systems of DE's to equivalent first order systems of DEs:

Example:
Consider this configuration of two coupled masses and springs:



Exercise 1) Use Newton's second law to derive a system of two second order differential equations for $x_1(t), x_2(t)$, the displacements of the respective masses from the equilibrium configuration. What initial value problem do you expect yields unique solutions in this case?

Exercise 2) Consider the IVP from Exercise 1, with the special values $m_1 = 2, m_2 = 1; k_1 = 4, k_2 = 2; F(t) = 40\sin(3t)$:

$$\begin{aligned}
x_1'' &= -3x_1 + x_2 \\
x_2'' &= 2x_1 - 2x_2 + 40\sin(3t) \\
x_1(0) &= b_1, x_1'(0) = b_2 \\
x_2(0) &= c_1, x_2'(0) = c_2 .
\end{aligned}$$

2a) Show that if $x_1(t), x_2(t)$ solve the IVP above, and if we define

$$\begin{aligned}
v_1(t) &:= x_1'(t) \\
v_2(t) &:= x_2'(t)
\end{aligned}$$

then $x_1(t), x_2(t), v_1(t), v_2(t)$ solve the first order system IVP

$$\begin{aligned}
x_1' &= v_1 \\
x_2' &= v_2 \\
v_1' &= -3x_1 + x_2 \\
v_2' &= 2x_1 - 2x_2 + 40\sin(3t) \\
x_1(0) &= b_1 \\
v_1(0) &= b_2 \\
x_2(0) &= c_1 \\
v_2(0) &= c_2 .
\end{aligned}$$

2b) Conversely, show that if $x_1(t), x_2(t), v_1(t), v_2(t)$ solve the IVP of four first order DE's, then $x_1(t), x_2(t)$ solve the original IVP for two second order DE's.

It is always the case that the natural initial value problems for single or systems of differential equations are equivalent to an initial value problem for a larger system of first order differential equations, as in the previous example.

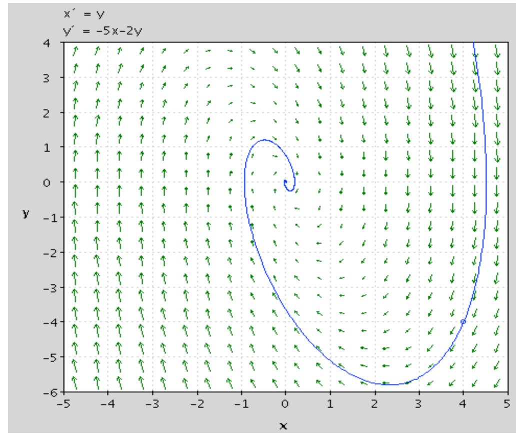
A special case of this fact is that if you have an IVP for single n^{th} order DE for $x(t)$, it is equivalent to an IVP for a system of n first-order DE's, in which the functions are

$$x_1(t) = x(t),\;\; x_2(t) := x'(t),\;\; x_3(t) := x''(t), \ldots,\;\; x_n(t) := x^{(n-1)}(t).$$

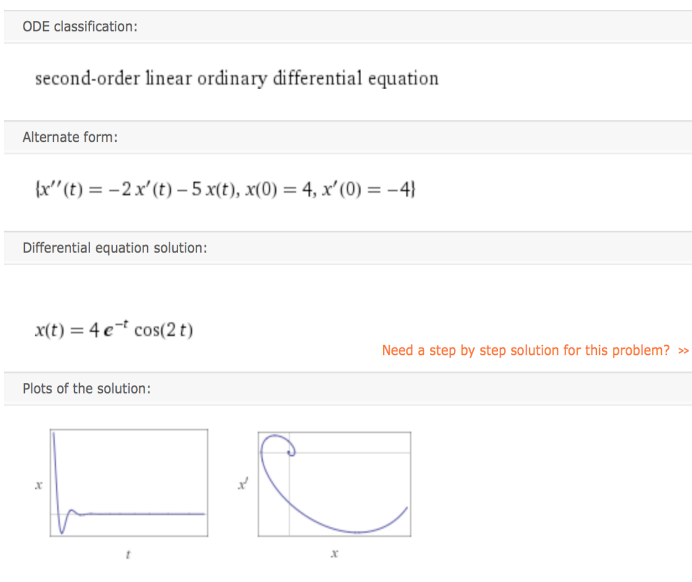
For example, consider this second order underdamped IVP for $x(t)$:

$$x'' + 2x' + 5x = 0$$
$$x(0) = 4$$
$$x'(0) = -4\ .$$

<u>Exercise 3)</u>
<u>3a)</u> Convert this single second order IVP into an equivalent first order system IVP for $x_1(t) := x(t)$ and $x_2(t) := x'(t)$.
<u>3b)</u> Solve the second order IVP in order to deduce a solution to the first order IVP. Use Chapter 5 methods even though you love Laplace transform more.
<u>3c)</u> How does the Chapter 5 "characteristic polynomial" in 3<u>b</u> compare with the Chapter 6 (eigenvalue) "characteristic polynomial" for the first order system matrix in 3<u>a</u>? hmmm.
<u>3d)</u> Is your analytic solution $[x(t), v(t)]$ in 3<u>b</u> consistent with the parametric curve shown on the next page, in a "pplane" screenshot? The picture is called a "phase portrait" for position and velocity.



If you've been using Wolfram alpha to solve second order differential equations you might have noticed pictures that look just like the one above, even though you might not have thought about them ... look at the plot at the lower right corner of this screenshot!



Theorem 1 For the IVP

$$\underline{\boldsymbol{x}}'(t) = \underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t))$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

If $\underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}})$ is continuous in the t-variable and differentiable in its $\underline{\boldsymbol{x}}$ variable, then there exists a unique solution to the IVP, at least on some (possibly short) time interval $t_0 - \delta < t < t_0 + \delta$.

Theorem 2 For the special case of the first order linear system of differential equations IVP

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{f}}(t)$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

If the matrix $A(t)$ and the vector function $\underline{\boldsymbol{f}}(t)$ are continuous on an open interval I containing t_0 then a solution $\underline{\boldsymbol{x}}(t)$ exists and is unique, on the entire interval.

Remark: The solutions to these systems of DE's may be approximated numerically using vectorized versions of Euler's method and the Runge Kutta method. The ideas are exactly the same as they were for scalar equations, except that they now use vectors. This is how commerical numerical DE solvers work. For example, with time-step h the Euler loop would increment as follows:

$$t_j = t_0 + h\,j$$
$$\underline{\boldsymbol{x}}_{j+1} = \underline{\boldsymbol{x}}_j + h\,\underline{\boldsymbol{F}}(t_j, \underline{\boldsymbol{x}}_j).$$

Remark: These theorems are the true explanation for why the n^{th}-order linear DE IVPs in Chapter 5 always have unique solutions - because each $n^{th} - order$ linear DE IVP is equivalent to an IVP for a first order system of n linear DE's. In fact, when software finds numerical approximations for solutions to higher order (linear or non-linear) DE IVPs that can't be found by the techniques of Chapter 5 or other mathematical formulas, it works by converting these IVPs to the equivalent first order system IVPs, and uses algorithms like Euler and Runge-Kutta to approximate the solutions.

Theorem 3) Vector space theory for first order systems of linear DEs (Notice the familiar themes...we can completely understand these facts if we take the intuitively reasonable existence-uniqueness Theorem 2 as fact.)

3.1) For vector functions $\underline{\mathbf{x}}(t)$ differentiable on an interval, the operator

$$L(\underline{\mathbf{x}}(t)) := \underline{\mathbf{x}}'(t) - A(t)\underline{x}(t)$$

is linear, i.e.

$$L(\underline{\mathbf{x}}(t) + \underline{\mathbf{z}}(t)) = L(\underline{\mathbf{x}}(t)) + L(\underline{\mathbf{z}}(t))$$
$$L(c\,\underline{\mathbf{x}}(t)) = c\,L(\underline{\mathbf{x}}(t))\,.$$

check!

3.2) Thus, by the fundamental theorem for linear transformations, the general solution to the non-homogeneous linear problem

$$\underline{\mathbf{x}}'(t) - A(t)\underline{\mathbf{x}}(t) = \underline{\boldsymbol{f}}(t)$$

$\forall\ t \in I$ is

$$\underline{\mathbf{x}}(t) = \underline{\mathbf{x}}_p(t) + \underline{\mathbf{x}}_H(t)$$

where $\underline{\mathbf{x}}_p(t)$ is any single particular solution and $\underline{\mathbf{x}}_H(t)$ is the general solution to the homogeneous problem

$$\underline{\mathbf{x}}'(t) - A(t)\underline{\mathbf{x}}(t) = \underline{\mathbf{0}}$$

We frequently write the homogeneous linear system of DE's as

$$\underline{\mathbf{x}}'(t) = A(t)\underline{\mathbf{x}}(t)\,.$$

3.3) For $A(t)_{n \times n}$ and $\underline{\boldsymbol{x}}(t) \in \mathbb{R}^n$ the solution space on the $t-$interval I to the homogeneous problem

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$

is n-dimensional. Here's why:

- Let $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), ...\underline{\boldsymbol{X}}_n(t)$ be any n solutions to the homogeneous problem chosen so that the Wronskian matrix at $t_0 \in I$

$$\left[W(\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \dots, \underline{\boldsymbol{X}}_n)\right](t_0) := \left[\underline{\boldsymbol{X}}_1(t_0) | \underline{\boldsymbol{X}}_2(t_0) | \cdots | \underline{\boldsymbol{X}}_n(t_0)\right]$$

is invertible. (By the existence theorem we can choose solutions for any collection of initial vectors - so for example, in theory we could pick the matrix above to actually equal the identity matrix. In practice we'll be happy with any invertible matrix.)

- Then for any $\underline{\boldsymbol{b}} \in \mathbb{R}^n$ the IVP

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{b}}$$

has solution $\underline{\boldsymbol{x}}(t) = c_1\underline{\boldsymbol{X}}_1(t) + c_2\underline{\boldsymbol{X}}_2(t) + \ldots + c_n\underline{\boldsymbol{X}}_n(t)$ where the linear combination coefficients are the solution to the Wronskian matrix equation

$$\begin{bmatrix} \underline{\boldsymbol{X}}_1(t_0) & \underline{\boldsymbol{X}}_2(t_0) & \cdots & \underline{\boldsymbol{X}}_n(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ : \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ : \\ b_n \end{bmatrix}.$$

Thus, because the Wronskian matrix at t_0 is invertible, every IVP can be solved with a linear combination of $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), ...\underline{\boldsymbol{X}}_n(t)$, and since each IVP has only one solution, $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), ...\underline{\boldsymbol{X}}_n(t)$ span the solution space. The same matrix equation shows that the only linear combination that yields the zero function (which has initial vector $\underline{\boldsymbol{b}} = \underline{\boldsymbol{0}}$) is the one with $\underline{\boldsymbol{c}} = \underline{\boldsymbol{0}}$. Thus $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), ...\underline{\boldsymbol{X}}_n(t)$ are also linearly independent. Therefore they are a basis for the solution space, and their number n is the dimension of the solution space.

7.3 Eigenvector method for finding the general solution to the homogeneous constant matrix first order system of differential equations

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$

Here's how: We look for a basis of solutions $\underline{\boldsymbol{x}}(t) = e^{\lambda t}\underline{\boldsymbol{v}}$, where $\underline{\boldsymbol{v}}$ is a constant vector. Substituting this form of potential solution into the system of DE's above yields the equation

$$\lambda\, e^{\lambda t}\underline{\boldsymbol{v}} = A\, e^{\lambda t}\underline{\boldsymbol{v}} = e^{\lambda t} A\,\underline{\boldsymbol{v}}\,.$$

Dividing both sides of this equation by the scalar function $e^{\lambda t}$ gives the condition

$$\lambda\,\underline{\boldsymbol{v}} = A\,\underline{\boldsymbol{v}}\,.$$

- We get a solution every time $\underline{\boldsymbol{v}}$ is an eigenvector of A with eigenvalue λ !
- If A is diagonalizable then there is an $\mathbb{R}^n$ basis of eigenvectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, ...\underline{\boldsymbol{v}}_n$ and solutions

$$\underline{\boldsymbol{X}}_1(t) = e^{\lambda_1 t}\,\underline{\boldsymbol{v}}_1,\ \underline{\boldsymbol{X}}_2(t) = e^{\lambda_2 t}\,\underline{\boldsymbol{v}}_2\,, ...,\ \underline{\boldsymbol{X}}_n(t) = e^{\lambda_n t}\,\underline{\boldsymbol{v}}_n$$

which are a basis for the solution space on the interval $I = \mathbb{R}$, because the Wronskian matrix at $t = 0$ is the invertible diagonalizing matrix

$$P = \left[\underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2 | \cdots | \underline{\boldsymbol{v}}_n\right]$$

that we considered in Chapter 6.

- If A has complex number eigenvalues and eigenvectors it may still be diagonalizable over $\mathbb{C}^n$, and we will still be able to extract a basis of real vector function solutions. If A is not diagonalizable over $\mathbb{R}^n$ or over $\mathbb{C}^n$ the situation is more complicated.

Exercise 4a) Use the method above to find the general homogeneous solution to

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

4b) Solve the IVP with

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

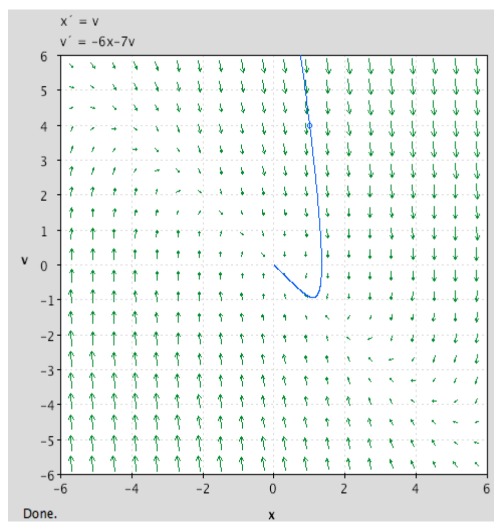
Exercise 5a) What second order overdamped initial value problem is equivalent to the first order system IVP on the previous page. And what is the solution function to this IVP?

5b) What do you notice about the Chapter 5 "Wronskian matrix" for the second order DE in 4a, and the Chapter 7 "Wronskian matrix" for the solution to the equivalent first order system?

5c) Since in the correspondence above, $x_2(t)$ equals the mass velocity $x'(t) = v(t)$, I've created the pplane phase portrait below using the lettering $[x(t), v(t)]^T$ rather than $[x_1(t), x_2(t)]^T$. Intepret the behavior of the overdamped mass-spring motion in terms of the pplane phase portrait.

5d) How do the eigenvectors show up in the phase portrait, in terms of the direction the origin is approached from as $t \to \infty$, and the direction solutions came from (as $t \to -\infty$)?

http://math.rice.edu/~dfield/dfpp.html



Math 2250-004
Tues Apr 18
7.1-7.3 Summary of what is covered in Monday's and Tuesday's notes:

• Any initial value problem for a differential equation or system of differential equations can be converted into an equivalent initial value problem for a system of first order differential equations.

• There is a "short-time" existence-uniqueness theorem for first order DE initial value problems

$$\underline{\boldsymbol{x}}'(t) = \underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t))$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

and solutions can be approximated using Euler or Runge-Kutta type algorithms.

• A special case of the IVP above is the one for a first order <u>linear system of differential equations</u> :

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{f}}(t)$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

If the matrix $A(t)$ and the vector function $\underline{\boldsymbol{f}}(t)$ are continuous on an open interval I containing t_0 then a solution $\underline{\boldsymbol{x}}(t)$ exists and is unique, <u>on the entire interval</u>.

• The general solution to an inhomogeneous linear system of DEs

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{f}}(t)$$
i.e.
$$\underline{\boldsymbol{x}}'(t) - A(t)\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{f}}(t)$$

will be of the form $\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}_P + \underline{\boldsymbol{x}}_H$ where $\underline{\boldsymbol{x}}_P$ is a particular solution, and $\underline{\boldsymbol{x}}_H$ is the general solution to the homogeneous system

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t) .$$

• For $A(t)_{n \times n}$ and $\underline{\boldsymbol{x}}(t) \in \mathbb{R}^n$ the solution space on the $t-$interval I to the homogeneous problem

$$\underline{\boldsymbol{x}}'(t) = A(t)\,\underline{\boldsymbol{x}}(t)$$
i.e.
$$\underline{\boldsymbol{x}}'(t) - A(t)\,\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{0}}$$

is <u>n-dimensional</u>.

• For $A_{n \times n}$ a constant matrix, we try to find a basis for the solution space to

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$

consisting of solutions of the form $\underline{\boldsymbol{x}}(t) = e^{\lambda t}\underline{\boldsymbol{v}}$, where $\underline{\boldsymbol{v}}$ is an eigenvector of A with eigenvalue λ. We will succeed as long as A is diagonalizable.

<u>Today:</u> We will continue using the eigenvalue-eigenvector method for finding the general solution to the homogeneous constant matrix first order system of differential equations

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$

that we discussed yesterday, and is in Monday's notes. Today we'll consider examples where the eigenvalues and eigenvectors are complex. There is such an example in the homework due Wednesday.... problem w13.3.

So far we've not considered the possibility of complex eigenvalues and eigenvectors. Linear algebra theory works the same with complex number scalars and vectors - one can talk about complex vector spaces, linear combinations, span, linear independence, reduced row echelon form, determinant, dimension, basis, etc. Then the model space is $\mathbb{C}^n$ rather than $\mathbb{R}^n$.

Definition: $\underline{v} \in \mathbb{C}^n$ ($\underline{v} \neq \underline{0}$) is a complex eigenvector of the matrix A, with eigenvalue $\lambda \in \mathbb{C}$ if $A\,\underline{v} = \boldsymbol{\lambda}\,\underline{v}$.

Just as before, you find the possibly complex eigenvalues by finding the roots of the characteristic polynomial $|A - \lambda I|$. Then find the eigenspace bases by reducing the corresponding matrix (using complex scalars in the elementary row operations).

The best way to see how to proceed in the case of complex eigenvalues/eigenvectors is to work an example. There is a general discussion on the page after this example that we will refer to along the way:

Glucose-insulin model (adapted from a discussion on page 339 of the text "Linear Algebra with Applications," by Otto Bretscher)

Let $G(t)$ be the excess glucose concentration (mg of G per 100 ml of blood, say) in someone's blood, at time t hours. Excess means we are keeping track of the difference between current and equilibrium ("fasting") concentrations. Similarly, Let $H(t)$ be the excess insulin concentration at time t hours. When blood levels of glucose rise, say as food is digested, the pancreas reacts by secreting insulin in order to utilize the glucose. Researchers have developed mathematical models for the glucose regulatory system. Here is a simplified (linearized) version of one such model, with particular representative matrix coefficients. It would be meant to apply between meals, when no additional glucose is being added to the system:

$$\begin{bmatrix} G'(t) \\ H'(t) \end{bmatrix} = \begin{bmatrix} -0.1 & -0.4 \\ 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix}$$

Exercise 3a) Understand why the signs of the matrix entries are reasonable.

Now let's solve the initial value problem, say right after a big meal, when

$$\begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

3b) The first step is to get the eigendata of the matrix. Do this, and compare with the Maple check on the next page.

> $with(LinearAlgebra):$

> $A := \begin{bmatrix} -\frac{1}{10} & -\frac{2}{5} \\ \frac{1}{10} & -\frac{1}{10} \end{bmatrix};$

$Eigenvectors(A);$

$$A := \begin{bmatrix} -\frac{1}{10} & -\frac{2}{5} \\ \frac{1}{10} & -\frac{1}{10} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{10} + \frac{1}{5}\,\mathrm{I} \\ -\frac{1}{10} - \frac{1}{5}\,\mathrm{I} \end{bmatrix}, \begin{bmatrix} 2\,\mathrm{I} & -2\,\mathrm{I} \\ 1 & 1 \end{bmatrix} \qquad \textbf{(1)}$$

Notice that Maple writes a capital $I = \sqrt{-1}$.

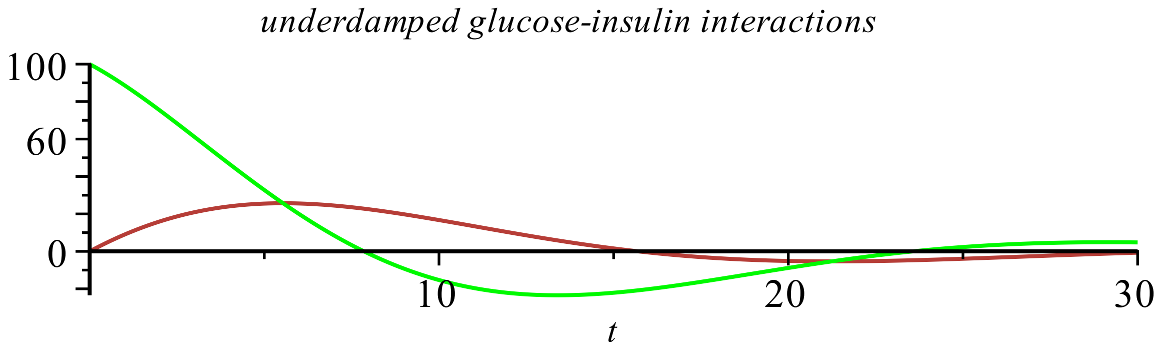
3c) Extract a basis for the solution space to his homogeneous system of differential equations from the eigenvector information above:

3d) Solve the initial value problem.

Here are some pictures to help understand what the model is predicting ... you could also construct these graphs using pplane.

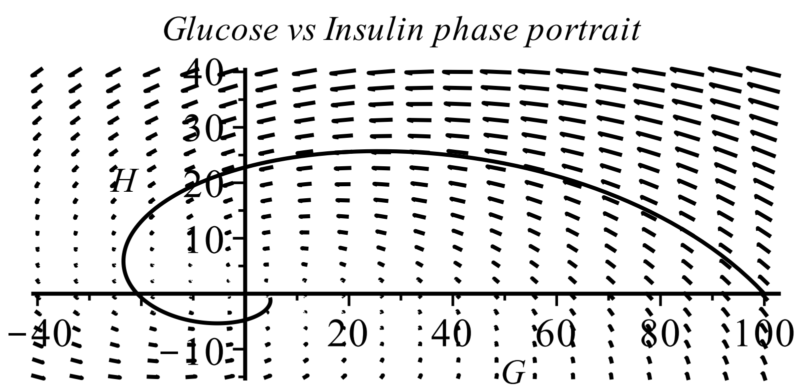
(1) Plots of glucose vs. insulin, at time t hours later:

```
> with(plots):
> G := t→100·exp(-.1·t)·cos(.2·t):
  H := t→50·exp(-.1·t)·sin(.2·t):
  plot1 := plot(G(t), t = 0..30, color = green):
  plot2 := plot(H(t), t = 0..30, color = brown):
  display({plot1, plot2}, title = `underdamped glucose-insulin interactions`);
```



2) A phase portrait of the glucose-insulin system:

```
> pict1 := fieldplot([-.1·G − .4·H, .1·G − .1·H], G = -40..100, H = -15..40):
  soltn := plot([G(t), H(t), t = 0..30], color = black):
  display({pict1, soltn}, title = `Glucose vs Insulin phase portrait`);
```



• The example we just worked is linear, and is vastly simplified. But mathematicians, doctors, bioengineers, pharmacists, are very interested in (especially more realistic) problems like these.

Solutions to homogeneous linear systems of DE's when matrix has complex eigenvalues:

$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}}$$

Let A be a real number matrix. Let

$$\lambda = a + b\,i \in \mathbb{C}$$
$$\underline{\mathbf{v}} = \underline{\boldsymbol{\alpha}} + i\,\underline{\boldsymbol{\beta}} \in \mathbb{C}^n$$

satisfy $A\,\underline{\mathbf{v}} = \boldsymbol{\lambda}\,\underline{\mathbf{v}}$, with $a, b \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}^n$.

- Then $\underline{\mathbf{z}}(t) = e^{\lambda t}\underline{\mathbf{v}}$ is a complex solution to

$$\underline{\mathbf{x}}'(t) = A\,\boldsymbol{x}$$

because $\underline{\mathbf{z}}'(t) = \lambda e^{\lambda t}\underline{\mathbf{v}}$ and this is equal to $A\,\underline{\mathbf{z}} = \boldsymbol{A}\,e^{\lambda t}\underline{\mathbf{v}} = e^{\lambda t}A\,\underline{\mathbf{v}} = e^{\lambda t}\lambda\,\underline{\mathbf{v}}$.

- But if we write $\underline{\mathbf{z}}(t)$ in terms of its real and imaginary parts,

$$\underline{\mathbf{z}}(t) = \underline{\mathbf{x}}(t) + i\,\underline{\mathbf{y}}(t)$$

then the equality

$$\underline{\mathbf{z}}'(t) = A\,\mathbf{z}$$
$$\Rightarrow \underline{\mathbf{x}}'(t) + i\,\underline{\mathbf{y}}'(t) = A(\underline{\mathbf{x}}(t) + i\,\underline{\mathbf{y}}(t)) = A\,\underline{\mathbf{x}}(t) + i\,A\,\underline{\mathbf{y}}(t)\,.$$

Equating the real and imaginary parts on each side yields

$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}}(t)$$
$$\underline{\mathbf{y}}'(t) = A\,\underline{\mathbf{y}}(t)$$

i.e. the real and imaginary parts of the complex solution are each real solutions.

- If $A\left(\underline{\boldsymbol{\alpha}} + i\,\underline{\boldsymbol{\beta}}\right) = (a + b\,i)\left(\underline{\boldsymbol{\alpha}} + i\,\underline{\boldsymbol{\beta}}\right)$ then it is straightforward to check that $A\left(\underline{\alpha} - i\,\underline{\boldsymbol{\beta}}\right) = (a - b\,i)\left(\underline{\alpha} - i\,\underline{\boldsymbol{\beta}}\right)$. Thus the complex conjugate eigenvalue yields the complex conjugate eigenvector. The corresponding complex solution to the system of DEs

$$e^{(a - i\,b)t}\left(\underline{\alpha} - i\,\underline{\boldsymbol{\beta}}\right) = \underline{\mathbf{x}}(t) - i\,\underline{\mathbf{y}}(t)$$

so yields the same two real solutions (except with a sign change on the second one). Another way to understand how we get the two real solutions is to take the two complex solutions

$$\underline{\mathbf{z}}(t) = \underline{\mathbf{x}}(t) + i\,\underline{\mathbf{y}}(t)$$
$$\underline{\mathbf{w}}(t) = \underline{\mathbf{x}}(t) - i\,\underline{\mathbf{y}}(t)$$

and recover $x(t), y(t)$ as linear combinations of these homogeneous solutions:

$$\underline{\mathbf{x}}(t) = \frac{1}{2}\left(\underline{\mathbf{z}}(t) + \underline{\mathbf{w}}(t)\right)$$
$$\underline{\mathbf{y}}(t) = \frac{1}{2\,i}\left(\underline{\mathbf{z}}(t) - \underline{\mathbf{w}}(t)\right).$$

Math 2250-004
Wed Apr 19

Summary of Chapter 7 so far:

- Any initial value problem for a differential equation or system of differential equations can be converted into an equivalent initial value problem for a system of first order differential equations.

- There is a "short-time" existence-uniqueness theorem for first order DE initial value problems

$$\underline{\boldsymbol{x}}'(t) = \underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t))$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

and solutions can be approximated using Euler or Runge-Kutta type algorithms.

- A special case of the IVP above is the one for a first order linear system of differential equations :

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{f}}(t)$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

If the matrix $A(t)$ and the vector function $\underline{\boldsymbol{f}}(t)$ are continuous on an open interval I containing t_0 then a solution $\underline{\boldsymbol{x}}(t)$ exists and is unique, on the entire interval.

- The general solution to an inhomogeneous linear system of DEs

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{f}}(t)$$
i.e.
$$\underline{\boldsymbol{x}}'(t) - A(t)\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{f}}(t)$$

will be of the form $\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}_P + \underline{\boldsymbol{x}}_H$ where $\underline{\boldsymbol{x}}_P$ is a particular solution, and $\underline{\boldsymbol{x}}_H$ is the general solution to the homogeneous system

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t) .$$

- For $A(t)_{n \times n}$ and $\underline{\boldsymbol{x}}(t) \in \mathbb{R}^n$ the solution space on the $t-$interval I to the homogeneous problem

$$\underline{\boldsymbol{x}}'(t) = A(t)\,\underline{\boldsymbol{x}}(t)$$
i.e.
$$\underline{\boldsymbol{x}}'(t) - A(t)\,\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{0}}$$

is n-dimensional. (This is also the "real" reason why the solution spaces to nth order homogeneous linear DE's are n-dimensional, since those DE's are equivalent to homogeneous sytems of n first order linear DE. s.)

- For $A_{n \times n}$ a constant matrix, we try to find a basis for the solution space to

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$

consisting of solutions of the form $\underline{\boldsymbol{x}}(t) = e^{\lambda t}\underline{\boldsymbol{v}}$, where $\underline{\boldsymbol{v}}$ is an eigenvector of A with eigenvalue λ. We will succeed as long as A is diagonalizable.

Finish 7.3 Applications of first order systems of differential equations

- Here are some of the details from Tuesdday's discussion of the **Glucose-insulin model**....

$$\begin{bmatrix} G'(t) \\ H'(t) \end{bmatrix} = \begin{bmatrix} -0.1 & -0.4 \\ 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix}$$

- Characteristic polynomial $|A - \lambda I| = (\lambda + .1)^2 + .04$ has roots $\lambda = -.1 \pm .2\,i$.
- For the eigenvalue $\lambda = -.1 + .2\,i$ we wish to solve the eigenvector system $(A - \lambda I)\underline{\boldsymbol{v}} = \underline{\mathbf{0}}$:

$$\left[\begin{array}{cc|c} -.2\,i & -0.4 & 0 \\ 0.1 & -.2\,i & 0 \end{array}\right]$$

$10\,R_2 \rightarrow R_1,\ -5\,R_1 \rightarrow R_2$:

$$\left[\begin{array}{cc|c} 1 & -2\,i & 0 \\ i & 2 & 0 \end{array}\right]$$

Although it doesn't look like it, the second row is a multiple of the first row, as it must be:
$-iR_1 + R_2 \rightarrow R_2$:

$$\left[\begin{array}{cc|c} 1 & -2\,i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So we may choose the eigenvector $\underline{\boldsymbol{v}} = [2\,i, 1]^T$. This yields a complex function solution

$$\underline{\boldsymbol{z}}(t) = e^{\lambda t}\underline{\boldsymbol{v}} = e^{(-.1 + 2\,i)t}\begin{bmatrix} 2\,i \\ 1 \end{bmatrix}.$$

If we rewrite $\underline{\boldsymbol{z}}(t) = \underline{\boldsymbol{x}}(t) + i\,\underline{\boldsymbol{y}}(t)$ with $\underline{\boldsymbol{x}}(t), \underline{\boldsymbol{y}}(t)$ real vector functions, then each of $\underline{\boldsymbol{x}}(t), \underline{\boldsymbol{y}}(t)$ will be real solutions to the system of differential equations, and will in fact be a basis. (See general discussion at end of Wednesday's notes.) At the end of class we did this decomposition of $\underline{\boldsymbol{z}}(t)$ into real and imaginary parts:

$$e^{(-.1 + 2\,i)t}\begin{bmatrix} 2\,i \\ 1 \end{bmatrix} = e^{-.1\,t}(\cos(.2\,t) + i\sin(.2\,t))\begin{bmatrix} 2\,i \\ 1 \end{bmatrix}$$

$$= e^{-.1\,t}\begin{bmatrix} (\cos(.2\,t) + i\sin(.2\,t))2\,i \\ (\cos(.2\,t) + i\sin(.2\,t))1 \end{bmatrix} = e^{-.1\,t}\begin{bmatrix} -2\sin(.2\,t) \\ \cos(.2\,t) \end{bmatrix} + i\,e^{-.1\,t}\begin{bmatrix} 2\cos(.2\,t) \\ \cos(.2\,t) \end{bmatrix}.$$

This gives real solutions

$$\underline{\boldsymbol{x}}(t) = e^{-.1\,t}\begin{bmatrix} -2\sin(.2\,t) \\ \cos(.2\,t) \end{bmatrix}, \underline{\boldsymbol{y}}(t) = e^{-.1\,t}\begin{bmatrix} 2\cos(.2\,t) \\ \sin(.2\,t) \end{bmatrix}$$

and general homogeneous solution using real functions:

$$\boldsymbol{x}_H(t) = c_1\underline{\boldsymbol{x}}(t) + c_2\underline{\boldsymbol{y}}(t) = c_1 e^{-.1\,t}\begin{bmatrix} -2\sin(.2\,t) \\ \cos(.2\,t) \end{bmatrix} + c_2 e^{-.1\,t}\begin{bmatrix} 2\cos(.2\,t) \\ \sin(.2\,t) \end{bmatrix}.$$

• The Glucose-insulin example is linearized, and is vastly simplified. But mathematicians, doctors, bioengineers, pharmacists, are very interested in (especially more realistic) problems like these. Prof. Fred Adler and a recent graduate student Chris Remien in the Math Department, and collaborating with the University Hospital recently modeled liver poisoning by acetominophen (brand name Tyleonol), by studying a non-linear system of 8 first order differential equations. They came up with a state of the art and very useful diagnostic test:

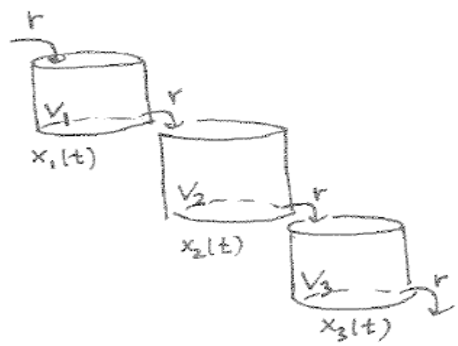
http://unews.utah.edu/news_releases/math-can-save-tylenol-overdose-patients-2/

Here's a link to their published paper. For fun, I copied and pasted the non-linear system of first order differential equations from a preprint of their paper, below:

http://onlinelibrary.wiley.com/doi/10.1002/hep.25656/full
http://www.math.utah.edu/~korevaar/2250spring12/adler-remien-preprint.pdf

APAP	$\frac{dA}{dt} = -\frac{\alpha}{H_{max}} AH - \delta_a A$
NAPQI	$\frac{dN}{dt} = \frac{qp\alpha}{H_{max}} A - \gamma NG$
GSH	$\frac{dG}{dt} = \kappa - \gamma NG - \delta_g G$
Functional Hepatocytes	$\frac{dH}{dt} = rH\left(1 - \frac{H+Z}{H_{max}}\right) - \eta NH$
Damaged Hepatocytes	$\frac{dZ}{dt} = \eta NH - \delta_z Z$
AST	$\frac{dS}{dt} = \frac{d_z \beta_s}{\theta H_{max}} Z - \delta_s (S - S_{min})$
ALT	$\frac{dL}{dt} = \frac{d_z \beta_l}{\theta H_{max}} Z - \delta_l (L - L_{min})$
Clotting Factor	$\frac{dF}{dt} = \beta_f \left(\frac{H}{H_{max}} - F\right)$

Example) consider the three component input-output model below:



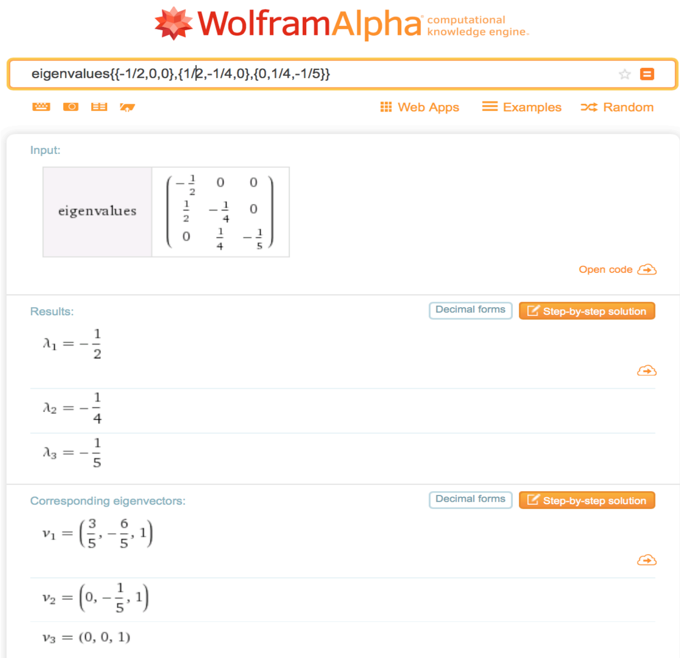
Exercise 1a) Derive the first order system for the tank cascade above.

1b) In case the tank volumes (in gallons) are $V_1 = 20$, $V_2 = 40$, $V_3 = 50$, the flow rate $r = 10 \, \frac{gal}{\min}$, and pure water (with no solute) is flowing into the first (top) tank, show that your system in (a) can be written as

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.25 & 0 \\ 0 & 0.25 & -0.2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

(This system is actually worked out in the text, page 422-434...but we'll modify the IVP, and then consider a second case as well.)

1c) Here's the eigenvector data for the matrix in b. You may want to check or derive parts of it by hand, especially if you're still not expert at finding eigenvalues and eigenvectors. I entered the matrix entries as rational numbers rather than decimals, although in other problems you'd want to use decimals. Use the eigendata to write down the general solution to the system in b.



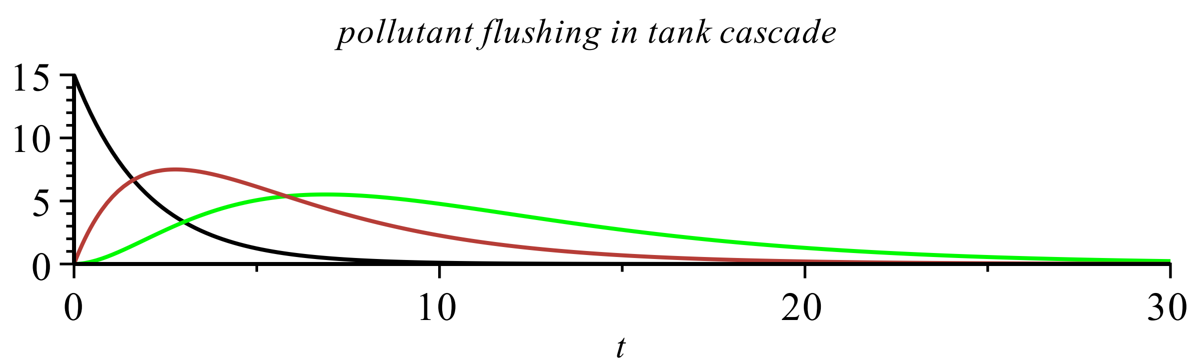
1d) Solve the IVP for this tank cascade, assuming that there are initially 15 *lb* of salt in the first tank, and no salt in the second and third tanks.

Your answer to d should be

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = 5\,\mathrm{e}^{-0.5\,t}\begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix} + 30\,\mathrm{e}^{-0.25\,t}\begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} + 125\,\mathrm{e}^{-0.2\,t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

1e) We can plot the amounts of salt in each tank to figure out what's going on. Make sure you understand how the formulas below are related to the vector equation above, and interpret the graphical results.

```
> with(plots) :
> x1 := t→15·exp(−.5·t) :
  plot1 := plot(x1(t), t=0..30, color=black) :
  x2 := t→ −30·exp(−.5·t) + 30·exp(−.25·t) :
  plot2 := plot(x2(t), t=0..30, color=brown) :
  x3 := t→25·exp(−.5·t) − 150·exp(−.25·t) + 125·exp(−.2·t) :
  plot3 := plot(x3(t), t=0..30, color=green) :
  display({plot1, plot2, plot3}, title=`pollutant flushing in tank cascade`);
```




```
>
```

Exercise 2) Use the same tank cascade. Only now, assume that there is initially 13 *lb* salt in the first tank, none in the others, and that when the water starts flowing the input pipe contains salty water, with concentration $0.5\ \frac{lb}{gal}$.

2a) Explain why this yields an IVP for an inhomogeneous system of linear DE's, namely

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.25 & 0 \\ 0 & 0.25 & -0.2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 13 \\ 0 \\ 0 \end{bmatrix}.$$

2b) Use a vector analog of "undetermined coefficients" to guess that there might be a particular solution that is a constant vector, i.e.

$$\underline{\boldsymbol{x}}_P(t) = \underline{\boldsymbol{d}}.$$

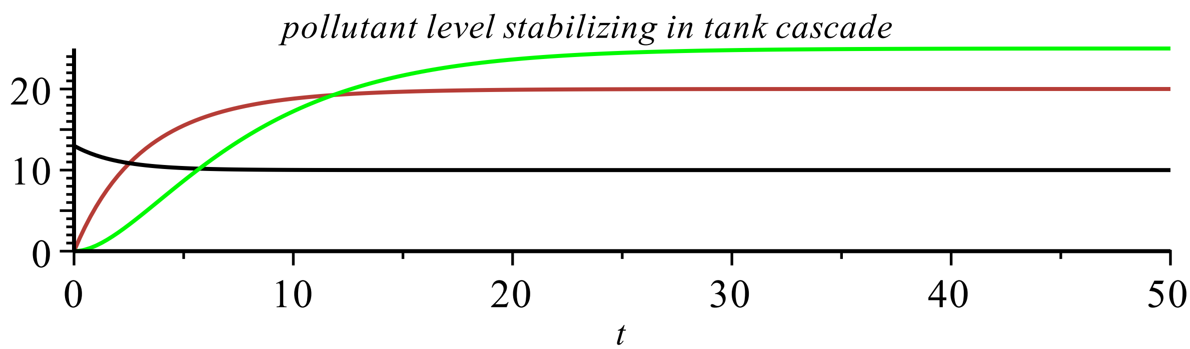
Plug this guess into the inhomgeneous system to deduce $\underline{\boldsymbol{d}}$. In other words, if we write the system with the "linear operator" part of the left as in Chapter 5, we'd write

$$\underline{\boldsymbol{x}}' - A\,\boldsymbol{x} = \underline{\boldsymbol{b}}$$

and we'd guess that since the right side was a constant vector there might be a particular solution that was also a constant vector (with undetermined entries).

2c) Use $\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{x}}_P(t) + \underline{\boldsymbol{x}}_H(t)$ to solve the IVP. Compare your solution to the plots below.

```
> x1 := t→10 + 3·exp(−.5·t) :
  plot1 := plot(x1(t), t=0 ..50, color=black) :
  x2 := t→20 − 6·exp(−.5·t)−14·exp(−.25·t) :
  plot2 := plot(x2(t), t=0 ..50, color=brown) :
  x3 := t→25 + 5·exp(−.5·t) + 70·exp(−.25·t) − 100·exp(−.2·t) :
  plot3 := plot(x3(t), t=0 ..50, color=green) :
  display({plot1, plot2, plot3}, title= `pollutant level stabilizing in tank cascade`);
```

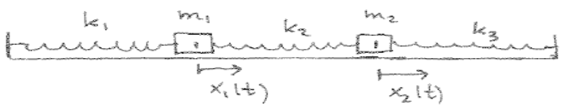


>

Fri Apr 21

7.4 Mass-spring systems and untethered mass-spring trains.

In your homework and lab for this week you study special cases of the spring systems below, with no damping. Although we draw the pictures horizontally, they would also hold in vertical configuration if we measure displacements from equilibrium in the underlying gravitational field.

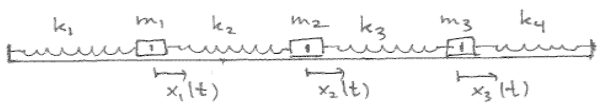


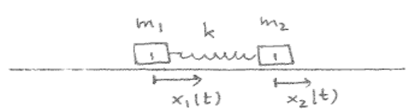
Let's make sure we understand why the natural system of DEs and IVP for this system is

$$m_1 x_1''(t) = -k_1 x_1 + k_2(x_2 - x_1)$$
$$m_2 x_2''(t) = -k_2(x_2 - x_1) - k_3 x_2$$
$$x_1(0) = a_1, \quad x_1'(0) = a_2$$
$$x_2(0) = b_1, \quad x_2'(0) = b_2$$

Exercise 1a) What is the dimension of the solution space to this homogeneous linear system of differential equations? Why?

1b) What if one had a configuration of n masses in series, rather than just 2 masses? What would the dimension of the homogeneous solution space be in this case? Why? Examples:





We can write the system of DEs for the system at the top of page 1 in matrix-vector form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 - k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We denote the diagonal matrix on the left as the "mass matrix" *M*, and the matrix on the right as the spring constant matrix *K* (although to be completely in sync with Chapter 5 it would be better to call the spring matrix $-K$). All of these configurations of masses in series with springs can be written as

$$M\,\underline{\mathbf{x}}''(t) = K\,\underline{\mathbf{x}}.$$

If we divide each equation by the reciprocal of the corresponding mass, we can solve for the vector of accelerations:

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{k_1 + k_2}{m_1} & \dfrac{k_2}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2 + k_3}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

which we write as

$$\underline{\mathbf{x}}''(t) = A\,\underline{\mathbf{x}}.$$

(You can think of *A* as the "acceleration" matrix.)
Notice that the simplification above is mathematically identical to the algebraic operation of multiplying the first matrix equation by the (diagonal) inverse of the diagonal mass matrix *M*. In all cases:

$$M\,\underline{\mathbf{x}}''(t) = K\,\underline{\mathbf{x}} \quad \Rightarrow \underline{\mathbf{x}}''(t) = A\,\underline{\mathbf{x}}\ \text{, with } A = M^{-1}K.$$

How to find a basis for the solution space to conserved-energy mass-spring systems of DEs

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}\ . \qquad (*)$$

Based on our previous experiences, the natural thing for this homogeneous system of linear differential equations is to try and find a basis of solutions of the form

$$\underline{\boldsymbol{x}}(t) = f(t)\underline{\boldsymbol{v}} \qquad (**)$$

You might guess that $f(t) = e^{\lambda t}$ but that turns out to not be the best way to go. Let's see what $f(t)$ should equal by substituting in our guess! (We would maybe also think about first converting the second order system to an equivalent first order system of twice as many DE's, one for for each position function and one for each velocity function, and then the exponential guess would work, but they'd end up being complex exponentials.) Substituting (**) into (*) yields

$$f''(t)\,\underline{\boldsymbol{v}} = A\,(f(t)\underline{v}) = f(t)\,A\,\underline{\boldsymbol{v}}\,.$$

Since for each t, the left side is a scalar multiple of the constant vector $\underline{\boldsymbol{v}}$, so must be the right side. So $\underline{\boldsymbol{v}}$ must be an eigenvector of A,

$$A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}\,,$$

and if $f(t)$ is a real function and if $\underline{\boldsymbol{v}}$ is a real (as opposed to complex) vector, then λ is also real. Then

$$f''(t)\,\underline{\boldsymbol{v}} = A\,(f(t)\underline{v}) = f(t)\,\lambda\,\underline{\boldsymbol{v}}\,.$$

So we must have

$$f''(t) - \lambda f(t) = 0.$$

So possible $f(t)$'s are

Case 1)

$$f(t) = c_1 + c_2 t \qquad \text{if } \lambda = 0$$

Case 2)

$$f(t) = c_1\cos(\omega t) + c_2\sin(\omega t) \qquad \text{if } \lambda < 0,\ \lambda = -\omega^2 \quad \omega = \sqrt{-\lambda}$$

Case 3)

$$f(t) = c_1 e^{\sqrt{\lambda}\,t} + c_2\,e^{-\sqrt{\lambda}\,t} \qquad \text{if } \lambda > 0\,.$$

Case 3 will never happen for our mass-spring configurations, because of conservation of energy!

This leads to the

Solution space algorithm: Consider a very special case of a homogeneous system of linear differential equations,

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}\ .$$

If $A_{n \times n}$ is a diagonalizable matrix and if all of its eigenvalues are non-positive then for each eigenpair $\left(\lambda_j, \underline{\boldsymbol{v}}_j\right)$ with $\lambda_j < 0$ there are two linearly independent sinusoidal solutions to $\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$ given by

$$\boldsymbol{x}_j(t) = \cos\left(\omega_j\, t\right)\underline{\boldsymbol{v}}_j \qquad \underline{\boldsymbol{y}}_j(t) = \sin\left(\omega_j\, t\right)\underline{\boldsymbol{v}}_j$$

with

$$\omega_j = \sqrt{-\lambda_j}\ .$$

And for an eigenpair $\left(\lambda_j, \underline{\boldsymbol{v}}_j\right)$ with $\lambda_j = 0$ there are two indpendent solutions given by constant and linear functions

$$\boldsymbol{x}_j(t) = \underline{\boldsymbol{v}}_j \qquad \underline{\boldsymbol{y}}_j(t) = t\,\underline{\boldsymbol{v}}_j$$

This procedure constructs $2\,n$ independent solutions to the system $\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$, i.e. a basis for the solution space.

Remark: What's amazing is that the fact that if the system is conservative, the acceleration matrix will always be diagonalizable, and all of its eigenvalues will be non-positive. In fact, if the system is tethered to at least one wall (as in the first two diagrams on page 1), all of the eigenvalues will be strictly negative, and the algorithm above will always yield a basis for the solution space. (If the system is not tethered and is free to move as a train, like the third diagram on page 1, then $\lambda = 0$ will be one of the eigenvalues, and will yield the constant velocity and displacement contribution to the solution space, $\left(c_1 + c_2\, t\right)\underline{\boldsymbol{v}}$, where $\underline{\boldsymbol{v}}$ is the corresponding eigenvector. Together with the solutions from strictly negative eigenvalues this will still lead to the general homogeneous solution.)

Exercise 2) Consider the special case of the configuration on page one for which $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$. In this case, the equation for the vector of the two mass accelerations reduces to

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{2\,k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2\,k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{k}{m} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

a) Find the eigendata for the matrix

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

b) Deduce the eigendata for the acceleration matrix A which is $\frac{k}{m}$ times this matrix.

c) Find the $4-$ dimensional solution space to this two-mass, three-spring system.

solution The general solution is a superposition of two "fundamental modes". In the slower mode both masses oscillate "in phase", with equal amplitudes, and with angular frequency $\omega_1 = \sqrt{\frac{k}{m}}$. In the faster mode, both masses oscillate "out of phase" with equal amplitudes, and with angular frequency $\omega_2 = \sqrt{\frac{3\,k}{m}}$. The general solution can be written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1\cos\left(\omega_1 t - \alpha_1\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2\cos\left(\omega_2 t - \alpha_2\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \left(c_1\cos\left(\omega_1 t\right) + c_2\sin\left(\omega_1 t\right)\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(c_3\cos\left(\omega_2 t\right) + c_4\sin\left(\omega_2 t\right)\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Exercise 3) Show that the general solution above lets you uniquely solve each IVP uniquely. This should reinforce the idea that the solution space to these two second order linear homgeneous DE's is four dimensional.

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{2\,k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2\,k}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1(0) = a_1, \quad x_1'(0) = a_2$$
$$x_2(0) = b_1, \quad x_2'(0) = b_2$$

Experiment: Although we probably won't have time in class to measure the spring constants, I've measured them earlier. We can predict the numerical values for the two fundamental modes of the vertical mass-spring configuration corresponding to Exercise 2, and then check our predictions like we did for the single mass-spring configuration, I have brought along a demonstration so that we can see these two vibrations.

> $Digits := 5:$

$k := \frac{.05 \cdot 9.806}{.153};$

$\omega 1 := \sqrt{\frac{k}{.05}};\ T1 := evalf\left(\frac{2 \cdot \pi}{\omega 1}\right);$

$\omega 2 := \sqrt{3.0} \cdot \omega 1;\ T2 := evalf\left(\frac{2 \cdot \pi}{\omega 2}\right);$

$$k := 3.2046$$

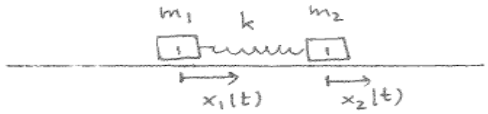
$$\omega 1 := 8.0057$$

$$T1 := 0.78483$$

$$\omega 2 := 13.867$$

$$T2 := 0.45311 \quad (2)$$

Exercise 4) Consider a train with two cars connected by a spring:



4a) Verify that the linear system of DEs that governs the dynamics of this configuration (it's actually a special case of what we did before, with two of the spring constants equal to zero) is

$$x_1'' = \frac{k}{m_1}\left(x_2 - x_1\right)$$

$$x_2'' = -\frac{k}{m_2}\left(x_2 - x_1\right)$$

4b). Use the eigenvalues and eigenvectors computed below to find the general solution. For $\lambda = 0$ and its corresponding eigenvector $\underline{\boldsymbol{v}}$ remember that you get two solutions

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{v}} \text{ and } \underline{\boldsymbol{x}}(t) = t\,\underline{\boldsymbol{v}},$$

rather than the expected $\cos(\omega t)\underline{\boldsymbol{v}}$, $\sin(\omega t)\underline{\boldsymbol{v}}$. Interpret these solutions in terms of train motions. You will use these ideas in some of your homework problems and in your lab exercise about molecular vibrations.

> *Eigenvectors* $\left(\begin{bmatrix} -\frac{k}{m_1} & \frac{k}{m_1} \\ \frac{k}{m_2} & -\frac{k}{m_2} \end{bmatrix}\right)$;

$$\begin{bmatrix} 0 \\ -\frac{k\,(m_1 + m_2)}{m_2\,m_1} \end{bmatrix}, \begin{bmatrix} 1 & -\frac{m_2}{m_1} \\ 1 & 1 \end{bmatrix} \tag{3}$$